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Positive solutions for singular coupled integral boundary value problems of nonlinear higher-order fractional q-difference equations

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Abstract

This paper investigates the positive solutions for the singular coupled integral boundary value problem of nonlinear higher-order fractional q-difference equations. By applying a mixed monotone method and Guo-Krasnoselskii fixed point theorem, sufficient conditions for the existence and uniqueness results of the problem are established. An interesting example is presented to illustrate the main results.

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Keywords: fractional q-difference systems; coupled integral boundary conditions; positive solution; Green's function; fixed point theorems

1 Introduction

Due to the intensive development of the theory of fractional calculus itself and its varied applications in many fields of science and engineering, the fractional differential equation has gained considerable popularity and importance for the last several decades. In fact, we can find numerous applications in physics, chemistry, aerodynamics, fitting of experimental data, control of dynamical systems, and signal and image processing, and so on. Therefore, there have been some papers dealing with the existence and multiplicity of solutions or positive solutions for boundary value problems involving nonlinear fractional differential equations; see [1–6] and references cited therein.

At the same time, we notice that boundary value problems for a coupled system of nonlinear fractional differential equations have been addressed by several researchers. For instance, for some results for the existence of solutions or positive solutions for a coupled system of nonlinear fractional differential equations, we refer the readers to [7–14] and references therein. Relying on the nonlinear alternative of Leray-Schauder type and Krasnoselskii's fixed point theorems, Yuan [15] studied the multiple positive solutions to the $(n-1,n)$-type integral boundary value problems for systems of nonlinear semipositone fractional differential equations. Under different conditions, Yuan *et al.* [16] and Jiang *et al.* [17] considered the positive solutions to the four-point coupled boundary value problems for systems of nonlinear semipositone fractional differential equations, respectively. Wang *et al.* [18] investigated the existence and uniqueness of positive solution of a $(n-1,n)$-type fractional differential system with coupled integral boundary conditions. Henderson and Luca [19] proved the existence of positive solutions for a system of nonlinear Riemann-Liouville fractional differential equations with coupled integral boundary conditions and a parameter.

Research on q-difference calculus or quantum calculus dates back to the beginning of the 20th century, when Jackson [20, 21] introduced the first definition of the q-difference. Then Al-Salam [22] and Agarwal [23] proposed the fractional q-difference calculus. Later, the theory of fractional q-difference calculus itself and nonlinear fractional q-difference equation boundary value problems have been extensively studied by many authors. For some recent developments on fractional q-difference calculus and boundary value problems of fractional q-difference equations, see [24–30] and the references therein. For example, by applying the generalized Banach contraction principle, the monotone iterative method, and Krasnoselskii's fixed point theorem Zhao *et al.* [31] showed some existence results of positive solutions to nonlocal q-integral boundary value problem of nonlinear fractional q-derivatives equation. Under different conditions, Graef and Kong [32, 33] investigated the existence of positive solutions for boundary value problems with fractional q-derivatives in terms of different ranges of λ, respectively. By applying some standard fixed point theorems, Agarwal *et al.* [34] and Ahmad *et al.* [35] showed some existence results for sequential q-fractional integrodifferential equations with q-antiperiodic boundary conditions and nonlocal four-point boundary conditions, respectively.

In [36], Ferreira considered the nonlinear fractional q-difference boundary value problem as follows:

$$\begin{aligned} &\bigl(D_q^\alpha u\bigr)(t) + f\bigl(u(t)\bigr) = 0, \quad t \in [0,1], \alpha \in (2,3], \\ &u(0) = (D_q u)(0) = 0, \qquad (D_q u)(1) = \beta \geq 0, \end{aligned}$$

where D_q^α is the q-derivative of Riemann-Liouville type of order α. By applying a fixed point theorem in cones, sufficient conditions for the existence of positive solutions were enunciated.

In [37], Zhao *et al.* dealt with following integral boundary value problem of nonlinear fractional q-difference equation:

$$\begin{aligned} &\bigl(D_q^\alpha u\bigr)(t) + f\bigl(t, u(t)\bigr) = 0, \quad t \in [0,1], \alpha \in (2,3], \\ &u(0) = (D_q u)(0) = 0, \qquad u(1) = \mu \int_0^1 u(s)\, d_q s, \quad 0 < \mu < [\alpha]_q. \end{aligned}$$

By using the fixed point index theorem, sufficient conditions for the existence of at least two and at least three positive solutions were obtained.

In [38], Ahmad *et al.* studied the following nonlocal boundary value problems of nonlinear fractional q-difference equations:

$$\begin{aligned} &\bigl({}^c D_q^\alpha u\bigr)(t) = f\bigl(t, u(t)\bigr), \quad t \in [0,1], \alpha \in (1,2], \\ &a_1 u(0) - b_1 (D_q u)(0) = c_1 u(\eta_1), \qquad a_2 u(1) + b_2 (D_q u)(1) = c_2 u(\eta_2), \end{aligned}$$

where ${}^c D_q^\alpha$ denotes the Caputo fractional q-derivative of order α, and $a_i, b_i, c_i, \eta_i \in \mathbb{R}$ ($i = 1, 2$). The existence of solutions for the problem was shown by applying some well-known tools of fixed point theory such as Banach contraction principle, the Krasnoselskii fixed point theorem, and the Leray-Schauder nonlinear alternative.

In [39], Zhou and Liu investigated the following fractional q-difference system:

$$\begin{aligned} &\left({}^cD_q^{\alpha}u\right)(t) = f\big(t, v(t)\big), \qquad \left({}^cD_q^{\beta}v\right)(t) = f\big(t, u(t)\big), \quad t \in [0,1], \alpha, \beta \in (1,2], \\ &a_1u(0) - b_1(D_qu)(0) = c_1u(\eta_1), \qquad a_2u(1) + b_2(D_qu)(1) = c_2u(\eta_2), \\ &a_3u(0) - b_3(D_qu)(0) = c_3u(\eta_3), \qquad a_4u(1) + b_4(D_qu)(1) = c_4u(\eta_4), \end{aligned}$$

where ${}^cD_q^{\alpha}$ and ${}^cD_q^{\alpha}$ denote the Caputo fractional q-derivative of order α and β, respectively. The uniqueness and existence of a solution were obtained based on the nonlinear alternative of Leray-Schauder type and Banach's fixed point theorem.

In [40], the author considered the following coupled integral boundary value problem for systems of nonlinear semipositone fractional q-difference equations:

$$\begin{aligned} &\left(D_q^{\alpha}u\right)(t) + \lambda f\big(t, u(t), v(t)\big) = 0, \qquad \left(D_q^{\beta}v\right)(t) + \lambda g\big(t, u(t), v(t)\big) = 0, \quad t \in (0,1), \lambda > 0, \\ &\left(D_q^{j}u\right)(0) = \left(D_q^{j}v\right)(0) = 0, \quad 0 \le j \le n-2, \\ &u(1) = \mu \int_0^1 v(s)\,d_qs, \qquad v(1) = \nu \int_0^1 u(s)\,d_qs, \end{aligned}$$

where λ, μ, ν are three parameters with $0 < \mu < [\beta]_q$ and $0 < \nu < [\alpha]_q$, $\alpha, \beta \in (n-1, n]$ are two real numbers and $n \ge 3$, D_q^{α}, D_q^{β} are the fractional q-derivative of the Riemann-Liouville type, and f, g are sign-changing continuous functions. By applying the nonlinear alternative of Leray-Schauder type and Krasnoselskii's fixed point theorems, sufficient conditions for the existence of one or a multiple of positive solutions were obtained.

To the best of our knowledge, there are few papers which deal with the positive solutions for systems of nonlinear fractional q-difference equations. Motivated by the wide applications of coupled boundary value problems and the results mentioned above, we consider the existence and uniqueness of positive solutions for the following singular fractional q-difference systems:

$$\left(D_q^{\alpha_1}u\right)(t) + f_1\big(t, u(t), v(t)\big) = 0, \qquad \left(D_q^{\alpha_2}v\right)(t) + f_2\big(t, u(t), v(t)\big) = 0, \quad t \in (0,1), \tag{1.1}$$

with the coupled integral boundary value conditions

$$\begin{aligned} &\left(D_q^{j_1}u\right)(0) = \left(D_q^{j_2}v\right)(0) = 0, \quad 0 \le j_i \le n_i - 2, \\ &u(1) = \mu_1 \int_0^1 g_1(s)v(s)\,d_qs, \qquad v(1) = \mu_2 \int_0^1 g_2(s)u(s)\,d_qs, \end{aligned} \tag{1.2}$$

where $\mu_i > 0$, $\alpha_i \in (n_i - 1, n_i]$ with $3 \le n_i \in \mathbb{N}$, D^{α_i} is the Riemann-Liouville type fractional q-derivative of fractional order α_i, $i = 1,2$; $f_1 : (0,1) \times [0,\infty) \times (0,\infty) \to [0,\infty)$ and $f_2 : (0,1) \times (0,\infty) \times [0,\infty) \to [0,\infty)$ are two continuous functions, and $f_1(t,x,y)$ may be singular at $t = 0,1$ and $y = 0$, where $f_2(t,x,y)$ may be singular at $t = 0,1$ and $x = 0$.

The organization of the rest is as follows. In Section 2, we present some preliminaries and lemmas that will be used to prove our main results. We obtain the corresponding Green's function and some of its properties. In Section 3, by applying a mixed monotone method and the Guo-Krasnoselskii fixed point theorem, we obtain the existence and uniqueness results of the singular coupled boundary value problem (1.1) and (1.2). Furthermore, an example is given to illustrate our main results in Section 4.

