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Boundedness character of a fourth-order system of difference equations

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Abstract

The boundedness character of positive solutions of the following system of difference equations: $x_{n+1} = A + \frac{y_n^p}{x_{n-3}^r}$, $y_{n+1} = A + \frac{x_n^p}{y_{n-3}^r}$, $n \in \mathbb{N}_0$, when $\min\{A, r\} > 0$ and $p \ge 0$, is studied.

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1 Introduction

Concrete nonlinear difference equations and systems, especially those which are not closely related to differential ones, have attracted a lot of attention recently (see, for example, [1–33] and the references therein). Among them, symmetric and close to symmetric systems of difference equations, whose study was essentially initiated by Papaschinopoulos and Schinas in the mid-1990s, have attracted a considerable interest (see, for example, [4, 7–11, 13, 14, 22–32]). For example, in [7] Papaschinopoulos and Schinas studied the oscillatory behavior, the boundedness character, and the global stability of positive solutions of the following close to symmetric system of difference equations:

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \qquad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n \in \mathbb{N}_0,$$

where $A > 0$ and $p, q \in \mathbb{N}$. It should be noted that the system is rational. On the other hand, for the case $p = q$ the system obviously becomes symmetric, that is, it is of the following form:

$$x_n = f(x_{n-k}, y_{n-l}), \qquad y_n = f(y_{n-k}, x_{n-l}), \quad n \in \mathbb{N}_0,$$

for some $k, l \in \mathbb{N}$.

On the other hand, a systematic study of positive solutions of nonlinear difference equations containing non-integer powers of their dependent variables began by Stević *et al.*, approximately since the publication of [15], where the first nontrivial results related to the following difference equation were given:

$$x_{n+1} = \alpha + \frac{x_{n-1}^p}{x_n^p}, \quad n \in \mathbb{N}_0, \tag{1}$$

where $\min\{\alpha, p\} > 0$.

A good prototype including (1) is the following difference equation:

$$x_n = \alpha + \frac{x_{n-k}^p}{x_{n-l}^r}, \quad n \in \mathbb{N}_0, \tag{2}$$

where $k, l \in \mathbb{N}$, $k \neq l$, $\min\{\alpha, r\} > 0$, and $p \geq 0$, which was proposed for studying by Stević at numerous talks. Some special cases of this, the corresponding max-type difference equation or related equations has been studied considerably (see, for example, [1, 2, 5, 12, 16–21, 30, 31] and the references therein).

Motivated by these two lines of investigations Stević has proposed recently studying symmetric and close to symmetric systems of difference equations which, among others, stem from special cases of (2).

Motivated by all above mentioned work, and especially by [19], here we investigate the boundedness character of the solutions of the next system of difference equations

$$x_{n+1} = A + \frac{y_n^p}{x_{n-3}^r}, \qquad y_{n+1} = A + \frac{x_n^p}{y_{n-3}^r}, \quad n \in \mathbb{N}_0, \tag{3}$$

when $\min\{A, r\} > 0$, $p \geq 0$, and $x_{-i}, y_{-i} > 0$, $i \in \{0, 1, 2, 3\}$. Our results extend and complement some results in [19].

By using the induction and the equations in (3) we see that if $x_{-i}, y_{-i} > 0$, $i \in \{0, 1, 2, 3\}$, then

$$\min\{x_n, y_n\} > 0, \quad n \geq -3,$$

which means that positive initial values generate positive solutions of system (3). Moreover, we have

$$\min\{x_n, y_n\} > A, \quad n \in \mathbb{N}. \tag{4}$$

The case $p = 0$ is simple. Namely, in this case by using (4) into (3) is obtained

$$\max\{x_{n+1}, y_{n+1}\} < A + \frac{1}{A^r}, \quad n \geq 4,$$

which means that all positive solutions of system (3) in this case are bounded. In fact, since

$$A < \min\{x_n, y_n\} \leq \max\{x_n, y_n\} < A + \frac{1}{A^r}, \quad n \geq 5,$$

they are persistent.

