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Some properties of Fibonacci numbers, Fibonacci octonions, and generalized Fibonacci-Lucas octonions

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Abstract

In this paper we determine some properties of Fibonacci octonions. Also, we introduce the generalized Fibonacci-Lucas octonions and we investigate some properties of these elements.

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1 Preliminaries

Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence:

$$f_0 = 0; \qquad f_1 = 1; \qquad f_n = f_{n-1} + f_{n-2}, \quad n \geq 2,$$

and let $(l_n)_{n\geq 0}$ be the Lucas sequence:

$$l_0 = 2; \qquad f_1 = 1; \qquad l_n = l_{n-1} + l_{n-2}, \quad n \geq 2.$$

Let $(h_n)_{n\geq 0}$ be the generalized Fibonacci sequence:

$$h_0 = p, \qquad h_1 = q, \qquad h_n = h_{n-1} + h_{n-2}, \quad n \geq 2,$$

where p and q are arbitrary integer numbers. The generalized Fibonacci numbers were introduced of Horadam in [1].

Later Horadam introduced the Fibonacci quaternions and generalized Fibonacci quaternions (in [2]). In [3], Flaut and Shpakivskyi and later in [4], Akyigit *et al.* gave some properties of the generalized Fibonacci quaternions. In [5] and [6], the authors introduced the Fibonacci symbol elements and Lucas symbol elements. Moreover, they proved that all these elements determine $\mathbb{Z}$-module structures. In [7] Kecilioglu and Akkus introduced the Fibonacci and Lucas octonions and they gave some identities and properties of them.

Quaternion algebras, symbol algebras, and octonion algebras have many properties and many applications, as the reader can find in [7–15]. In [13], Kecilioglu and Akkus gave some properties of the split Fibonacci and Lucas octonions in the octonion algebra $\mathcal{O}(1,1,-1)$.

In this paper we study the Fibonacci octonions in certain generalized octonion algebras.

In [12], we introduced the generalized Fibonacci-Lucas quaternions and we determined some properties of these elements. In this paper we introduce the generalized Fibonacci-Lucas octonions and we prove that these elements have similar properties to the properties of the generalized Fibonacci-Lucas quaternions.

2 Properties of the Fibonacci and Lucas numbers

The following properties of Fibonacci and Lucas numbers are well known.

Proposition 2.1 ([16]) *Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence and let $(l_n)_{n\geq 0}$ be the Lucas sequence. The following properties hold*:

(i)
$$f_n + f_{n+2} = l_{n+1}, \quad \forall n \in \mathbb{N};$$

(ii)
$$l_n + l_{n+2} = 5f_{n+1}, \quad \forall n \in \mathbb{N};$$

(iii)
$$f_n^2 + f_{n+1}^2 = f_{2n+1}, \quad \forall n \in \mathbb{N};$$

(iv)
$$l_n^2 + l_{n+1}^2 = l_{2n} + l_{2n+2} = 5f_{2n+1}, \quad \forall n \in \mathbb{N};$$

(v)
$$l_n^2 = l_{2n} + 2(-1)^n, \quad \forall n \in \mathbb{N}^*;$$

(vi)
$$l_{2n} = 5f_n^2 + 2(-1)^n, \quad \forall n \in \mathbb{N}^*;$$

(vii)
$$l_n + f_n = 2f_{n+1}.$$

Proposition 2.2 ([5, 6]) *Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence and let $(l_n)_{n\geq 0}$ be the Lucas sequence. Then*:

(i)
$$f_n + f_{n+3} = 2f_{n+2}, \quad \forall n \in \mathbb{N};$$

(ii)
$$f_n + f_{n+4} = 3f_{n+2}, \quad \forall n \in \mathbb{N};$$

(iii)
$$f_n + f_{n+6} = 2l_{n+3}, \quad \forall n \in \mathbb{N};$$

(iv)
$$f_{n+4} - f_n = l_{n+2}, \quad \forall n \in \mathbb{N}.$$

In the following proposition, we will give other properties of the Fibonacci and Lucas numbers, which will be necessary in the next proofs.