2 Preliminaries

For the convenience of the reader, we present some necessary definitions and lemmas of fractional q-calculus theory to facilitate analysis of the semipositone boundary value problem (1.1). These details can be found in the recent literature; see [41] and references therein.

Let $q \in (0,1)$ and define

$$[a]_q = \frac{q^a - 1}{q - 1}, \quad a \in \mathbb{R}.$$

The q-analog of the power $(a-b)^n$ with $n \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ is

$$(a-b)^{(0)} = 1, \qquad (a-b)^{(n)} = \prod_{k=0}^{n-1}\bigl(a - bq^k\bigr), \quad n \in \mathbb{N}_0, a, b \in \mathbb{R}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^\alpha$. Here we point out that the following equality holds:

$$(a-b)^{(\alpha)} = \bigl(a - bq^{\alpha-1}\bigr)(a-b)^{(\alpha-1)}.$$

The q-gamma function is defined by

$$\Gamma_q(x) = (1-q)^{(x-1)}(1-q)^{1-x}, \quad x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is here defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad (D_q f)(0) = \lim_{x \to 0}(D_q f)(x),$$

and q-derivatives of higher order by

$$\bigl(D_q^0 f\bigr)(x) = f(x) \quad \text{and} \quad \bigl(D_q^n f\bigr)(x) = D_q\bigl(D_q^{n-1} f\bigr)(x), \quad n \in \mathbb{N}.$$

The q-integral of a function f defined in the interval $[0,b]$ is given by

$$(I_q f)(x) = \int_0^x f(t)\, d_q t = x(1-q)\sum_{n=0}^{\infty} f\bigl(xq^n\bigr)q^n, \quad x \in [0,b].$$

If $a \in [0,b]$ and f is defined in the interval $[0,b]$, its integral from a to b is defined by

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t.$$

Similarly to derivatives, an operator I_q^n can be defined, namely,

$$\left(I_q^0 f\right)(x) = f(x) \quad \text{and} \quad \left(I_q^n f\right)(x) = I_q\left(I_q^{n-1} f\right)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, *i.e.*,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in [41]. We now point out five formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i):

$$\int_a^b f(s)(D_q g)(s)\,d_q s = \left[f(s)g(s)\right]_{s=a}^{s=b} - \int_a^b (D_q f)(s)g(qs)\,d_q s \quad (q\text{-integration by parts}),$$
$$\left[a(t-s)\right]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}, \qquad {}_tD_q(t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)},$$
$${}_sD_q(t-s)^{(\alpha)} = -[\alpha]_q (t-qs)^{(\alpha-1)},$$
$$\left({}_xD_q \int_0^x f(x,t)\,d_q t\right)(x) = \int_0^x {}_xD_q f(x,t)\,d_q t + f(qx,x).$$

Note that if $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$ [42].

Definition 2.1 ([23]) Let $\alpha \geq 0$ and f be function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is $I_q^0 f(x) = f(x)$ and

$$\left(I_q^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t)\,d_q t, \quad \alpha > 0, x \in [0,1].$$

Definition 2.2 ([29]) The fractional q-derivative of the Riemann-Liouville type of order $\alpha \geq 0$ is defined by $D_q^0 f(x) = f(x)$ and

$$\left(D_q^\alpha f\right)(x) = \left(D_q^m I_q^{m-\alpha} f\right)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Definition 2.3 ([29]) The fractional q-derivative of the Caputo type of order $\alpha \geq 0$ is defined by

$$\left({}^c D_q^\alpha f\right)(x) = \left(I_q^{m-\alpha} D_q^m f\right)(x), \quad \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Lemma 2.1 ([23]) *Let* $\alpha, \beta \geq 0$ *and* f *be a function defined on* $[0,1]$. *Then the following formulas hold*:

(1) $(I_q^\beta I_q^\alpha f)(x) = I_q^{\alpha+\beta} f(x)$,

(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 2.2 ([42]) *Let $\alpha > 0$ and p be a positive integer. Then the following equality holds*:

$$\big(I_q^\alpha D_q^p f\big)(x) = \big(D_q^p I_q^\alpha f\big)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \big(D_q^k f\big)(0).$$

Now we derive the corresponding Green's function for boundary value problem (1.1), and obtain some properties of the Green's function. For the sake of simplicity, we always assume that the following condition (H) holds.

(H) $g_1, g_2 : [0,1] \to [0,\infty)$ are two continuous functions and satisfy

$$\nu_1 = \int_0^1 s^{\alpha_2-1} g_1(s)\, d_q s, \qquad \nu_2 = \int_0^1 s^{\alpha_1-1} g_2(s)\, d_q s, \quad 1 - \mu_1\mu_2\nu_1\nu_2 > 0.$$

Lemma 2.3 *Assume that* (H) *holds. Then, for $x, y \in C[0,1]$, the boundary value problem*

$$\begin{aligned} &\big(D_q^{\alpha_1} u\big)(t) + x(t) = 0, \qquad \big(D_q^{\alpha_2} v\big)(t) + y(t) = 0, \quad t \in (0,1), \\ &\big(D_q^j u\big)(0) = \big(D_q^j v\big)(0) = 0, \quad 0 \le j \le n-2, \\ &u(1) = \mu_1 \int_0^1 g_1(s) v(s)\, d_q s, \qquad v(1) = \mu_2 \int_0^1 g_2(s) u(s)\, d_q s, \end{aligned} \tag{2.1}$$

has an integral representation

$$\begin{cases} u(t) = \int_0^1 K_1(t,qs) x(s)\, d_q s + \int_0^1 H_1(t,qs) y(s)\, d_q s, \\ v(t) = \int_0^1 K_2(t,qs) y(s)\, d_q s + \int_0^1 H_2(t,qs) x(s)\, d_q s, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} K_1(t,s) &= G_1(t,s) + \frac{\mu_1\mu_2\nu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2} \int_0^1 g_2(\tau) G_1(\tau,s)\, d_q\tau, \\ H_1(t,s) &= \frac{\mu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2} \int_0^1 g_1(\tau) G_2(\tau,s)\, d_q\tau, \end{aligned} \tag{2.3}$$

$$\begin{aligned} K_2(t,s) &= G_2(t,s) + \frac{\mu_1\mu_2\nu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2\nu_1\nu_2} \int_0^1 g_1(\tau) G_2(\tau,s)\, d_q\tau, \\ H_2(t,s) &= \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2\nu_1\nu_2} \int_0^1 g_2(\tau) G_1(\tau,s)\, d_q\tau, \end{aligned} \tag{2.4}$$

$$G_i(t,s) = \frac{1}{\Gamma_q(\alpha_i)} \begin{cases} t^{\alpha_i-1}(1-s)^{(\alpha_i-1)} - (t-s)^{(\alpha_i-1)}, & 0 \le s \le t \le 1, \\ t^{\alpha_i-1}(1-s)^{(\alpha_i-1)}, & 0 \le t \le s \le 1, \end{cases} \quad i = 1,2. \tag{2.5}$$

Proof In view of Definition 2.1 and Lemma 2.1, we see that

$$\begin{aligned} \big(D_q^{\alpha_1} u\big)(t) = -x(t) \quad &\Longleftrightarrow \quad \big(I_q^{\alpha_1} D_q^n I_q^{n-\alpha_1} u\big)(t) = -\big(I_q^{\alpha_1} x\big)(t), \\ \big(D_q^{\alpha_2} u\big)(t) = -y(t) \quad &\Longleftrightarrow \quad \big(I_q^{\alpha_2} D_q^n I_q^{n-\alpha_2} u\big)(t) = -\big(I_q^{\alpha_2} y\big)(t). \end{aligned} \tag{2.6}$$

From (2.6) and Lemma 2.2, we can reduce (2.1) to the following equivalent integral equations:

$$\begin{aligned} u(t) &= c_{11}t^{\alpha_1-1} + c_{12}t^{\alpha_1-2} + \cdots + c_{1n}t^{\alpha_1-n} - \int_0^t \frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}x(s)\,d_qs, \\ v(t) &= c_{21}t^{\alpha_2-1} + c_{22}t^{\alpha_2-2} + \cdots + c_{2n}t^{\alpha_2-n} - \int_0^t \frac{(t-qs)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}y(s)\,d_qs. \end{aligned} \tag{2.7}$$

From $D_q^j u(0) = D_q^j v(0) = 0$, $0 \le j \le n-2$, we have $c_{in} = c_{i(n-1)} = \cdots = c_{i2} = 0$ $(i = 1,2)$. Thus, (2.7) reduces to

$$\begin{aligned} u(t) &= c_{11}t^{\alpha_1-1} - \int_0^t \frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}x(s)\,d_qs, \\ v(t) &= c_{21}t^{\alpha_2-1} - \int_0^t \frac{(t-qs)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}y(s)\,d_qs. \end{aligned} \tag{2.8}$$

Using the boundary conditions $u(1) = \mu_1 \int_0^1 g_1(s)v(s)\,d_qs$ and $v(1) = \mu_2 \int_0^1 g_2(s)u(s)\,d_qs$, from (2.8), we obtain

$$\begin{aligned} c_{11} &= \mu_1 \int_0^1 g_1(s)v(s)\,d_qs + \int_0^1 \frac{(1-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}x(s)\,d_qs, \\ c_{21} &= \mu_2 \int_0^1 g_2(s)u(s)\,d_qs + \int_0^1 \frac{(1-qs)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}y(s)\,d_qs. \end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9), we have

$$\begin{aligned} u(t) &= \mu_1 t^{\alpha_1-1} \int_0^1 g_1(s)v(s)\,d_qs + \int_0^1 G_1(t,qs)x(s)\,d_qs, \\ v(t) &= \mu_2 t^{\alpha_2-1} \int_0^1 g_2(s)u(s)\,d_qs + \int_0^1 G_2(t,qs)y(s)\,d_qs. \end{aligned} \tag{2.10}$$