For a solution $(x_n, y_n)_{n \geq -3}$ of system (3) it is said that it is *unbounded* if

$$\sup_{n \geq -3} \|(x_n, y_n)\|_{\mathbb{R}^2} = \sup_{n \geq -3} \sqrt{x_n^2 + y_n^2} = +\infty.$$

Otherwise, the solution is *bounded*, that is, if there is a nonnegative constant M such that

$$\sup_{n\ge -3}\big\|(x_n,y_n)\big\|_{\mathbb{R}^2}\le M<+\infty.$$

2 Main results

In this section we prove the main results in this paper, all of which are related to the boundedness character, that is, the boundedness of all positive solutions of system (3) or the existence of an unbounded solution of the system depending on the values of parameters A, p, and r.

Theorem 1 *Assume that* $\min\{A,p,r\}>0$ *and* $27p^4<256r$. *Then all positive solutions of system* (3) *are bounded.*

Proof Using the equations in (3), we have

$$\begin{aligned}
x_{n+1} &= A+\frac{y_n^p}{x_{n-3}^r}\\
&= A+\left(\frac{y_n}{x_{n-3}^{\frac{r}{p}}}\right)^p\\
&= A+\left(\frac{A}{x_{n-3}^{\frac{r}{p}}}+\frac{x_{n-1}^p}{x_{n-3}^{\frac{r}{p}}y_{n-4}^r}\right)^p\\
&= A+\left(\frac{A}{x_{n-3}^{\frac{r}{p}}}+\left(\frac{x_{n-1}}{x_{n-3}^{\frac{r}{p^2}}y_{n-4}^{\frac{r}{p}}}\right)^p\right)^p\\
&= A+\left(\frac{A}{x_{n-3}^{\frac{r}{p}}}+\left(\frac{A}{x_{n-3}^{\frac{r}{p^2}}y_{n-4}^{\frac{r}{p}}}+\frac{y_{n-2}^p}{x_{n-3}^{\frac{r}{p^2}}y_{n-4}^{\frac{r}{p}}x_{n-5}^r}\right)^p\right)^p\\
&= A+\left(\frac{A}{x_{n-3}^{\frac{r}{p}}}+\left(\frac{A}{x_{n-3}^{\frac{r}{p^2}}y_{n-4}^{\frac{r}{p}}}+\left(\frac{y_{n-2}}{x_{n-3}^{\frac{r}{p^3}}y_{n-4}^{\frac{r}{p^2}}x_{n-5}^{\frac{r}{p}}}\right)^p\right)^p\right)^p\\
&= A+\left(\frac{A}{x_{n-3}^{\frac{r}{p}}}+\left(\frac{A}{x_{n-3}^{\frac{r}{p^2}}y_{n-4}^{\frac{r}{p}}}+\left(\frac{A}{x_{n-3}^{\frac{r}{p^3}}y_{n-4}^{\frac{r}{p^2}}x_{n-5}^{\frac{r}{p}}}+\frac{x_{n-3}^{p-\frac{r}{p^3}}}{y_{n-4}^{\frac{r}{p^2}}x_{n-5}^{\frac{r}{p}}y_{n-6}^r}\right)^p\right)^p\right)^p \qquad (5)\\
&= A+\left(\frac{A}{x_{n-3}^{\frac{r}{p}}}+\left(\frac{A}{x_{n-3}^{\frac{r}{p^2}}y_{n-4}^{\frac{r}{p}}}+\left(\frac{A}{x_{n-3}^{\frac{r}{p^3}}y_{n-4}^{\frac{r}{p^2}}x_{n-5}^{\frac{r}{p}}}\right.\right.\right.\\
&\quad\left.\left.\left.+\left(\frac{x_{n-3}}{y_{n-4}^{\frac{r}{p^2}/(p-\frac{r}{p^3})}x_{n-5}^{\frac{r}{p}/(p-\frac{r}{p^3})}y_{n-6}^{r/(p-\frac{r}{p^3})}}\right)^{p-\frac{r}{p^3}}\right)^p\right)^p\right)^p\\
&= A+\left(\frac{A}{x_{n-3}^{\frac{r}{p}}}+\left(\frac{A}{x_{n-3}^{\frac{r}{p^2}}y_{n-4}^{\frac{r}{p}}}+\left(\frac{A}{x_{n-3}^{\frac{r}{p^3}}y_{n-4}^{\frac{r}{p^2}}x_{n-5}^{\frac{r}{p}}}+\left(\frac{x_{n-3}}{y_{n-4}^{a_1}x_{n-5}^{b_1}y_{n-6}^{c_1}}\right)^{p-\frac{r}{p^3}}\right)^p\right)^p\right)^p, \qquad (6)
\end{aligned}$$