Proposition 2.3 *Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence and $(l_n)_{n\geq 0}$ be the Lucas sequence Then*:

(i)
$$l_{n+4} + l_n = 3l_{n+2}, \quad \forall n \in \mathbb{N}.$$

(ii)
$$l_{n+4} - l_n = 5f_{n+2}, \quad \forall n \in \mathbb{N}.$$

(iii)
$$f_n + f_{n+8} = 7f_{n+4}, \quad \forall n \in \mathbb{N}.$$

Proof (i) Using Proposition 2.1(i) we have

$$l_{n+4} + l_n = f_{n+3} + f_{n+5} + f_{n-1} + f_{n+1}.$$

From Proposition 2.2(ii) and Proposition 2.1(i), we obtain

$$l_{n+4} + l_n = 3f_{n+1} + 3f_{n+3} = 3l_{n+2}.$$

(ii) Applying Proposition 2.1(ii), we have

$$l_{n+4} - l_n = (l_{n+4} + l_{n+2}) - (l_{n+2} + l_n) = 5f_{n+3} - 5f_{n+1} = 5f_{n+2}.$$

(iii) We have

$$f_n + f_{n+8} = (f_n + f_{n+4}) + (f_{n+8} - f_{n+4}).$$

Using Proposition 2.2(ii), (iv), we have

$$f_n + f_{n+8} = 3f_{n+2} + l_{n+6}.$$

From Proposition 2.1(i) and the Fibonacci recurrence, we obtain

$$f_n + f_{n+8} = 3f_{n+2} + f_{n+5} + f_{n+7} = 3f_{n+2} + 2f_{n+5} + f_{n+6} = 3f_{n+2} + 3f_{n+5} + f_{n+4}.$$

Using Proposition 2.2(i), we obtain

$$f_n + f_{n+8} = 6f_{n+4} + f_{n+4} = 7f_{n+4}.$$ □

3 Fibonacci octonions

Let $\mathcal{O}_{\mathbb{R}}(\alpha, \beta, \gamma)$ be the generalized octonion algebra over $\mathbb{R}$ with basis $\{1, e_1, e_2, \ldots, e_7\}$. It is well known that this algebra is an eight-dimensional non-commutative and non-associative algebra.

The multiplication table for the basis of $\mathcal{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$ is

$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-\alpha$	e_3	$-\alpha e_2$	e_5	$-\alpha e_4$	$-e_7$	αe_6
e_2	e_2	$-e_3$	$-\beta$	βe_1	e_6	e_7	$-\beta e_4$	$-\beta e_5$
e_3	e_3	αe_2	$-\beta e_1$	$-\alpha\beta$	e_7	$-\alpha e_6$	βe_5	$-\alpha\beta e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-\gamma$	γe_1	γe_2	γe_3
e_5	e_5	αe_4	$-e_7$	αe_6	$-\gamma e_1$	$-\alpha\gamma$	$-\gamma e_3$	$\alpha\gamma e_2$
e_6	e_6	e_7	βe_4	$-\beta e_5$	$-\gamma e_2$	γe_3	$-\beta\gamma$	$-\beta\gamma e_1$
e_7	e_7	$-\alpha e_6$	βe_5	$\alpha\beta e_4$	$-\gamma e_3$	$-\alpha\gamma e_2$	$\beta\gamma e_1$	$-\alpha\beta\gamma$

Let $x \in \mathcal{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$, $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7$ and its conjugate $\overline{x} = x_0 - x_1e_1 - x_2e_2 - x_3e_3 - x_4e_4 - x_5e_5 - x_6e_6 - x_7e_7$, the norm of x is $n(x) = x\overline{x} = x_0^2 + \alpha x_1^2 + \beta x_2^2 + \alpha\beta x_3^2 + \gamma x_4^2 + \alpha\gamma x_5^2 + \beta\gamma x_6^2 + \alpha\beta\gamma x_7^2 \in \mathbb{R}$.