Multiplying both sides of the first and second equations of (2.10) by $g_2(t)$ and $g_1(t)$, respectively, and integrating the resulting equations obtained with respect to t from 0 to 1, we obtain

$$\begin{aligned} &\int_0^1 g_2(t)u(t)\,d_qt \\ &\quad = \mu_1 \int_0^1 t^{\alpha_1-1}g_2(t)\,d_qt \int_0^1 g_1(s)v(s)\,d_qs + \int_0^1 g_2(t)\int_0^1 G_1(t,qs)x(s)\,d_qs\,d_qt, \\ &\int_0^1 g_1(t)v(t)\,d_qt \\ &\quad = \mu_2 \int_0^1 t^{\alpha_2-1}g_1(t)\,d_qt \int_0^1 g_2(s)u(s)\,d_qs + \int_0^1 g_1(t)\int_0^1 G_2(t,qs)y(s)\,d_qs\,d_qt. \end{aligned}$$

Solving for $\int_0^1 g_1(s)v(s)\,d_qs$ and $\int_0^1 g_2(s)u(s)\,d_qs$, we have

$$\begin{aligned} \int_0^1 g_1(s)v(s)\,d_qs = \frac{1}{1-\mu_1\mu_2\nu_1\nu_2}\biggl(&\int_0^1 g_1(t)\int_0^1 G_2(t,qs)y(s)\,d_qs\,d_qt \\ &+ \mu_2\nu_1 \int_0^1 g_2(t)\int_0^1 G_1(t,qs)x(s)\,d_qs\,d_qt\biggr) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-\mu_1\mu_2\nu_1\nu_2}\left(\int_0^1 y(s)\int_0^1 g_1(\tau)G_2(\tau,qs)\,d_q\tau\,d_qs\right. \\
&\quad \left. + \mu_2\nu_1\int_0^1 x(s)\int_0^1 g_2(\tau)G_1(\tau,qs)\,d_q\tau\,d_qs\right), \qquad (2.11)\\
\int_0^1 g_2(s)u(s)\,d_qs &= \frac{1}{1-\mu_1\mu_2\nu_1\nu_2}\left(\int_0^1 g_2(t)\int_0^1 G_1(t,qs)x(s)\,d_qs\,d_qt\right. \\
&\quad \left. + \mu_1\nu_2\int_0^1 g_1(t)\int_0^1 G_2(t,qs)y(s)\,d_qs\,d_qt\right) \\
&= \frac{1}{1-\mu_1\mu_2\nu_1\nu_2}\left(\int_0^1 x(s)\int_0^1 g_2(\tau)G_1(\tau,qs)\,d_q\tau\,d_qs\right. \\
&\quad \left. + \mu_1\nu_2\int_0^1 y(s)\int_0^1 g_1(\tau)G_2(\tau,qs)\,d_q\tau\,d_qs\right).
\end{aligned}
$$

Combining (2.10) and (2.11), we get

$$
\begin{aligned}
u(t) &= \int_0^1 G_1(t,qs)x(s)\,d_qs + \frac{\mu_1\mu_2\nu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 x(s)\int_0^1 g_2(\tau)G_1(\tau,qs)\,d_q\tau\,d_qs \\
&\quad + \frac{\mu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 y(s)\int_0^1 g_1(\tau)G_2(\tau,qs)\,d_q\tau\,d_qs \\
&= \int_0^1 K_1(t,qs)x(s)\,d_qs + \int_0^1 H_1(t,qs)y(s)\,d_qs, \\
v(t) &= \int_0^1 G_2(t,qs)y(s)\,d_qs + \frac{\mu_1\mu_2\nu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 y(s)\int_0^1 g_1(\tau)G_2(\tau,qs)\,d_q\tau\,d_qs \\
&\quad + \frac{\mu_2 t^{\alpha_2-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 x(s)\int_0^1 g_2(\tau)G_1(\tau,qs)\,d_q\tau\,d_qs \\
&= \int_0^1 K_2(t,qs)y(s)\,d_qs + \int_0^1 H_2(t,qs)x(s)\,d_qs.
\end{aligned}
$$

This completes the proof of the lemma. □

Lemma 2.4 *The function $G_i(t,s)$ defined by* (2.5) *has the following properties*:

(I) *$G_i(t,s)$ is continuous function on $(t,s)\in[0,1]\times[0,1]$ and $G_i(t,qs)>0$, for $t,s\in(0,1)$;*

(II) *$q^{\alpha_i-2}\psi_i(t)\varphi_i(qs)\le\Gamma_q(\alpha_i)G_i(t,s)\le[\alpha_i-1]_q\varphi_i(qs)$, for $t,s\in[0,1]$;*

(III) *$q^{\alpha_i-2}\psi_i(t)\varphi_i(qs)\le\Gamma_q(\alpha_i)G_i(t,s)\le[\alpha_i-1]_q\psi_i(t)$, for $t,s\in[0,1]$,*

where $\psi_i(t)=t^{\alpha_i-1}(1-t)$ and $\varphi_i(s)=(1-s)^{(\alpha_i-1)}s$.

Proof The continuity of G_i is easily checked. For $0\le qs\le t\le 1$, we have

$$
\begin{aligned}
\Gamma_q(\alpha_i)G_i(t,qs) &= t^{\alpha_i-1}(1-qs)^{(\alpha_i-1)}-(t-qs)^{(\alpha_i-1)} \\
&= [\alpha_i-1]_q\int_{t-qs}^{t(1-qs)} D_qx^{(\alpha_i-2)}\,d_qx \\
&\le [\alpha_i-1]_q t^{\alpha_i-2}(1-qs)^{(\alpha_i-2)}\big[(t-tqs)-(t-qs)\big] \\
&= [\alpha_i-1]_q t^{\alpha-2}(1-qs)^{(\alpha_i-2)}(1-t)qs \le [\alpha_i-1]_q(1-qs)^{(\alpha_i-2)}(1-qs)qs
\end{aligned}
$$

$$\begin{aligned}&\leq [\alpha_i - 1]_q (1-qs)^{(\alpha_i-2)}\left(1-q^{\alpha_i-1}s\right)qs \\ &= [\alpha_i - 1]_q (1-qs)^{(\alpha_i-1)}qs = [\alpha_i-1]_q\varphi_i(qs)\end{aligned}$$

and

$$\begin{aligned}&\Gamma_q(\alpha_i)G_i(t,qs) \\ &\quad = t^{\alpha_i-1}(1-qs)^{(\alpha_i-1)} - (t-qs)^{(\alpha_i-1)} \\ &\quad = (t-tqs)^{(\alpha_i-2)}\left(t-tq^{\alpha_i-1}s\right) - (t-qs)^{(\alpha_i-2)}\left(t-q^{\alpha_i-1}s\right) \\ &\quad \geq t^{\alpha_i-2}(1-qs)^{(\alpha_i-2)}\left[\left(t-tq^{\alpha_i-1}s\right)-\left(t-q^{\alpha_i-1}s\right)\right] = q^{\alpha_i-1}t^{\alpha_i-2}(1-t)(1-qs)^{(\alpha_i-2)}s \\ &\quad \geq q^{\alpha_i-2}t^{\alpha_i-1}(1-t)(1-qs)^{(\alpha_i-2)}\left(1-q^{\alpha_i-1}s\right)qs = q^{\alpha_i-2}\varrho_i(t)\varphi_i(qs).\end{aligned}$$

For $0 \leq t \leq qs \leq 1$, one verifies that

$$\begin{aligned}\Gamma_q(\alpha_i)G_i(t,qs) &= t^{\alpha_i-1}(1-qs)^{(\alpha_i-1)} = t^{\alpha_i-2}(1-qs)^{\alpha_i-1}t \\ &\leq [\alpha_i-1]_q(1-qs)^{q-1}qs = [\alpha_i-1]_q\varphi_i(qs)\end{aligned}$$

and

$$\Gamma_q(\alpha_i)G_i(t,qs) = t^{\alpha_i-1}(1-qs)^{(\alpha_i-1)} \geq q^{\alpha_i-2}t^{\alpha_i-1}(1-t)(1-qs)^{(\alpha_i-1)}qs = q^{\alpha_i-2}\psi_i(t)\varphi_i(qs).$$

Next, we prove the right side of (III). For $0 \leq qs \leq t \leq 1$, we can state that

$$\begin{aligned}\Gamma_q(\alpha_i)G_i(t,qs) &\leq [\alpha_i-1]_q t^{\alpha-2}(1-qs)^{(\alpha_i-2)}(1-t)qs \\ &\leq [\alpha_i-1]_q t^{\alpha-2}(1-t)t = [\alpha_i-1]_q\psi_i(t).\end{aligned}$$