where

$$a_1:=\frac{r}{p^2}\Big/\left(p-\frac{r}{p^3}\right),\qquad b_1=\frac{r}{p}\Big/\left(p-\frac{r}{p^3}\right),\qquad c_1:=r\Big/\left(p-\frac{r}{p^3}\right).$$

Now using the first equation in (3) in (6) we get

$$
\begin{aligned}
x_{n+1} &= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \left(\frac{A}{x_{n-3}^{\frac{r}{p^2}} y_{n-4}^{\frac{r}{p}}} + \left(\cdots + \left(\frac{A}{y_{n-4}^{a_1} x_{n-5}^{b_1} y_{n-6}^{c_1}} + \frac{y_{n-4}^{p-a_1}}{x_{n-5}^{b_1} y_{n-6}^{c_1} x_{n-7}^{r}} \right)^{p-\frac{r}{p^3}} \right)^{p} \right)^{p} \right)^{p} \\
&= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \left(\frac{A}{x_{n-3}^{\frac{r}{p^2}} y_{n-4}^{\frac{r}{p}}} + \left(\cdots + \left(\frac{A}{y_{n-4}^{a_1} x_{n-5}^{b_1} y_{n-6}^{c_1}} \right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. + \left(\frac{y_{n-4}}{x_{n-5}^{\frac{b_1}{p-a_1}} y_{n-6}^{\frac{c_1}{p-a_1}} x_{n-7}^{\frac{r}{p-a_1}}} \right)^{p-a_1} \right)^{p-\frac{r}{p^3}} \right)^{p} \right)^{p} \right)^{p} \\
&= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \left(\frac{A}{x_{n-3}^{\frac{r}{p^2}} y_{n-4}^{\frac{r}{p}}} + \left(\cdots + \left(\frac{A}{y_{n-4}^{a_1} x_{n-5}^{b_1} y_{n-6}^{c_1}} \right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. + \left(\frac{y_{n-4}}{x_{n-5}^{a_2} y_{n-6}^{b_2} x_{n-7}^{c_2}} \right)^{p-a_1} \right)^{p-\frac{r}{p^3}} \right)^{p} \right)^{p} \right)^{p},
\end{aligned} \tag{7}
$$

where

$$
a_2 := \frac{b_1}{p-a_1}, \qquad b_2 = \frac{c_1}{p-a_1}, \qquad c_2 := \frac{r}{p-a_1}.
$$

Assume that for some $k \geq 2$ we have proved that the following equalities hold:

$$
x_{n+1} = A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \cdots \left(\frac{x_{n-2k+1}}{y_{n-2k}^{a_{2k-3}} x_{n-2k-1}^{b_{2k-3}} y_{n-2k-2}^{c_{2k-3}}} \right)^{p-a_{2k-4}} \cdots \right)^{p} \tag{8}
$$

$$
= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \cdots \left(\frac{y_{n-2k}}{x_{n-2k-1}^{a_{2k-2}} y_{n-2k-2}^{b_{2k-2}} x_{n-2k-3}^{c_{2k-2}}} \right)^{p-a_{2k-3}} \cdots \right)^{p}, \tag{9}
$$

where the sequences a_k, b_k, and c_k are defined by

$$
a_{k+1} = \frac{b_k}{p-a_k}, \qquad b_{k+1} = \frac{c_k}{p-a_k}, \qquad c_{k+1} = \frac{r}{p-a_k}, \tag{10}
$$