If, for $x \in \mathcal{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$, we have $n(x) = 0$ if and only if $x = 0$, then the octonion algebra $\mathcal{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$ is called a division algebra. Otherwise $\mathcal{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$ is called a split algebra.

Let K be an algebraic number field. The following criterion is well known to decide if an octonion algebra is a division algebra.

Proposition 3.1 ([17]) *A generalized octonion algebra $\mathcal{O}_K(\alpha,\beta,\gamma)$ is a division algebra if and only if the quaternion algebra $\mathbb{H}_K(\alpha,\beta)$ is a division algebra and the equation $n(x) = -\gamma$ does not have solutions in the quaternion algebra $\mathbb{H}_K(\alpha,\beta)$.*

It is well known that the octonion algebra $\mathcal{O}_{\mathbb{R}}(1,1,1)$ is a division algebra and the octonion algebra $\mathcal{O}_{\mathbb{R}}(1,1,-1)$ is a split algebra (see [13, 18]). In [18] appears the following result, which allows us to decide if an octonion algebra over $\mathbb{R}$, $\mathcal{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$ is a division algebra or a split algebra.

Proposition 3.2 ([18]) *We consider the generalized octonion algebra $\mathcal{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$, with $\alpha,\beta,\gamma \in \mathbb{R}^*$. Then there are the following isomorphisms*:

(i) *if $\alpha,\beta,\gamma > 0$, then the octonion algebra $\mathcal{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$ is isomorphic to the octonion algebra $\mathcal{O}_{\mathbb{R}}(1,1,1)$;*

(ii) *if $\alpha,\beta > 0$, $\gamma < 0$ or $\alpha,\gamma > 0$, $\beta < 0$ or $\alpha < 0$, $\beta,\gamma > 0$ or $\alpha > 0$, $\beta,\gamma < 0$ or $\alpha,\gamma < 0$, $\beta > 0$ or $\alpha,\beta < 0$, $\gamma > 0$ or $\alpha,\beta,\gamma < 0$ then the octonion algebra $\mathcal{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$ is isomorphic to the octonion algebra $\mathcal{O}_{\mathbb{R}}(1,1,-1)$.*

Let n be an integer, $n \geq 0$. In [7], Kecilioglu and Akkus introduced the Fibonacci octonions:

$$F_n = f_n + f_{n+1}e_1 + f_{n+2}e_2 + f_{n+3}e_3 + f_{n+4}e_4 + f_{n+5}e_5 + f_{n+6}e_6 + f_{n+7}e_7,$$

where f_n is nth Fibonacci number.

Now, we consider the generalized octonion algebra $\mathcal{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$, with α, β, γ in arithmetic progression, $\alpha = a+1$, $\beta = 2a+1$, $\gamma = 3a+1$, where $a \in \mathbb{R}$.

In the following, we calculate the norm of a Fibonacci octonion in this octonion algebra.

Proposition 3.3 *Let a be a real number and let F_n be the nth Fibonacci octonion. Then the norm of F_n in the generalized octonion algebra $\mathbb{O}_{\mathbb{R}}(a+1, 2a+1, 3a+1,)$ is*

$$n(F_n) = f_{2n+6}\left(79a^2 + 46a + \frac{174a^3 - 4a}{5}\right) + f_{2n+7}\left(130a^2 + 84a + 21 + \frac{282a^3 + 8a}{5}\right) + (-1)^n\left(4a^2 + \frac{12a^3 + 8a}{5}\right).$$