For $\alpha \in (n, n+1]$ with $1 \leq n \in \mathbb{N}$, we have $(a-b)^{(\alpha)} \leq (a-b)^{(n)}$. In fact, according to the definitions of $(a-b)^{(\alpha)}$ and $(a-b)^{(n)}$, we get

$$\begin{aligned}(1-s)^{(\alpha)} &= \prod_{k=0}^{\infty}\frac{1-sq^k}{1-sq^{\alpha+k}} \\ &= \frac{(1-s)(1-sq)\cdots(1-sq^k)\cdots(1-sq^{n-1})(1-sq^n)(1-sq^{n+1})\cdots}{(1-sq^{\alpha})(1-sq^{\alpha+1})\cdots(1-sq^{\alpha+k-1})(1-sq^{\alpha+k})\cdots} \\ &\leq \frac{(1-s)(1-sq)\cdots(1-sq^k)\cdots(1-sq^{n-1})(1-sq^n)(1-sq^{n+1})\cdots}{(1-sq^{n})(1-sq^{n+1})\cdots(1-sq^{n+k-1})(1-sq^{n+k})\cdots} \\ &= (1-s)(1-sq)\cdots\left(1-sq^k\right)\cdots\left(1-sq^{n-1}\right) = \prod_{k=0}^{n-1}\left(1-sq^k\right) = (1-s)^{(n)}.\end{aligned}$$

For $0 \leq t \leq qs \leq 1$, from the above inequality and $\alpha_i \in (n_i-1, n_i]$, we have

$$\begin{aligned}\Gamma_q(\alpha_i)G_i(t,qs) &= t^{\alpha_i-1}(1-qs)^{(\alpha_i-1)} \leq t^{\alpha_i-1}(1-qs)^{(n_i-2)} = t^{\alpha_i-1}\prod_{k=0}^{n_i-3}\left(1-sq^{k+1}\right) \\ &\leq t^{\alpha_i-1}(1-sq) \leq [\alpha_i-1]_q t^{\alpha_i-1}(1-sq) \\ &\leq [\alpha_i-1]_q t^{\alpha-1}(1-t) = [\alpha_i-1]_q\psi_i(t).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.5 *The functions $K_i(t,s)$ and $H_i(t,s)$ $(i = 1,2)$ defined by* (2.3) *and* (2.4) *satisfy the following conditions*:

(a) *$K_i(t,s)$ and $H_i(t,s)$ are continuous functions on $(t,s) \in [0,1] \times [0,1]$ and $K_i(t,qs) \geq 0$ and $H_i(t,qs) \geq 0$ for $(t,s) \in [0,1] \times [0,1]$, $i = 1,2$;*

(b) *$\varrho t^{\alpha_i-1}\varphi_i(qs) \leq K_i(t,qs) \leq \rho\varphi_i(qs)$, $K_i(t,qs) \leq \rho t^{\alpha_i-1}$, $\varrho t^{\alpha_1-1}\varphi_2(qs) \leq H_1(t,qs) \leq \rho\varphi_2(qs)$, $\varrho t^{\alpha_2-1}\varphi_1(qs) \leq H_2(t,qs) \leq \rho\varphi_1(qs)$, and $H_i(t,qs) \leq \rho t^{\alpha_i-1}$ for $(t,s) \in [0,1] \times [0,1]$, $i = 1,2$, where φ_1, φ_2 are defined as Lemma* 2.4, *$\varrho = \min\{\varrho_1,\varrho_2,\varrho_3,\varrho_4\}$, $\rho = \max\{\rho_1,\rho_2,\rho_3,\rho_4\}$, and*

$$\begin{aligned}
\varrho_1 &= \frac{q^{\alpha_1-2}\mu_1\mu_2\nu_1}{\Gamma_q(\alpha_1)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_2(\tau)\psi_1(\tau)\,d_q\tau, \\
\varrho_2 &= \frac{q^{\alpha_2-2}\mu_1}{\Gamma_q(\alpha_2)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_1(\tau)\psi_2(\tau)\,d_q\tau, \\
\varrho_3 &= \frac{q^{\alpha_2-2}\mu_1\mu_2\nu_2}{\Gamma_q(\alpha_2)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_1(\tau)\psi_2(\tau)\,d_q\tau, \\
\varrho_4 &= \frac{q^{\alpha_1-2}\mu_2}{\Gamma_q(\alpha_1)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_2(\tau)\psi_1(\tau)\,d_q\tau, \\
\rho_1 &= \frac{[\alpha_1-1]_q}{\Gamma_q(\alpha_1)}\left(1+\frac{\mu_1\mu_2\nu_1}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)\,d_q\tau\right), \\
\rho_2 &= \frac{\mu_1[\alpha_2-1]_q}{\Gamma_q(\alpha_2)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_1(\tau)\,d_q\tau, \\
\rho_3 &= \frac{[\alpha_2-1]_q}{\Gamma_q(\alpha_2)}\left(1+\frac{\mu_1\mu_2\nu_2}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_1(\tau)\,d_q\tau\right), \\
\rho_4 &= \frac{\mu_2[\alpha_1-1]_q}{\Gamma_q(\alpha_1)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_2(\tau)\,d_q\tau.
\end{aligned}$$

Proof The continuity of K_i and H_i $(i = 1,2)$ is easily checked. According to the property (II) of Lemma 2.4 and (2.3), we have

$$\begin{aligned}
K_1(t,qs) &= G_1(t,qs) + \frac{\mu_1\mu_2\nu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)G_1(\tau,qs)\,d_q\tau \\
&\geq \frac{\mu_1\mu_2\nu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)G_1(\tau,qs)\,d_q\tau \\
&\geq \frac{\mu_1\mu_2\nu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)\frac{q^{\alpha_1-2}\psi_1(\tau)\varphi_1(qs)}{\Gamma_q(\alpha_1)}\,d_q\tau \\
&= \frac{q^{\alpha_1-2}\mu_1\mu_2\nu_1}{\Gamma_q(\alpha_1)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_2(\tau)\psi_1(\tau)\,d_q\tau\, t^{\alpha_1-1}\varphi_1(qs) = \varrho_1 t^{\alpha_1-1}\varphi_1(qs), \\
K_1(t,qs) &= G_1(t,qs) + \frac{\mu_1\mu_2\nu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)G_1(\tau,qs)\,d_q\tau \\
&\leq \frac{[\alpha_1-1]_q\varphi_1(qs)}{\Gamma_q(\alpha_1)} + \frac{\mu_1\mu_2\nu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)\frac{[\alpha_1-1]_q\varphi_1(qs)}{\Gamma_q(\alpha_1)}\,d_q\tau \\
&= \frac{[\alpha_1-1]_q}{\Gamma_q(\alpha_1)}\left(1+\frac{\mu_1\mu_2\nu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)\,d_q\tau\right)\varphi_1(qs) \\
&\leq \frac{[\alpha_1-1]_q}{\Gamma_q(\alpha_1)}\left(1+\frac{\mu_1\mu_2\nu_1}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)\,d_q\tau\right)\varphi_1(qs) = \rho_1\varphi_1(qs),
\end{aligned}$$

$$\begin{aligned} H_1(t,qs) &= \frac{\mu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_1(\tau)G_2(\tau,qs)\,d_q\tau \\ &\geq \frac{\mu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_1(\tau)\frac{q^{\alpha_2-2}\psi_2(\tau)\varphi_2(qs)}{\Gamma_q(\alpha_2)}\,d_q\tau \\ &= \frac{q^{\alpha_2-2}\mu_1}{\Gamma_q(\alpha_2)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_1(\tau)\psi_2(\tau)\,d_q\tau\, t^{\alpha_1-1}\varphi_2(qs) = \varrho_2 t^{\alpha_1-1}\varphi_2(qs), \end{aligned}$$

and

$$\begin{aligned} H_1(t,qs) &= \frac{\mu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_1(\tau)G_2(\tau,qs)\,d_q\tau \\ &\leq \frac{\mu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_1(\tau)\frac{[\alpha_2-1]_q\varphi_2(qs)}{\Gamma_q(\alpha_2)}\,d_q\tau \\ &= \frac{\mu_1[\alpha_2-1]_q t^{\alpha_1-1}}{\Gamma_q(\alpha_2)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_1(\tau)\,d_q\tau\,\varphi_2(qs) \\ &\leq \frac{\mu_1[\alpha_2-1]_q}{\Gamma_q(\alpha_2)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_1(\tau)\,d_q\tau\,\varphi_2(qs) = \rho_2\varphi_2(qs). \end{aligned}$$

Similarly, from the property (II) of Lemma 2.4 and (2.4), we get

$$\begin{aligned} K_2(t,qs) &\geq \frac{q^{\alpha_2-2}\mu_1\mu_2\nu_2}{\Gamma_q(\alpha_2)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_1(\tau)\psi_2(\tau)\,d_q\tau\, t^{\alpha_2-1}\varphi_2(qs) = \varrho_3 t^{\alpha_2-1}\varphi_2(qs), \\ K_2(t,qs) &\leq \frac{[\alpha_2-1]_q}{\Gamma_q(\alpha_2)}\left(1+\frac{\mu_1\mu_2\nu_2}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_1(\tau)\,d_q\tau\right)\varphi_2(qs) = \rho_3\varphi_2(qs), \\ H_2(t,qs) &\geq \frac{q^{\alpha_1-2}\mu_2}{\Gamma_q(\alpha_1)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_2(\tau)\psi_1(\tau)\,d_q\tau\, t^{\alpha_2-1}\varphi_1(qs) = \varrho_4 t^{\alpha_2-1}\varphi_1(qs), \\ H_2(t,qs) &\leq \frac{\mu_2[\alpha_1-1]_q}{\Gamma_q(\alpha_1)(1-\mu_1\mu_2\nu_1\nu_2)}\int_0^1 g_2(\tau)\,d_q\tau\,\varphi_1(qs) = \rho_4\varphi_1(qs). \end{aligned}$$