with

$$
a_0 = \frac{r}{p^3}, \qquad b_0 = \frac{r}{p^2}, \qquad c_0 = \frac{r}{p}.
$$

Using again the equations in (3) and the recurrent relations in (10), we have

$$
\begin{aligned}
x_{n+1} &= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \cdots \left(\frac{y_{n-2k}}{x_{n-2k-1}^{a_{2k-2}} y_{n-2k-2}^{b_{2k-2}} x_{n-2k-3}^{c_{2k-2}}} \right)^{p-a_{2k-3}} \cdots \right)^{p} \\
&= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \cdots \left(\frac{A}{x_{n-2k-1}^{a_{2k-2}} y_{n-2k-2}^{b_{2k-2}} x_{n-2k-3}^{c_{2k-2}}} \right.\right. \\
&\quad \left.\left. + \frac{x_{n-2k-1}^{p-a_{2k-2}}}{y_{n-2k-2}^{b_{2k-2}} x_{n-2k-3}^{c_{2k-2}} y_{n-2k-4}^{r}} \right)^{p-a_{2k-3}} \cdots \right)^{p}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \cdots \left(\frac{A}{x_{n-2k-1}^{a_{2k-2}} y_{n-2k-2}^{b_{2k-2}} x_{n-2k-3}^{c_{2k-2}}}\right.\right. \\
&\quad \left.\left. + \left(\frac{x_{n-2k-1}}{y_{n-2k-2}^{\frac{b_{2k-2}}{p-a_{2k-2}}} x_{n-2k-3}^{\frac{c_{2k-2}}{p-a_{2k-2}}} y_{n-2k-4}^{\frac{r}{p-a_{2k-2}}}}\right)^{p-a_{2k-2}}\right)^{p-a_{2k-3}} \cdots \right)^{p} \\
&= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \cdots \left(\frac{A}{x_{n-2k-1}^{a_{2k-2}} y_{n-2k-2}^{b_{2k-2}} x_{n-2k-3}^{c_{2k-2}}}\right.\right. \\
&\quad \left.\left. + \left(\frac{x_{n-2k-1}}{y_{n-2k-2}^{a_{2k-1}} x_{n-2k-3}^{b_{2k-1}} y_{n-2k-4}^{c_{2k-1}}}\right)^{p-a_{2k-2}}\right)^{p-a_{2k-3}} \cdots \right)^{p}, \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
x_{n+1} &= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \cdots \left(\frac{A}{y_{n-2k-2}^{a_{2k-1}} x_{n-2k-3}^{b_{2k-1}} y_{n-2k-4}^{c_{2k-1}}}\right.\right. \\
&\quad \left.\left. + \frac{y_{n-2k-2}^{p-a_{2k-1}}}{x_{n-2k-3}^{b_{2k-1}} y_{n-2k-4}^{c_{2k-1}} x_{n-2k-5}^{r}}\right)^{p-a_{2k-2}} \cdots \right)^{p} \qquad (13) \\
&= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \cdots \left(\frac{y_{n-2k-2}}{x_{n-2k-3}^{\frac{b_{2k-1}}{p-a_{2k-1}}} y_{n-2k-4}^{\frac{c_{2k-1}}{p-a_{2k-1}}} x_{n-2k-5}^{\frac{r}{p-a_{2k-1}}}}\right)^{p-a_{2k-1}} \cdots \right)^{p} \\
&= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \cdots \left(\frac{y_{n-2k-2}}{x_{n-2k-3}^{a_{2k}} y_{n-2k-4}^{b_{2k}} x_{n-2k-5}^{c_{2k}}}\right)^{p-a_{2k-1}} \cdots \right)^{p}. \qquad (14)
\end{aligned}
$$

From (6), (7), (12), (14), and the method of induction it follows that (8) and (9) hold for every $k \geq 2$, and for every $n \geq 2k$.