Proof

$$\begin{aligned} n(F_n) &= f_n^2 + (a+1)f_{n+1}^2 + (2a+1)f_{n+2}^2 + (a+1)(2a+1)f_{n+3}^2 \\ &\quad + (3a+1)f_{n+4}^2 + (a+1)(3a+1)f_{n+5}^2 \\ &\quad + (2a+1)(3a+1)f_{n+6}^2 + (a+1)(2a+1)(3a+1)f_{n+7}^2 \\ &= f_n^2 + f_{n+1}^2 + f_{n+2}^2 + f_{n+3}^2 + f_{n+4}^2 + f_{n+5}^2 + f_{n+6}^2 + f_{n+7}^2 \\ &\quad + a\left(f_{n+1}^2 + 2f_{n+2}^2 + 3f_{n+3}^2 + 3f_{n+4}^2 + 4f_{n+5}^2 + 5f_{n+6}^2 + 6f_{n+7}^2\right) \\ &\quad + a^2\left(2f_{n+3}^2 + 3f_{n+5}^2 + 6f_{n+6}^2 + 11f_{n+7}^2\right) + 6a^3f_{n+7}^2 \\ &= S_1 + S_2 + S_3 + 6a^3f_{n+7}^2, \end{aligned} \tag{3.1}$$

where we denoted $S_1 = f_n^2 + f_{n+1}^2 + f_{n+2}^2 + f_{n+3}^2 + f_{n+4}^2 + f_{n+5}^2 + f_{n+6}^2 + f_{n+7}^2$, $S_2 = a(f_{n+1}^2 + 2f_{n+2}^2 + 3f_{n+3}^2 + 3f_{n+4}^2 + 4f_{n+5}^2 + 5f_{n+6}^2 + 6f_{n+7}^2)$, $S_3 = a^2(2f_{n+3}^2 + 3f_{n+5}^2 + 6f_{n+6}^2 + 11f_{n+7}^2)$.

Now, we calculate S_1, S_2, S_3.

Using [7], p.3, we have

$$S_1 = f_8f_{2n+7} = 21f_{2n+7.} \tag{3.2}$$

Applying Proposition 2.1(iii) and Proposition 2.1(i), we have

$$\begin{aligned} S_2 &= a\left(f_{n+1}^2 + 2f_{n+2}^2 + 3f_{n+3}^2 + 3f_{n+4}^2 + 4f_{n+5}^2 + 5f_{n+6}^2 + 6f_{n+7}^2\right) \\ &= 6a\left(f_{n+6}^2 + f_{n+7}^2\right) - a\left(f_{n+6}^2 + f_{n+5}^2\right) + 5a\left(f_{n+5}^2 + f_{n+4}^2\right) - 2a\left(f_{n+4}^2 + f_{n+3}^2\right) \\ &\quad + 4af_{n+3}^2 + a\left(f_{n+2}^2 + f_{n+3}^2\right) + a\left(f_{n+1}^2 + f_{n+2}^2\right) \\ &= a \cdot \left(6f_{2n+13} - f_{2n+11} + 5f_{2n+9} - 2f_{2n+7} + 4f_{n+3}^2 + f_{2n+5} + f_{2n+3}\right) \\ &= a \cdot \left[6f_{2n+13} - f_{2n+11} + 7f_{2n+9} - 2(f_{2n+7} + f_{2n+9}) + 4f_{n+3}^2 + l_{2n+4}\right] \\ &= a \cdot \left[6(f_{2n+9} + f_{2n+13}) + (f_{2n+9} - f_{2n+11}) + l_{2n+4} - 2l_{2n+8} + 4 \cdot \frac{l_{2n+6} - 2(-1)^{n+3}}{5}\right]. \end{aligned}$$

From Proposition 2.2(ii), Proposition 2.3(ii), and Proposition 2.1(i), we have

$$\begin{aligned} S_2 &= a \cdot \left[18f_{2n+11} - f_{2n+10} - (l_{2n+8} - l_{2n+4}) - l_{2n+8} + 4 \cdot \frac{l_{2n+6} + 2(-1)^n}{5}\right] \\ &= a \cdot \left[18f_{2n+11} - f_{2n+10} - 5f_{2n+6} - f_{2n+7} - f_{2n+9} + 4 \cdot \frac{l_{2n+6} + 2(-1)^n}{5}\right]. \end{aligned}$$