On the other hand, according to the property (III) of Lemma 2.4 and (2.3), we obtain

$$\begin{aligned} K_1(t,qs) &= G_1(t,qs) + \frac{\mu_1\mu_2\nu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)G_1(\tau,qs)\,d_q\tau \\ &\leq \frac{[\alpha_1-1]_q t^{\alpha_1-1}(1-t)}{\Gamma_q(\alpha_1)} + \frac{\mu_1\mu_2\nu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)\frac{[\alpha_1-1]_q t^{\alpha_1-1}(1-t)}{\Gamma_q(\alpha_1)}\,d_q\tau \\ &\leq \frac{[\alpha_1-1]_q t^{\alpha_1-1}}{\Gamma_q(\alpha_1)} + \frac{\mu_1\mu_2\nu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)\frac{[\alpha_1-1]_q}{\Gamma_q(\alpha_1)}\,d_q\tau \\ &= \frac{[\alpha_1-1]_q}{\Gamma_q(\alpha_1)}\left(1+\frac{\mu_1\mu_2\nu_1}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_2(\tau)\,d_q\tau\right)t^{\alpha_1-1} = \rho_1 t^{\alpha_1-1} \end{aligned}$$

and

$$\begin{aligned} H_1(t,qs) &= \frac{\mu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_1(\tau)G_2(\tau,qs)\,d_q\tau \\ &\leq \frac{\mu_1 t^{\alpha_1-1}}{1-\mu_1\mu_2\nu_1\nu_2}\int_0^1 g_1(\tau)\frac{[\alpha_2-1]_q t^{\alpha_2-1}(1-t)}{\Gamma_q(\alpha_2)}\,d_q\tau \end{aligned}$$

$$= \frac{\mu_1[\alpha_2 - 1]_q t^{\alpha_1-1}}{\Gamma_q(\alpha_2)(1 - \mu_1\mu_2\nu_1\nu_2)} \int_0^1 g_1(\tau)\, d_q\tau$$

$$= \frac{\mu_1[\alpha_2 - 1]_q}{\Gamma_q(\alpha_2)(1 - \mu_1\mu_2\nu_1\nu_2)} \int_0^1 g_1(\tau)\, d_q\tau\, t^{\alpha_1-1} = \rho_2 t^{\alpha_1-1}.$$

Similarly, from the property (III) of Lemma 2.4 and (2.4), we get

$$K_2(t,qs) \le \frac{[\alpha_2 - 1]_q}{\Gamma_q(\alpha_2)}\left(1 + \frac{\mu_1\mu_2\nu_2}{1 - \mu_1\mu_2\nu_1\nu_2}\int_0^1 g_1(\tau)\, d_q\tau\right) t^{\alpha_2-1} = \rho_3 t^{\alpha_2-1},$$

$$H_2(t,qs) \le \frac{\mu_2[\alpha_1 - 1]_q}{\Gamma_q(\alpha_1)(1 - \mu_1\mu_2\nu_1\nu_2)} \int_0^1 g_2(\tau)\, d_q\tau\, t^{\alpha_2-1} = \rho_4 t^{\alpha_2-1}.$$

This completes the proof of the lemma. □

Remark 2.1 From Lemmas 2.5, for $t, \tau, s \in [0,1]$, we have

$$K_1(t,qs) \ge \omega t^{\alpha_1-1} H_2(\tau,qs), \qquad K_2(t,qs) \ge \omega t^{\alpha_2-1} H_1(\tau,qs),$$
$$H_1(t,qs) \ge \omega t^{\alpha_1-1} K_2(\tau,qs), \qquad H_2(t,qs) \ge \omega t^{\alpha_2-1} K_1(\tau,qs),$$
$$K_i(t,qs) \ge \omega t^{\alpha_i-1} K_i(\tau,qs), \qquad H_i(t,qs) \ge \omega t^{\alpha_i-1} H_i(\tau,qs), \quad i = 1,2,$$

where $\omega = \varrho/\rho$, ϱ, ρ are defined as Lemma 2.5, $0 < \omega < 1$.

In order to obtain the main results in this paper, we will use the following cone compression and expansion fixed point theorem.

Lemma 2.6 ([43]) *Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that either*

(a) $\|Sw\| \le \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_2$, *or*

(b) $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \le \|w\|$, $w \in P \cap \partial\Omega_2$.

Then S has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Main results

In this section, let $X = C[0,1] \times C[0,1]$, then X is a Banach space with the norm

$$\|(u,v)\| = \max\{\|u\|, \|v\|\}, \qquad \|u\| = \max_{t\in[0,1]}|u(t)|, \qquad \|v\| = \max_{t\in[0,1]}|v(t)|.$$

Denote

$$P = \{(u,v) \in X : u(t) \ge \omega t^{\alpha_1-1}\|(u,v)\|, v(t) \ge \omega t^{\alpha_2-1}\|(u,v)\|, t \in [0,1]\},$$

where ω is defined as Remark 2.1. It is easy to see that P is a positive cone in X. It can easily be seen that P is a cone in X. For any real constants r and R with $0 < r < R$, define

$$P_r = \{(u,v) \in P : \|(u,v)\| < r\}, \qquad P_{[r,R]} = \{(u,v) \in P : r \le \|(u,v)\| \le R\}.$$

In what follows, we first list the following assumptions for convenience.

(A1) $f_1 : (0,1) \times [0,\infty) \times (0,\infty) \to [0,\infty)$ is continuous, $f_1(t,u,v)$ is nondecreasing in u and nonincreasing in v, and there exist two constants $\theta_1, \vartheta_1 \in [0,1)$ such that

$$\begin{aligned} &\kappa^{\theta_1} f_1(t,u,v) \le f_1(t,\kappa u,v), \\ &f_1(t,u,\kappa v) \le \kappa^{-\vartheta_1} f_1(t,u,v), \quad \forall u,v>0, \kappa \in (0,1); \end{aligned} \tag{3.1}$$

$f_2 : (0,1) \times (0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous, $f_2(t,u,v)$ is nonincreasing in u and nondecreasing in v, and there exist two constants $\theta_2, \vartheta_2 \in [0,1)$ such that

$$\begin{aligned} &\kappa^{\theta_2} f_2(t,u,v) \le f_2(t,u,\kappa v), \\ &f_2(t,\kappa u,v) \le \kappa^{-\vartheta_2} f_2(t,u,v), \quad \forall u,v>0, \kappa \in (0,1). \end{aligned} \tag{3.2}$$

(A2) The following inequalities hold:

$$0 < \int_0^1 \varphi_1(qs) f_1\bigl(s,1,s^{\alpha_2-1}\bigr)\,d_q s < +\infty, \qquad 0 < \int_0^1 \varphi_2(qs) f_2\bigl(s,s^{\alpha_1-1},1\bigr)\,d_q s < +\infty,$$

where φ_1 and φ_2 are defined as Lemma 2.4.

Remark 3.1 From assumption (A1), we have

$$f_1\bigl(s,s^{\alpha_2-1},1\bigr) \le f_1\bigl(s,1,s^{\alpha_2-1}\bigr), \qquad f_2\bigl(s,1,s^{\alpha_1-1}\bigr) \le f_2\bigl(s,s^{\alpha_1-1},1\bigr).$$

This together with (A2) yields

$$\begin{aligned} &0 < \int_0^1 \varphi_1(qs) f_1\bigl(s,s^{\alpha_2-1},1\bigr)\,d_q s \le \int_0^1 \varphi_1(qs) f_1\bigl(s,1,s^{\alpha_2-1}\bigr)\,d_q s < +\infty, \\ &0 < \int_0^1 \varphi_2(qs) f_2\bigl(s,1,s^{\alpha_1-1}\bigr)\,d_q s \le \int_0^1 \varphi_2(qs) f_2\bigl(s,s^{\alpha_1-1},1\bigr)\,d_q s < +\infty. \end{aligned}$$

Remark 3.2 The inequalities (3.1) and (3.2) imply that

$$f_1(t,\kappa u,v) \le \kappa^{\theta_1} f_1(t,u,v), \qquad f_1(t,u,v) \le \kappa^{\vartheta_1} f_1(t,u,\kappa v), \quad \forall u,v>0, \kappa \in (0,1); \tag{3.3}$$

$$f_2(t,u,\kappa v) \le \kappa^{\theta_2} f_2(t,u,v), \qquad f_2(t,u,v) \le \kappa^{\vartheta_2} f_2(t,\kappa u,v), \quad \forall u,v>0, \kappa \in (0,1), \tag{3.4}$$

respectively. Conversely, we have (3.3) and (3.4) and (3.1) and (3.2), respectively.

From the above assumptions (A1) and (A2), for any $(u,v) \in P \setminus \{(0,0)\}$, we define an integral operator $T : P \setminus \{(0,0)\} \to P$ by

$$T(u,v)(t) = \bigl(T_1(u,v)(t), T_2(u,v)(t)\bigr), \quad t \in [0,1],$$

where $T_1, T_2 : P \setminus \{(0,0)\} \to Q = \{x(t) \in C[0,1] : x(t) \ge 0, t \in [0,1]\}$ are defined by

$$T_1(u,v)(t) = \int_0^1 K_1(t,qs) f_1\bigl(s,u(s),v(s)\bigr)\,d_q s + \int_0^1 H_1(t,qs) f_2\bigl(s,u(s),v(s)\bigr)\,d_q s, \quad t \in [0,1],$$

$$T_2(u,v)(t) = \int_0^1 K_2(t,qs)f_2\big(s,u(s),v(s)\big)\,d_qs + \int_0^1 H_2(t,qs)f_1\big(s,u(s),v(s)\big)\,d_qs, \quad t\in[0,1].$$

Obviously, (u,v) is a positive solutions of the coupled boundary value problem (1.1) and (1.2) if and only if (u,v) is a fixed point of T in $P\setminus\{(0,0)\}$.