If $p^4 > r$, then

$$
\begin{aligned}
a_0 &= \frac{r}{p^3} < \frac{r}{p^2} \Big/ \left(p - \frac{r}{p^3}\right) = a_1, \\
b_0 &= r/p^2 < \frac{r}{p} \Big/ \left(p - \frac{r}{p^3}\right) = b_1, \\
c_0 &= r/p < r \Big/ \left(p - \frac{r}{p^3}\right) = c_1.
\end{aligned}
$$

From this and by using recurrent relations (10), it follows that a_k, b_k, and c_k increase, as far as $a_k < p$. On the other hand, (10) implies

$$a_{k+1} = \frac{r}{(p - a_k)(p - a_{k-1})(p - a_{k-2})}, \quad k \geq 2.$$

Hence, if $a_k < p$ for every $k \in \mathbb{N}$ we see that there is a finite limit $\lim_{k\to\infty} a_k = x^* \in (0, p]$, and that x^* is a solution of the equation

$$f(x) = x(p - x)^3 - r = 0.$$

We have $f(0) = f(p) = -r$ and $f'(x) = (x - p)^2(p - 4x)$. Hence $\max_{x\in[0,p]} f(x) = f(p/4)$. Since by a condition of the theorem

$$f(p/4) = \frac{27p^4}{256} - r < 0,$$

we arrive at a contradiction.

This guarantees the existence of the smallest $l \in \mathbb{N}$ such that $a_{l-1} < p$ and $a_l \geq p$. This, along with (13) with $l = 2k_0 - 1$, implies that

$$\begin{aligned} x_{n+1} &= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \cdots \left(\frac{A}{y_{n-2k_0-2}^{a_{2k_0-1}} x_{n-2k_0-3}^{b_{2k_0-1}} y_{n-2k_0-4}^{c_{2k_0-1}}} \right.\right. \\ &\quad \left.\left. + \frac{y_{n-2k_0-2}^{p-a_{2k_0-1}}}{x_{n-2k_0-3}^{b_{2k_0-1}} y_{n-2k_0-4}^{c_{2k_0-1}} x_{n-2k_0-5}^{r}} \right)^{p-a_{2k_0-2}} \cdots \right)^p \\ &\leq A + \left(\frac{1}{A^{\frac{r}{p}-1}} + \cdots \left(\frac{1}{A^{a_{2k_0-1}+b_{2k_0-1}+c_{2k_0-1}-1}} \right.\right. \\ &\quad \left.\left. + \frac{1}{A^{a_{2k_0-1}+b_{2k_0-1}+c_{2k_0-1}+r-p}} \right)^{p-a_{2k_0-2}} \cdots \right)^p, \end{aligned}$$

for $n \geq 2k_0 + 6$, which implies the boundedness of x_n in this case. Due to the symmetry of system (3) the boundedness of y_n follows and consequently the boundedness of the solution. If $l = 2k_0 - 2$, then from (11) it follows that

$$\begin{aligned} x_{n+1} &= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \cdots \left(\frac{A}{x_{n-2k_0-1}^{a_{2k_0-2}} y_{n-2k_0-2}^{b_{2k_0-2}} x_{n-2k_0-3}^{c_{2k_0-2}}} \right.\right. \\ &\quad \left.\left. + \frac{x_{n-2k_0-1}^{p-a_{2k_0-2}}}{y_{n-2k_0-2}^{b_{2k_0-2}} x_{n-2k_0-3}^{c_{2k_0-2}} y_{n-2k_0-4}^{r}} \right)^{p-a_{2k_0-3}} \cdots \right)^p \\ &\leq A + \left(\frac{1}{A^{\frac{r}{p}-1}} + \cdots \left(\frac{1}{A^{a_{2k_0-2}+b_{2k_0-2}+c_{2k_0-2}-1}} \right.\right. \\ &\quad \left.\left. + \frac{1}{A^{a_{2k_0-2}+b_{2k_0-2}+c_{2k_0-2}+r-p}} \right)^{p-a_{2k_0-3}} \cdots \right)^p, \end{aligned}$$

for $n \geq 2k_0 + 5$, which implies the boundedness of x_n in this case.