Using several times the recurrence of Fibonacci sequence and Proposition 2.1(vii), we obtain

$$S_2 = a \cdot \left[46f_{2n+6} + 84f_{2n+7} + 4 \cdot \frac{2f_{2n+7} - f_{2n+6} + 2(-1)^n}{5}\right]. \tag{3.3}$$

Applying Proposition 2.1(iii) and Proposition 2.1(vi), (i), we have

$$\begin{aligned} S_3 &= a^2 \cdot \left(2f_{n+3}^2 + 3f_{n+5}^2 + 6f_{n+6}^2 + 11f_{n+7}^2\right) \\ &= a^2 \cdot \left[2\left(f_{n+3}^2 + f_{n+4}^2\right) - 2\left(f_{n+4}^2 + f_{n+5}^2\right) + 5\left(f_{n+5}^2 + f_{n+6}^2\right) + \left(f_{n+6}^2 + f_{n+7}^2\right) + 10f_{n+7}^2\right] \\ &= a^2 \cdot \left[2f_{2n+7} - 2f_{2n+9} + 5f_{2n+11} + f_{2n+13} + 2l_{2n+14} - 4(-1)^{n+7}\right] \\ &= a^2 \cdot \left[-2f_{2n+8} + 5f_{2n+11} + 3f_{2n+13} + 2f_{2n+15} + 4(-1)^n\right]. \end{aligned}$$

From Proposition 2.3(iii) and the recurrence of the Fibonacci sequence, we have

$$\begin{aligned} S_3 &= a^2 \cdot \left[-2f_{2n+7} - 2f_{2n+6} + 5f_{2n+11} + 3f_{2n+13} + 14f_{2n+11} - 2f_{2n+7} + 4(-1)^n\right] \\ &= a^2 \cdot \left[-4f_{2n+7} - 2f_{2n+6} + 19f_{2n+11} + 3f_{2n+13} + 4(-1)^n\right] \\ &= a^2 \cdot \left[-4f_{2n+7} - 2f_{2n+6} + 134f_{2n+7} + 81f_{2n+6} + 4(-1)^n\right]. \end{aligned}$$

Therefore, we obtain

$$S_3 = a^2 \cdot \left[79f_{2n+6} + 130f_{2n+7} + 4(-1)^n\right]. \tag{3.4}$$

From Proposition 2.1(vi), (i), we have

$$6a^3 f_{n+7}^2 = \frac{6a^3}{5} \cdot \left[l_{2n+14} - 2 \cdot (-1)^{n+7}\right] = \frac{6a^3}{5} \cdot \left[f_{2n+13} + f_{2n+15} + 2 \cdot (-1)^n\right].$$

Applying Proposition 2.3(iii) and the recurrence of the Fibonacci sequence many times, we have

$$\begin{aligned} 6a^3 f_{n+7}^2 &= \frac{6a^3}{5} \cdot \left[7f_{2n+9} - f_{2n+5} + 7f_{2n+11} - f_{2n+7} + 2 \cdot (-1)^n\right] \\ &= \frac{6a^3}{5} \cdot \left[29f_{2n+6} + 47f_{2n+7} + 2 \cdot (-1)^n\right]. \end{aligned} \tag{3.5}$$

From (3.1), (3.2), (3.3), (3.4), and (3.5), we have

$$\begin{aligned} n(F_n) = 21f_{2n+7} &+ a \cdot \left[46f_{2n+6} + 84f_{2n+7} + 4 \cdot \frac{2f_{2n+7} - f_{2n+6} + 2(-1)^n}{5}\right] \\ &+ a^2 \cdot \left[79f_{2n+6} + 130f_{2n+7} + 4(-1)^n\right] \\ &+ \frac{6a^3}{5} \cdot \left[29f_{2n+6} + 47f_{2n+7} + 2 \cdot (-1)^n\right]. \end{aligned}$$