Lemma 3.1 *Assume that* (H), (A1) *and* (A2) *hold. For any* $0<r_1<r_2<+\infty$, $T: P_{[r_1,r_2]}\to P$ *is a completely continuous operator.*

Proof For any $(u,v)\in P\setminus\{(0,0)\}$, we can see that

$$\omega t^{\alpha_1-1}\big\|(u,v)\big\| \le u(t)\le \big\|(u,v)\big\|, \qquad \omega t^{\alpha_2-1}\big\|(u,v)\big\|\le v(t)\le\big\|(u,v)\big\|, \quad t\in[0,1]. \tag{3.5}$$

Let κ be a positive number such that $\|(u,v)\|/\kappa<1$, $\kappa>1$. From (A1) and (3.5), we have

$$\begin{aligned} f_1\big(t,u(t),v(t)\big) &\le f_1\big(t,\kappa,\omega t^{\alpha_2-1}\big\|(u,v)\big\|\big)\le \kappa^{\theta_1}f_1\left(t,1,\frac{\omega\|(u,v)\|}{\kappa}t^{\alpha_2-1}\right)\\ &\le \kappa^{\theta_1+\vartheta_1}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_1}f_1\big(t,1,t^{\alpha_2-1}\big),\\ f_2\big(t,u(t),v(t)\big) &\le f_2\big(t,\omega t^{\alpha_1-1}\big\|(u,v)\big\|,\kappa\big)\le \kappa^{\theta_2}f_2\left(t,\frac{\omega\|(u,v)\|}{\kappa}t^{\alpha_1-1},1\right)\\ &\le \kappa^{\theta_2+\vartheta_2}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_2}f_2\big(t,t^{\alpha_1-1},1\big). \end{aligned} \tag{3.6}$$

Hence, for any $t\in[0,1]$, by Lemma 2.5 and (3.6), we get

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,qs)f_1\big(s,u(s),v(s)\big)\,d_qs + \int_0^1 H_1(t,qs)f_2\big(s,u(s),v(s)\big)\,d_qs\\ &\le \rho\left(\kappa^{\theta_1+\vartheta_1}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_1}\int_0^1\varphi_1(qs)f_1\big(s,1,s^{\alpha_2-1}\big)\,d_qs\right.\\ &\quad\left.+\kappa^{\theta_2+\vartheta_2}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_2}\int_0^1\varphi_2(qs)f_2\big(s,s^{\alpha_1-1},1\big)\,d_qs\right)<+\infty,\\ T_2(u,v)(t) &= \int_0^1 K_2(t,qs)f_2\big(s,u(s),v(s)\big)\,d_qs + \int_0^1 H_2(t,qs)f_1\big(s,u(s),v(s)\big)\,d_qs\\ &\le \rho\left(\kappa^{\theta_1+\vartheta_1}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_1}\int_0^1\varphi_1(qs)f_1\big(s,1,s^{\alpha_2-1}\big)\,d_qs\right.\\ &\quad\left.+\kappa^{\theta_2+\vartheta_2}\big(\omega\big\|(u,v)\big\|\big)^{-\vartheta_2}\int_0^1\varphi_2(qs)f_2\big(s,s^{\alpha_1-1},1\big)\,d_qs\right)<+\infty. \end{aligned}$$

Together with the continuity of $K_i(t,s)$ and $H_i(t,s)$ $(i=1,2)$, it is easy to see that $T_i\in C[0,1]$. Therefore, $T: P\setminus\{(0,0)\}\to P$ is well defined.

For any $(u,v)\in P_{[r_1,r_2]}$ and $t,\tau\in[0,1]$, by Remark 2.1, we obtain

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,qs)f_1\big(s,u(s),v(s)\big)\,d_qs + \int_0^1 H_1(t,qs)f_2\big(s,u(s),v(s)\big)\,d_qs\\ &\ge \int_0^1 \omega t^{\alpha_1-1}K_1(\tau,qs)f_1\big(s,u(s),v(s)\big)\,d_qs\\ &\quad+\int_0^1\omega t^{\alpha_1-1}H_1(\tau,qs)f_2\big(s,u(s),v(s)\big)\,d_qs \end{aligned}$$

$$
\begin{aligned}
&= \omega t^{\alpha_1-1}\left(\int_0^1 K_1(\tau,qs)f_1\big(s,u(s),v(s)\big)\,d_qs + \int_0^1 H_1(\tau,qs)f_2\big(s,u(s),v(s)\big)\,d_qs\right)\\
&= \omega t^{\alpha_1-1}T_1(u,v)(\tau),\\
T_1(u,v)(t) &= \int_0^1 K_1(t,qs)f_1\big(s,u(s),v(s)\big)\,d_qs + \int_0^1 H_1(t,qs)f_2\big(s,u(s),v(s)\big)\,d_qs\\
&\geq \int_0^1 \omega t^{\alpha_1-1}H_2(\tau,qs)f_1\big(s,u(s),v(s)\big)\,d_qs\\
&\quad + \int_0^1 \omega t^{\alpha_1-1}K_2(\tau,qs)f_2\big(s,u(s),v(s)\big)\,d_qs\\
&= \omega t^{\alpha_1-1}\left(\int_0^1 K_2(\tau,qs)f_2\big(s,u(s),v(s)\big)\,d_qs + \int_0^1 H_2(\tau,qs)f_1\big(s,u(s),v(s)\big)\,d_qs\right)\\
&= \omega t^{\alpha_1-1}T_2(u,v)(\tau).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&T_1(u,v)(t) \geq \omega t^{\alpha_1-1}\big\|T_1(u,v)\big\|, \qquad T_2(u,v)(t) \geq \omega t^{\alpha_1-1}\big\|T_2(u,v)\big\|,\\
&\quad \textit{i.e.}, T_1(u,v)(t) \geq \omega t^{\alpha_1-1}\big\|\big(T_1(u,v),T_2(u,v)\big)\big\|.
\end{aligned}
$$

In the same way, we can prove that

$$
\begin{aligned}
&T_2(u,v)(t) \geq \omega t^{\alpha_2-1}\big\|T_2(u,v)\big\|, \qquad T_2(u,v)(t) \geq \omega t^{\alpha_2-1}\big\|T_1(u,v)\big\|,\\
&\quad \textit{i.e.}, T_2(u,v)(t) \geq \omega t^{\alpha_2-1}\big\|\big(T_1(u,v),T_2(u,v)\big)\big\|.
\end{aligned}
$$

Therefore, we have $T(P_{[r_1,r_2]}) \subseteq T(P)$. According to the Ascoli-Arzela theorem, we easily see that $T : P_{[r_1,r_2]} \to P$ is completely continuous. This completes the proof of the lemma. □

Theorem 3.1 *Assume that* (H), (A1) *and* (A2) *hold. Then the coupled boundary value problem* (1.1) *and* (1.2) *has at least one positive solution* (u^*,v^*), *and there exists a real number* $0 < l < 1$ *satisfying*

$$
lt^{\alpha_1-1} \leq u^*(t) \leq l^{-1}t^{\alpha_1-1}, \qquad lt^{\alpha_2-1} \leq v^*(t) \leq l^{-1}t^{\alpha_2-1}, \quad t \in [0,1].
$$

Proof First, we show that the coupled boundary value problem (1.1) and (1.2) has at least one positive solution.

Choose r and R such that

$$
\begin{aligned}
&0 < r \leq \min\left\{\left(\varrho c^{\alpha_1-1}\omega^{\theta_1}\int_0^1 \varphi_1(qs)f_1\big(s,s^{\alpha_1-1},1\big)\,d_qs\right)^{\frac{1}{1-\theta_1}}, \frac{1}{2}\right\}, \quad c \in \left(0,\frac{1}{2}\right),\\
&R \geq \max\left\{\left(\rho\int_0^1 \varphi_1(qs)f_1\big(s,1,s^{\alpha_2-1}\big)\,d_qs\right.\right.\\
&\qquad \left.\left. + \rho\int_0^1 \varphi_2(qs)f_2\big(s,s^{\alpha_1-1},1\big)\,d_qs\right)^{\frac{1}{1-\max\{\theta_1,\theta_2\}}}, \frac{1}{\omega}, 2\right\}.
\end{aligned}
$$

For any $(u,v) \in \partial K_r$, we have

$$r\omega t^{\alpha_1-1} \leq u(t) \leq r, \qquad r\omega t^{\alpha_2-1} \leq v(t) \leq r, \quad t \in [0,1].$$

By Lemma 2.5, Remark 3.1, and (A1), for any $(u,v) \in \partial P_r$, we get

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,qs)f_1\big(s,u(s),v(s)\big)\,d_qs + \int_0^1 H_1(t,qs)f_2\big(s,u(s),v(s)\big)\,d_qs \\ &\geq \int_0^1 K_1(t,qs)f_1\big(s,u(s),v(s)\big)\,d_qs \geq \varrho t^{\alpha_1-1}\int_0^1 \varphi_1(qs)f_1\big(s,r\omega s^{\alpha_1-1},r\big)\,d_qs \\ &\geq \varrho t^{\alpha_1-1}\int_0^1 \varphi_1(qs)f_1\big(s,r\omega s^{\alpha_1-1},1\big)\,d_qs \\ &\geq \varrho t^{\alpha_1-1}(r\omega)^{\theta_1}\int_0^1 \varphi_1(qs)f_1\big(s,s^{\alpha_1-1},1\big)\,d_qs \\ &\geq \varrho c^{\alpha_1-1}\omega^{\theta_1}\int_0^1 \varphi_1(qs)f_1\big(s,s^{\alpha_1-1},1\big)\,d_qs r^{\theta_1} \geq r = \big\|(u,v)\big\|, \quad t \in [c,1-c]. \end{aligned}$$