Due to the symmetry of system (3) we also have

$$y_{n+1} \leq A + \left(\frac{1}{A^{\frac{r}{p}-1}} + \cdots \left(\frac{1}{A^{a_{2k_0-2}+b_{2k_0-2}+c_{2k_0-2}-1}} + \frac{1}{A^{a_{2k_0-2}+b_{2k_0-2}+c_{2k_0-2}+r-p}} \right)^{p-a_{2k_0-3}} \cdots \right)^p,$$

for $n \geq 2k_0 + 5$, which along with the previous inequality implies the boundedness of the solution.

If $p^4 \leq r$, then using (4) in (5) we get

$$\begin{aligned} x_{n+1} &= A + \left(\frac{A}{x_{n-3}^{\frac{r}{p}}} + \left(\frac{A}{x_{n-3}^{\frac{r}{p^2}} y_{n-4}^{\frac{r}{p}}} + \left(\frac{A}{x_{n-3}^{\frac{r}{p^3}} y_{n-4}^{\frac{r}{p^2}} x_{n-5}^{\frac{r}{p}}} + \frac{x_{n-3}^{p-\frac{r}{p^3}}}{y_{n-4}^{\frac{r}{p^2}} x_{n-5}^{\frac{r}{p}} y_{n-6}^{r}} \right)^p \right)^p \right)^p \\ &\leq A + \left(\frac{1}{A^{\frac{r}{p}-1}} + \left(\frac{1}{A^{\frac{r}{p^2}+\frac{r}{p}-1}} + \left(\frac{1}{A^{\frac{r}{p^3}+\frac{r}{p^2}+\frac{r}{p}-1}} + \frac{1}{A^{\frac{r}{p^3}+\frac{r}{p^2}+\frac{r}{p}+r-p}} \right)^p \right)^p \right)^p, \end{aligned}$$

for $n \geq 7$, from which the boundedness follows in the case.

Due to the symmetry of system (3) we see that the inequality

$$y_{n+1} \leq A + \left(\frac{1}{A^{\frac{r}{p}-1}} + \left(\frac{1}{A^{\frac{r}{p^2}+\frac{r}{p}-1}} + \left(\frac{1}{A^{\frac{r}{p^3}+\frac{r}{p^2}+\frac{r}{p}-1}} + \frac{1}{A^{\frac{r}{p^3}+\frac{r}{p^2}+\frac{r}{p}+r-p}} \right)^p \right)^p \right)^p,$$

holds for $n \geq 7$, from which along with the previous inequality the boundedness of the solution follows. □

Remark 1 Note that if $a_k = p$ for some $k \in \mathbb{N}$, then a_{k+1}, b_{k+1}, and c_{k+1} are not defined. However, if this happens then above mentioned index l is chosen to be this k. For such chosen l is obtained an upper bound for positive solutions of system (3) in the way described in the proof of Theorem 1.

Theorem 2 *Assume that* $\min\{A, p, r\} > 0$, $27p^4 \geq 256r$, *and* $p \geq 4/3$ (*where at least one of these two inequalities is strict*), *or* $r < p - 1 < 1/3$. *Then system* (3) *has positive unbounded solutions*.

Proof Assume that $(x_n, y_n)_{n\geq -3}$ is a positive solution of (3). Then we have

$$x_{n+1} \geq \frac{y_n^p}{x_{n-3}^r}, \tag{15}$$

$$y_{n+1} \geq \frac{x_n^p}{y_{n-3}^r}, \tag{16}$$

for $n \in \mathbb{N}_0$.