Therefore, we get

$$n(F_n) = f_{2n+6}\left(79a^2 + 46a + \frac{174a^3 - 4a}{5}\right) + f_{2n+7}\left(130a^2 + 84a + 21 + \frac{282a^3 + 8a}{5}\right) + (-1)^n\left(4a^2 + \frac{12a^3 + 8a}{5}\right). \qquad \square$$

We obtain immediately the following remark.

Remark 3.1 If a is a real number, $a < -1$, then the generalized octonion algebra $\mathcal{O}_{\mathbb{R}}(a + 1, 2a + 1, 3a + 1)$ is a split algebra.

Proof Using Proposition 3.2(ii) and the fact that the octonion algebra $\mathcal{O}_{\mathbb{R}}(1, 1, -1)$ is a split algebra, as a result we see that, if $a < -1$, the generalized octonion algebra $\mathcal{O}_{\mathbb{R}}(a + 1, 2a + 1, 3a + 1)$ is a split algebra. $\square$

For example, for $a = -4$ we obtain the generalized octonion algebra $\mathcal{O}_{\mathbb{R}}(-3, -7, -11)$. From Remark 3.1 as a result we see that this is a split algebra (another way to prove that this algebra is a split algebra is to remark that the equation $n(x) = 11$ has solutions in the quaternion algebra $\mathbb{H}_K(-3, -7)$ and then to apply Proposition 3.1.

Now, we want to determine how many Fibonacci octonions invertible are in the octonion algebra $\mathcal{O}_{\mathbb{R}}(-3, -7, -11)$. Applying Proposition 3.3, we obtain $n(F_n) = -1144f_{2n+6} - 1851f_{2n+7} - 96(-1)^n$, $n \in \mathbb{N}$. Using that $f_{2n+6}, f_{2n+7} > 0$, $(\forall)n \in \mathbb{N}$, as a result $n(F_n) < 0$, $(\forall)n \in \mathbb{N}$, therefore, in the split octonion algebra $\mathcal{O}_{\mathbb{R}}(-3, -7, -11)$ all Fibonacci octonions are invertible.

For $a = -2$, after a few calculations, we also find that in the split octonion algebra $\mathcal{O}_{\mathbb{R}}(-1, -3, -5)$ all Fibonacci octonions are invertible.

From the above, the following question arises: how many invertible Fibonacci octonions are therein the octonion algebra $\mathcal{O}_{\mathbb{R}}(a + 1, 2a + 1, 3a + 1)$, with $a < -1$? We get the following result.

Proposition 3.4 *Let a be a real number, $a \leq -2$ and let $\mathcal{O}_{\mathbb{R}}(a + 1, 2a + 1, 3a + 1)$ be a generalized octonion algebra. Then, in this algebra, all Fibonacci octonions are invertible elements.*

Proof It is sufficient to prove that $n(F_n) \neq 0$, $(\forall)n \in \mathbb{N}$. Using Proposition 3.3, we have

$$n(F_n) = f_{2n+6}\left(79a^2 + 46a + \frac{174a^3 - 4a}{5}\right) + f_{2n+7}\left(130a^2 + 84a + 21 + \frac{282a^3 + 8a}{5}\right) + (-1)^n\left(4a^2 + \frac{12a^3 + 8a}{5}\right) \quad \Leftrightarrow$$

$$n(F_n) = f_{2n+6} \cdot \frac{174a^3 + 395a^2 + 226a}{5} + f_{2n+7} \cdot \frac{282a^3 + 650a^2 + 428a + 105}{5} + (-1)^n \cdot \frac{12a^3 + 20a^2 + 8a}{5}.$$