This guarantees that

$$\big\|T(u,v)\big\| \geq \big\|(u,v)\big\|, \quad \forall (u,v) \in \partial P_r. \tag{3.7}$$

On the other hand, for any $(u,v) \in \partial P_R$, we have

$$R\omega t^{\alpha_1-1} \leq u(t) \leq R, \qquad R\omega t^{\alpha_2-1} \leq v(t) \leq R, \quad t \in [0,1].$$

By Lemma 2.5, (A1), and (A2), for any $(u,v) \in \partial P_R$, we get

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 K_1(t,qs)f_1\big(s,u(s),v(s)\big)\,d_qs + \int_0^1 H_1(t,qs)f_2\big(s,u(s),v(s)\big)\,d_qs \\ &\leq \rho\int_0^1 \varphi_1(qs)f_1\big(s,R,R\omega s^{\alpha_2-1}\big)\,d_qs + \rho\int_0^1 \varphi_2(qs)f_2\big(s,R\omega s^{\alpha_1-1},R\big)\,d_qs \\ &\leq \rho\int_0^1 \varphi_1(qs)f_1\big(s,R,s^{\alpha_2-1}\big)\,d_qs + \rho\int_0^1 \varphi_2(qs)f_2\big(s,s^{\alpha_1-1},R\big)\,d_qs \\ &\leq \rho R^{\theta_1}\int_0^1 \varphi_1(qs)f_1\big(s,1,s^{\alpha_2-1}\big)\,d_qs + \rho R^{\theta_2}\int_0^1 \varphi_2(qs)f_2\big(s,s^{\alpha_1-1},1\big)\,d_qs \\ &\leq \rho R^{\max\{\theta_1,\theta_2\}}\bigg(\int_0^1 \varphi_1(qs)f_1\big(s,1,s^{\alpha_2-1}\big)\,d_qs + \int_0^1 \varphi_2(qs)f_2\big(s,s^{\alpha_1-1},1\big)\,d_qs\bigg) \\ &\leq R = \big\|(u,v)\big\|. \end{aligned}$$

In the same way, we have $T_2(u,v)(t) \leq R = \|(u,v)\|$, for all $(u,v) \in \partial P_R$. So we have

$$\big\|T(u,v)\big\| \leq \big\|(u,v)\big\|, \quad \forall (u,v) \in \partial P_R. \tag{3.8}$$

By the complete continuity of T, (3.7) and (3.8), and Lemma 2.6, we find that T has a fixed point $(u^*,v^*) \in P_{[r,R]}$. Consequently, the coupled boundary value problem (1.1) and (1.2) has a positive solution $(u^*,v^*) \in P_{[r,R]}$.

Next, we show there exists a real number $0 < l < 1$ satisfying

$$lt^{\alpha_1-1} \leq u^*(t) \leq l^{-1}t^{\alpha_1-1}, \qquad lt^{\alpha_2-1} \leq v^*(t) \leq l^{-1}t^{\alpha_2-1}, \quad t \in [0,1].$$

From Lemma 3.1, we know $(u^*, v^*) \in P \setminus \{(0,1)\}$. So, we have

$$\begin{aligned}
&\omega t^{\alpha_1-1}\big\|(u^*,v^*)\big\| \leq u^*(t) \leq \big\|(u^*,v^*)\big\|, \\
&\omega t^{\alpha_2-1}\big\|(u^*,v^*)\big\| \leq v^*(t) \leq \big\|(u^*,v^*)\big\|, \quad t \in [0,1].
\end{aligned}$$

Choose κ, such that $\|(u^*,v^*)\|/\kappa < 1$, $\kappa > 1/\omega$. By Lemma 2.5 and (A1), for $t \in [0,1]$, we have

$$\begin{aligned}
u^*(t) &= \int_0^1 K_1(t,qs)f_1\big(s,u^*(s),v^*(s)\big)\,d_qs + \int_0^1 H_1(t,qs)f_2\big(s,u^*(s),v^*(s)\big)\,d_qs \\
&\leq \int_0^1 \rho t^{\alpha_1-1}f_1\big(s,\kappa,\omega s^{\alpha_2-1}\big\|(u^*,v^*)\big\|\big)\,d_qs + \int_0^1 \rho t^{\alpha_1-1}f_2\big(s,\omega s^{\alpha_1-1}\big\|(u^*,v^*)\big\|,\kappa\big)\,d_qs \\
&\leq \rho t^{\alpha_1-1}\int_0^1 f_1\left(s,\kappa,\frac{\omega\|(u^*,v^*)\|}{\kappa}s^{\alpha_2-1}\right)d_qs \\
&\quad + \rho t^{\alpha_1-1}\int_0^1 f_2\left(s,\frac{\omega\|(u^*,v^*)\|}{\kappa}s^{\alpha_1-1},\kappa\right)d_qs \\
&\leq \rho t^{\alpha_1-1}\left(\kappa^{\theta_1+\vartheta_1}\big(\omega\big\|(u^*,v^*)\big\|\big)^{-\vartheta_1}\int_0^1 \varphi_1(qs)f_1\big(s,1,s^{\alpha_2-1}\big)\,d_qs\right. \\
&\quad \left. + \kappa^{\theta_2+\vartheta_2}\big(\omega\big\|(u^*,v^*)\big\|\big)^{-\vartheta_2}\int_0^1 \varphi_2(qs)f_2\big(s,s^{\alpha_1-1},1\big)\,d_qs\right) \\
&\leq \rho t^{\alpha_1-1}\left(\kappa^{\theta_1+\vartheta_1}(\omega R)^{-\vartheta_1}\int_0^1 \varphi_1(qs)f_1\big(s,1,s^{\alpha_2-1}\big)\,d_qs\right. \\
&\quad \left. + \kappa^{\theta_2+\vartheta_2}(\omega R)^{-\vartheta_2}\int_0^1 \varphi_2(qs)f_2\big(s,s^{\alpha_1-1},1\big)\,d_qs\right).
\end{aligned}$$

In the same way, for $t \in [0,1]$, we also have

$$\begin{aligned}
v^*(t) \leq \rho t^{\alpha_2-1}&\left(\kappa^{\theta_1+\vartheta_1}(\omega R)^{-\vartheta_1}\int_0^1 \varphi_1(qs)f_1\big(s,1,s^{\alpha_2-1}\big)\,d_qs\right. \\
&\left. + \kappa^{\theta_2+\vartheta_2}(\omega R)^{-\vartheta_2}\int_0^1 \varphi_2(qs)f_2\big(s,s^{\alpha_1-1},1\big)\,d_qs\right).
\end{aligned}$$

Choose

$$\begin{aligned}
l = \min\Bigg\{&\omega r,\left(\rho\kappa^{\theta_1+\vartheta_1}(\omega R)^{-\vartheta_1}\int_0^1 \varphi_1(qs)f_1\big(s,1,s^{\alpha_2-1}\big)\,d_qs\right. \\
&\left. + \rho\kappa^{\theta_2+\vartheta_2}(\omega R)^{-\vartheta_2}\int_0^1 \varphi_2(qs)f_2\big(s,s^{\alpha_1-1},1\big)\,d_qs\right)^{-1},\frac{1}{2}\Bigg\},
\end{aligned}$$

then we have

$$lt^{\alpha_1-1} \leq u^*(t) \leq l^{-1}t^{\alpha_1-1}, \qquad lt^{\alpha_2-1} \leq v^*(t) \leq l^{-1}t^{\alpha_2-1}, \quad t \in [0,1].$$

This completes the proof of Theorem 3.1. □

Theorem 3.2 *Assume that* (H), (A1) *and* (A2) *hold. Furthermore, assume* $\theta_1 + \vartheta_1 < 1$ *and* $\theta_2 + \vartheta_2 < 1$. *Then the coupled boundary value problem* (1.1) *and* (1.2) *has a unique positive solution on* $[0,1]$.

Proof Assume that the coupled boundary value problem (1.1) and (1.2) has two different positive solutions (u_1, v_1) and (u_2, v_2). By Theorem 3.1, there exist $0 < l_1 < 1$ and $0 < l_2 < 1$ such that

$$\begin{aligned} &l_1 t^{\alpha_1 - 1} \leq u_1(t) \leq l_1^{-1} t^{\alpha_1 - 1}, \qquad l_1 t^{\alpha_2 - 1} \leq v_1(t) \leq l_1^{-1} t^{\alpha_2 - 1}, \quad t \in [0,1], \\ &l_2 t^{\alpha_1 - 1} \leq u_2(t) \leq l_2^{-1} t^{\alpha_1 - 1}, \qquad l_2 t^{\alpha_2 - 1} \leq v_2(t) \leq l_2^{-1} t^{\alpha_2 - 1}, \quad t \in [0,1]. \end{aligned} \tag{3.9}$$

Thus, from (3.9), we have

$$l_1 l_2 u_2(t) \leq u_1(t) \leq (l_1 l_2)^{-1} u_2(t), \qquad l_1 l_2 v_2(t) \leq v_1(t) \leq (l_1 l_2)^{-1} v_2(t), \quad t \in [0,1].$$

Obviously, one has $l_1 l_2 \neq 1$. Put

$$L = \sup\bigl\{l : l u_2(t) \leq u_1(t) \leq l^{-1} u_2(t), l v_2(t) \leq v_1(t) \leq l^{-1} v_2(t), t \in [0,1]\bigr\}.$$

It is easy to see that $0 < l_1 l_2 \leq L < 1$, and

$$L u_2(t) \leq u_1(t) \leq L^{-1} u_2(t), \qquad L v_2(t) \leq v_1(t) \leq L^{-1} v_2(t), \quad t \in [0,1]. \tag{3.10}$$