Let

$$z_n = \ln(x_n y_n), \quad n \geq -3.$$

Taking the logarithm of the both sides in (15), (16), then summing such obtained inequalities, it follows that

$$z_{n+1} - pz_n + rz_{n-3} \geq 0, \quad n \in \mathbb{N}_0. \tag{17}$$

Let

$$P(\lambda) = \lambda^4 - p\lambda^3 + r. \tag{18}$$

Then $P(0) = r$ and

$$P'(\lambda) = \lambda^2(4\lambda - 3p),$$

from which it follows that the polynomial $P(\lambda)$ has a local minimum at $\lambda = 3p/4$, and according to the conditions of the theorem

$$P(3p/4) = -27p^4/256 + r \leq 0. \tag{19}$$

If $p > 4/3$, then $3p/4 > 1$. From this, (19) and since

$$\lim_{\lambda\to+\infty} P(\lambda) = +\infty, \tag{20}$$

it follows that there is $\lambda_1 > 1$ such that $P(\lambda_1) = 0$. If $p = 4/3$, inequality (19) is strict, $3p/4 = 1$, and since (20) holds, we also see that there is $\lambda_1 > 1$ such that $P(\lambda_1) = 0$.

Now assume that $r < p - 1 < 1/3$. Then $P(1) = 1 - p + r < 0$ and since (20) holds, we again see that there is $\lambda_1 > 1$ such that $P(\lambda_1) = 0$.

Let

$$P_2(\lambda) = P(\lambda)/(\lambda - \lambda_1) = \lambda^3 + a\lambda^2 + b\lambda + c$$

and

$$u_n = z_n + az_{n-1} + bz_{n-2} + cz_{n-3}.$$

Then inequality (17) can be written in the following form:

$$u_{n+1} - \lambda_1 u_n \geq 0. \tag{21}$$

Choose $x_{-i}, y_{-i}, i \in \{0, 1, 2, 3\}$, such that $u_0 > 0$. For example, to get $u_0 > 0$, it is enough to choose $x_{-i}, y_{-i}, i \in \{0, 1, 2, 3\}$, such that

$$x_0 y_0 > |a||x_{-1}y_{-1}| + |b||x_{-2}y_{-2}| + |c||x_{-3}y_{-3}|.$$

From this and (21) it follows that

$$u_n \geq \lambda_1^n u_0. \tag{22}$$

Since $u_0 > 0$ and $\lambda_1 > 1$, by letting $n \to \infty$ in (22) we obtain $u_n \to +\infty$ as $n \to \infty$. If z_n were bounded then u_n would be also bounded, which would be a contradiction. Hence z_n is unbounded. From this and since

$$\sqrt{x_n^2 + y_n^2} \geq 2x_n y_n = 2e_n^z, \quad n \in \mathbb{N}_0,$$

we have

$$\sup_{n \geq -3} \sqrt{x_n^2 + y_n^2} = +\infty,$$

that is, the solution of system (3) is unbounded, completing the proof of the theorem. □

Theorem 3 *Assume that* $\min\{A, p, r\} > 0$ *and* $p = r + 1$. *Then system* (3) *has positive unbounded solutions.*

Proof Let

$$x_0 y_0 > x_{-1} y_{-1} > x_{-2} y_{-2} > x_{-3} y_{-3} > 0. \tag{23}$$

Since $p = r + 1$, system (3) is

$$x_{n+1} = A + \frac{y_n^{r+1}}{x_{n-3}^r}, \qquad y_{n+1} = A + \frac{x_n^{r+1}}{y_{n-3}^r}, \quad n \in \mathbb{N}_0. \tag{24}$$

Multiplying these two equations we easily obtain

$$\frac{x_{n+1}y_{n+1}}{x_n y_n} > \left(\frac{x_n y_n}{x_{n-3}y_{n-3}}\right)^r, \tag{25}$$

from which with $n = 0$, it follows that

$$\frac{x_1 y_1}{x_0 y_0} > \left(\frac{x_0 y_0}{x_{-3}y_{-3}}\right)^r > 1.$$

Assume that we have proved

$$x_k y_k > x_{k-1}y_{k-1}, \quad \text{for } -2 \le k \le n. \tag{26}$$

Then from (25) and (26) we have

$$\frac{x_{n+1}y_{n+1}}{x_n y_n} > \left(\frac{x_n y_n}{x_{n-3}y_{n-3}}\right)^r > 1, \quad n \in \mathbb{N}_0. \tag{27}$$