After a few calculations, we obtain

$$\begin{aligned} n(F_n) &= f_{2n+6} \cdot \frac{a(a+2)(174a+47)+132a}{5} \\ &\quad + f_{2n+7} \cdot \frac{2(a+2)(141a^2+43a+126)-407}{5} + (-1)^n \cdot \frac{4a(a+2)(3a-1)+16a}{5}. \end{aligned}$$

We remark that $141a^2 + 43a + 126 > 0$, $(\forall)a \in \mathbb{R}$ (since $\Delta < 0$) and

$$\begin{aligned} &\frac{a(a+2)(174a+47)+132a}{5} < 0, \quad (\forall)a \le -2, \\ &\frac{2(a+2)(141a^2+43a+126)-407}{5} < 0, \quad (\forall)a \le -2, \\ &\frac{4a(a+2)(3a-1)+16a}{5} < 0, \quad (\forall)a \le -2. \end{aligned}$$

Since $f_{2n+6}, f_{2n+7} > 0$, $(\forall)n \in \mathbb{N}$, we obtain that $n(F_n) < 0$, $(\forall)a \le -2$, $n \in \mathbb{N}$ (even if n is an odd number). This implies that, in the generalized octonion algebra $\mathbb{O}_{\mathbb{R}}(a+1, 2a+1, 3a+1)$, with $a \le -2$, all Fibonacci octonions are invertible. □

Now, we wonder: what happens with the Fibonacci octonions in the generalized octonion algebra $\mathbb{O}_{\mathbb{R}}(a+1, 2a+1, 3a+1)$, when $a \in (-2, -1)$? Are all Fibonacci octonions in a such octonion algebra invertible or are there Fibonacci octonions zero divisors?

For example, for $a = -\frac{3}{2}$, using Proposition 3.3, we see that the norm of a Fibonacci octonion in the octonion algebra $\mathbb{O}_{\mathbb{R}}(-\frac{1}{2}, -2, -\frac{7}{2})$ is $n(F_n) = -\frac{7}{2} f_{2n+6} + \frac{831}{2} f_{2n+7} - \frac{3}{2} \cdot (-1)^n > 0$, $(\forall)n \in \mathbb{N}^*$. This implies that in the generalized octonion algebra $\mathbb{O}_{\mathbb{R}}(-\frac{1}{2}, -2, -\frac{7}{2})$ all Fibonacci octonions are invertible.

In the future, we will study if this fact is true in each generalized octonion algebra $\mathbb{O}_{\mathbb{R}}(a+1, 2a+1, 3a+1)$, with $a \in (-2, -1)$.

4 Generalized Fibonacci-Lucas octonions

In [12], we introduced the generalized Fibonacci-Lucas numbers, namely: if n is an arbitrary positive integer and p, q be two arbitrary integers, the sequence $(g_n)_{n\ge 1}$, where

$$g_{n+1} = pf_n + ql_{n+1}, \quad n \ge 0,$$

is called *the sequence of the generalized Fibonacci-Lucas numbers*. To not induce confusion, we will use the notation $g_n^{p,q}$ instead of g_n.

Let $\mathbb{O}_{\mathbb{Q}}(\alpha, \beta, \gamma)$ be the generalized octonion algebra over $\mathbb{Q}$ with the basis $\{1, e_1, e_2, \ldots, e_7\}$. We define the *nth generalized Fibonacci-Lucas octonion* to be the element of the form

$$G_n^{p,q} = g_n^{p,q} \cdot 1 + g_{n+1}^{p,q} \cdot e_1 + g_{n+2}^{p,q} \cdot e_2 + g_{n+3}^{p,q} \cdot e_3 + g_{n+4}^{p,q} \cdot e_4 + g_{n+5}^{p,q} \cdot e_5 + g_{n+6}^{p,q} \cdot e_6 + g_{n+7}^{p,q} \cdot e_7.$$

We wonder what algebraic structure determines the generalized Fibonacci-Lucas octonions. First, we make the following remark.