By (A1) and (3.10), we get

$$\begin{aligned} f_1\bigl(t, u_1(t), v_1(t)\bigr) &\geq f_1\bigl(t, L u_2(t), L^{-1} v_2(t)\bigr) \geq L^{\theta_1 + \vartheta_1} f_1\bigl(t, u_2(t), v_2(t)\bigr) \\ &\geq L^{\sigma} f_1\bigl(t, u_2(t), v_2(t)\bigr), \\ f_2\bigl(t, u_1(t), v_1(t)\bigr) &\geq f_2\bigl(t, L u_2(t), L^{-1} v_2(t)\bigr) \geq L^{\theta_2 + \vartheta_2} f_2\bigl(t, u_2(t), v_2(t)\bigr) \\ &\geq L^{\sigma} f_2\bigl(t, u_2(t), v_2(t)\bigr), \end{aligned} \tag{3.11}$$

where $\sigma = \max\{\theta_1 + \vartheta_1, \theta_2 + \vartheta_2\}$ such that $\sigma < 1$. Similarly, by (A1) and (3.10), we have

$$\begin{aligned} f_1\bigl(t, u_2(t), v_2(t)\bigr) &\geq f_1\bigl(t, L u_1(t), L^{-1} v_1(t)\bigr) \geq L^{\theta_1 + \vartheta_1} f_1\bigl(t, u_1(t), v_1(t)\bigr) \\ &\geq L^{\sigma} f_1\bigl(t, u_1(t), v_1(t)\bigr), \\ f_2\bigl(t, u_2(t), v_2(t)\bigr) &\geq f_2\bigl(t, L u_1(t), L^{-1} v_1(t)\bigr) \geq L^{\theta_2 + \vartheta_2} f_2\bigl(t, u_1(t), v_1(t)\bigr) \\ &\geq L^{\sigma} f_2\bigl(t, u_1(t), v_1(t)\bigr). \end{aligned} \tag{3.12}$$

From (3.11), for $t \in [0,1]$, we have

$$\begin{aligned} u_1(t) = T_1(u_1, v_1)(t) &= \int_0^1 K_1(t, qs) f_1\bigl(s, u_1(s), v_1(s)\bigr)\, d_q s + \int_0^1 H_1(t, qs) f_2\bigl(s, u_1(s), v_1(s)\bigr)\, d_q s \\ &\geq \int_0^1 K_1(t, qs) L^{\sigma} f_1\bigl(s, u_2(s), v_2(s)\bigr)\, d_q s + \int_0^1 H_1(t, qs) L^{\sigma} f_2\bigl(s, u_2(s), v_2(s)\bigr)\, d_q s \\ &= L^{\sigma} T_1(u_2, v_2)(t) = u_2(t), \end{aligned} \tag{3.13}$$

$$\begin{aligned}
v_1(t) &= T_2(u_1,v_1)(t)\\
&= \int_0^1 K_2(t,qs)f_2\big(s,u_1(s),v_1(s)\big)\,d_qs + \int_0^1 H_2(t,qs)f_1\big(t,u_1(s),v_1(s)\big)\,d_qs\\
&\geq \int_0^1 K_2(t,qs)L^\sigma f_2\big(s,u_2(s),v_2(s)\big)\,d_qs + \int_0^1 H_2(t,qs)L^\sigma f_1\big(s,u_2(s),v_2(s)\big)\,d_qs\\
&= L^\sigma T_2(u_2,v_2)(t) = v_2(t).
\end{aligned}$$

Similarly, from (3.12), for $t \in [0,1]$, we have

$$\begin{aligned}
u_2(t) &= T_1(u_2,v_2)(t)\\
&= \int_0^1 K_1(t,qs)f_1\big(s,u_2(s),v_2(s)\big)\,d_qs + \int_0^1 H_1(t,qs)f_2\big(s,u_2(s),v_2(s)\big)\,d_qs\\
&\geq \int_0^1 K_1(t,qs)L^\sigma f_1\big(s,u_1(s),v_1(s)\big)\,d_qs + \int_0^1 H_1(t,qs)L^\sigma f_2\big(s,u_1(s),v_1(s)\big)\,d_qs\\
&= L^\sigma T_1(u_1,v_1)(t) = u_1(t),\\
v_2(t) &= T_2(u_2,v_2)(t)\\
&= \int_0^1 K_2(t,qs)f_2\big(s,u_2(s),v_2(s)\big)\,d_qs + \int_0^1 H_2(t,qs)f_1\big(t,u_2(s),v_2(s)\big)\,d_qs\\
&\geq \int_0^1 K_2(t,qs)L^\sigma f_2\big(s,u_1(s),v_1(s)\big)\,d_qs + \int_0^1 H_2(t,qs)L^\sigma f_1\big(s,u_1(s),v_1(s)\big)\,d_qs\\
&= L^\sigma T_2(u_1,v_1)(t) = v_1(t).
\end{aligned} \tag{3.14}$$

Combining (3.13) and (3.14), we can obtain

$$L^\sigma u_2(t) \leq u_1(t) \leq \big(L^\sigma\big)^{-1} u_2(t), \qquad L^\sigma v_2(t) \leq v_1(t) \leq \big(L^\sigma\big)^{-1} v_2(t), \quad t \in [0,1].$$

Noticing that $0 < L, \sigma < 1$, we get to a contradiction with the maximality of L. Thus, the coupled boundary value problem (1.1) and (1.2) has a unique positive solution (u^*, v^*). This completes the proof of Theorem 3.2. □

4 An example

In this section, we give an example to illustrate the usefulness of our main results.

Example 4.1 Consider the singular fractional q-difference system with coupled boundary integral conditions

$$\begin{aligned}
&\big(D_{0.5}^{2.5}u\big)(t) + \frac{\sqrt{u}}{\sqrt[3]{t^2(1-t)v}} = 0, \qquad \big(D_{0.5}^{2.5}v\big)(t) + \frac{\sqrt[3]{v}}{\sqrt{t(1-t)u}} = 0, \quad t \in (0,1),\\
&\big(D_{0.5}^{j_1}u\big)(0) = \big(D_{0.5}^{j_2}v\big)(0) = 0, \quad j_i = 0,1,\\
&u(1) = \frac{2}{3}\int_0^1 sv(s)\,d_qs, \qquad v(1) = \frac{4}{5}\int_0^1 su(s)\,d_qs.
\end{aligned} \tag{4.1}$$

Obviously, we have $q = 0.5$, $\alpha_1 = \alpha_2 = 2.5$, $\mu_1 = 2/3$, $\mu_2 = 4/5$, $g_1(t) = t$, and $g_2(t) = 1/\sqrt{t}$. By simple computation, we have

$$\nu_1 = \int_0^1 s^{\alpha_2-1} g_1(s)\, d_q s = \int_0^1 s^{2.5}\, d_q s \approx 0.548479,$$
$$\nu_2 = \int_0^1 s^{\alpha_1-1} g_2(s)\, d_q s = \int_0^1 s\, d_q s \approx 0.666667, \quad 1 - \mu_1\mu_2\nu_1\nu_2 \approx 0.804985 > 0.$$

So, the condition (H) holds. We have

$$f_1(t,u,v) = \frac{\sqrt{u}}{\sqrt[3]{t^2(1-t)v}}, \qquad f_2(t,u,v) = \frac{\sqrt[3]{v}}{\sqrt{t(1-t)u}}.$$

It is easy to see that $f_1 : (0,1) \times [0,\infty) \times (0,\infty) \to [0,\infty)$ is continuous, $f_1(t,u,v)$ is nondecreasing in u and nonincreasing in v, $f_2 : (0,1) \times (0,\infty) \times [0,\infty) \to [0,\infty)$ is continuous, $f_2(t,u,v)$ is nonincreasing in u and nondecreasing in v. Take

$$\theta_1 = \frac{11}{20}, \qquad \vartheta_1 = \frac{2}{5}, \qquad \theta_2 = \frac{3}{5}, \qquad \vartheta_2 = \frac{1}{5}.$$

Then we know that the condition (A1) holds. As

$$\begin{aligned}\int_0^1 \varphi_1(qs) f_1\bigl(s,1,s^{\alpha_2-1}\bigr)\, d_q s &= \int_0^1 \frac{(1-qs)^{(1.5)} q s^{0.5}}{\sqrt[3]{s^2(1-s)}}\, d_q s \\ &\le q \int_0^1 \frac{1}{\sqrt[3]{s^2(1-s)}}\, d_q s \approx 3.05253 < +\infty, \\ \int_0^1 \varphi_2(qs) f_2\bigl(s,s^{\alpha_1-1},1\bigr)\, d_q s &= \int_0^1 \frac{(1-qs)^{(1.5)} q s^{0.25}}{\sqrt{s(1-s)}}\, d_q s \\ &\le q \int_0^1 \frac{1}{\sqrt{s(1-s)}}\, d_q s \approx 1.69963 < +\infty,\end{aligned}$$

the condition (A2) is also satisfied. Therefore, by Theorem 3.1, we see that the coupled boundary value problem (4.1) has at least one positive solution (u^*, v^*). Furthermore,

$$\theta_1 + \vartheta_1 = \frac{11}{20} + \frac{2}{5} = \frac{19}{20} < 1, \qquad \theta_2 + \vartheta_2 = \frac{3}{5} + \frac{1}{5} = \frac{4}{5} < 1.$$

By Theorem 3.2, we see that (u^*, v^*) is the unique positive solution of the coupled boundary value problem (4.1).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed to each part of this work equally and read and approved the final version of the manuscript.

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