Hence

$$x_{n+1}y_{n+1} > x_n y_n,$$

for every $n \ge -3$. If $x_n y_n$ was bounded, then there would be a finite positive $\lim_{n\to\infty} x_n y_n = c$. Letting $n \to \infty$ in the product of equations in (24) we would obtain $c \ge A^2 + c$, which would be a contradiction. Hence, all the solutions of (24) satisfying (23) are unbounded. □

Theorem 4 *Assume that* $\min\{A, r\} > 0$ *and* $p \in (0, 1)$. *Then every positive solution of system* (3) *is bounded*.

Proof Since $x_n > A$, $n \in \mathbb{N}$, we have

$$x_{n+1} \le A + \frac{y_n^p}{A^r}, \qquad y_{n+1} \le A + \frac{x_n^p}{A^r},$$

for $n \ge 4$, where $(x_n, y_n)_{n\ge -3}$ is an arbitrary positive solution of system (3).

Hence

$$x_{n+1} + y_{n+1} \le 2A + \frac{2(x_n + y_n)^p}{A^r}, \tag{28}$$

for $n \ge 4$.

Let $(z_n)_{n\ge 4}$ be the solution of the equation

$$z_{n+1} = 2A + \frac{2z_n^p}{A^r}, \quad n \ge 4, \tag{29}$$

such that $z_4 = x_4 + y_4$.

Since

$$f(x) = 2A + \frac{2x^p}{A^r}$$

is increasing on $\mathbb{R}_+$, a simple inductive argument shows that

$$x_n + y_n \le z_n, \quad \text{for } n \ge 4. \tag{30}$$

Since $p \in (0,1)$ function f is concave, which implies that there is a unique fixed point x^* of f and that the next condition

$$\bigl(f(x) - x\bigr)\bigl(x - x^*\bigr) < 0, \quad x \in (0,\infty) \setminus \bigl\{x^*\bigr\}, \tag{31}$$

holds.

If $z_4 \in (0, x^*]$ condition (31) implies that $(z_n)_{n\ge4}$ is nondecreasing and bounded above by x^*, and if $z_4 \ge x^*$ that it is nonincreasing and bounded below by x^*. Hence $(z_n)_{n\ge4}$ is bounded, which along with (30) implies the boundedness of $(x_n)_{n\ge4}$ and $(y_n)_{n\ge4}$, from which the result easily follows. □

In the next theorem we use the fact that the comparison equation is a linear first order difference equation, which is solvable in closed form. For recent application of this and related equations see, for example, [4, 22, 23, 25–29, 33].

Theorem 5 *Assume that* $p = 1$, $r > 0$, *and* $A > \sqrt[r]{2}$. *Then every positive solution of system* (3) *is bounded*.

Proof From the proof of Theorem 4 we see that any positive solution $(x_n, y_n)_{n\ge-3}$ of system (3) satisfies (28) with $p = 1$.

Let $(z_n)_{n\ge4}$ be the solution of the equation

$$z_{n+1} = 2A + \frac{2z_n}{A^r}, \quad n \ge 4, \tag{32}$$

such that $z_4 = x_4 + y_4$. Then clearly (30) also holds.

It is well known that (32) is solvable. Using its solution in closed form is easily proved that

$$\lim_{n\to\infty} z_n = \frac{2A^{r+1}}{A^r - 2},$$

from which the boundedness of $(z_n)_{n\ge4}$ follows. This fact along with (30) implies the boundedness of $(x_n)_{n\ge4}$ and $(y_n)_{n\ge4}$, from which the result easily follows. □

Remark 2 The boundedness character of positive solutions of system (3) in the following two cases:

(a) $r \le 27p^4/256$, $1 < p < r + 1$, $r < 1/3$;
(b) $r \le 27p^4/256$, $p = 1$, $A \in (0, \sqrt[r]{2}]$,

is not known to us. Hence, we leave the cases to the interested reader.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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