Remark 4.1 Let n, p, q three arbitrary positive integers, $p, q \ge 0$. Then the nth generalized Fibonacci-Lucas octonion $G_n^{p,q} = 0$ if and only if $p = q = 0$.

Proof $\Rightarrow$ If $G_n^{p,q} = 0$, it results $g_n^{p,q} = g_{n+1}^{p,q} = \cdots = g_{n+7}^{p,q} = 0$. This implies that $g_{n-1}^{p,q} = \cdots = g_1^{p,q} = 0$. We obtain immediately $q = 0$ and $p = 0$.

$\Leftarrow$ is trivial. □

In [12], we proved the following properties of the generalized Fibonacci-Lucas numbers.

Remark 4.2 Let $n, m \in \mathbb{N}^*$, $a, b, p, q, p', q' \in \mathbb{Z}$. Then we have

(i)

$$ag_n^{p,q} + bg_m^{p',q'} = g_n^{ap,aq} + g_m^{bp',bq'};$$

(ii)

$$5g_n^{p,q} \cdot 5g_m^{p',q'} = 5g_{m+n-2}^{5p'q,pp'} + 5g_{m+n-1}^{5p'q,0} + 5g_{n-m}^{5p'q\cdot(-1)^m,pp'\cdot(-1)^m} + 5g_{n-m+1}^{5p'q\cdot(-1)^m,0} + 5g_{m+n}^{5pq',5qq'} + 5g_{n-m}^{5pq'\cdot(-1)^m,5qq'\cdot(-1)^m}.$$

Using this remark we can prove the following.

Theorem 4.1 *Let A and B be the sets*

$$A = \left\{ \sum_{i=1}^{n} G_{n_i}^{p_i,q_i} \,\middle|\, n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Z}, (\forall) i = \overline{1,n} \right\},$$

$$B = \left\{ \sum_{i=1}^{n} 5G_{n_i}^{p_i,q_i} \,\middle|\, n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Q}, (\forall) i = \overline{1,n} \right\} \cup \{1\}.$$

Then the following statements are true:

(i) *A is a free $\mathbb{Z}$-submodule of rank* 8 *of the generalized octonions algebra $\mathcal{O}_{\mathbb{Q}}(\alpha, \beta, \gamma)$*;

(ii) *B with octonions addition and multiplication, is a unitary non-associative subalgebra of the generalized octonions algebra $\mathcal{O}_{\mathbb{Q}}(\alpha, \beta, \gamma)$.*

Proof (i) Using Remark 4.2, as a result we immediately have

$$aG_n^{p,q} + bG_m^{p',q'} = G_n^{ap,aq} + G_m^{bp',bq'}, \quad (\forall) m, n \in \mathbb{N}^*, a, b, p, q, p', q' \in \mathbb{Z}.$$

Moreover, applying Remark 4.1, as a result $0 \in A$.

These implies that A is a $\mathbb{Z}$-submodule of the generalized octonions algebra $\mathcal{O}_{\mathbb{Q}}(\alpha, \beta, \gamma)$. Since $\{1, e_1, e_2, \ldots, e_7\}$ is a basis of A, as a result A is a free $\mathbb{Z}$-module of rank 8.

(ii) From Remark 4.2(ii), we immediately see that $5G_m^{p,q} \cdot 5G_n^{p',q'} \in B$, $(\forall) m, n \in \mathbb{N}^*$, $p, q, p', q' \in \mathbb{Z}$. Using this fact and a similar reason to that in the proof of (i), we see that B is a unitary non-associative subalgebra of the generalized octonions algebra $\mathcal{O}_{\mathbb{Q}}(\alpha, \beta, \gamma)$. □

Competing interests
The author declares that she has no competing interests.

Author's contributions
The author performed all tasks of this research: drafting, thinking of the study, writing, and revision of paper.

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