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Random attractor for second-order stochastic delay lattice sine-Gordon equation

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Abstract

In this paper, we prove the existence of random $\mathcal{D}$-attractor for the second-order stochastic delay sine-Gordon equation on infinite lattice with certain dissipative conditions, and then establish the upper bound of Kolmogorov ε-entropy for the random $\mathcal{D}$-attractor.

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1 Introduction

This paper deals with the following second-order stochastic delay lattice sine-Gordon equation:

$$\begin{cases} \ddot{u}_i + \alpha \dot{u}_i + (Au)_i + \lambda u_i + \beta \sin u_i = f_i(u_{it}) + g_i + \epsilon_i \dot{w}_i, & t > 0, \\ u_{i0} = u_i(\tau), \qquad \dot{u}_{i0} = \dot{u}_i(\tau), & \tau \in [-h, 0], \end{cases} \tag{1.1}$$

where $i \in \mathbb{Z}^k$ with $\mathbb{Z}$ being the set of integers and $k \in \mathbb{N}$ a fixed positive integer, $u_{it} = u_{it}(\tau) = u_i(t + \tau)$ is the delay term with the interval of delay time $[-h, 0]$, and $u_{i0} = u_i(\tau)$, $\dot{u}_{i0} = \dot{u}_i(\tau)$ is the initial data, A is a linear operator defined by (3.1), $u = (u_i)_{i \in \mathbb{Z}^k}$, $g = (g_i)_{i \in \mathbb{Z}^k} \in l^2$, α, β, h are positive constants, $\epsilon = (\epsilon_i)_{i \in \mathbb{Z}^k} \in l^2$, f_i is a smooth function satisfying some dissipative conditions (see the hypotheses H_1–H_3 in Sect. 3), $\{w_i(t) : i \in \mathbb{Z}^k\}$ is independent two-sided real valued standard Wiener processes.

Lattice dynamical systems, whose the spatial structure has a discrete character, arise from a variety of applications such as electrical engineering [1], biology [2, 3], chemical reaction [4], and pattern formation [5]. As a matter of fact, systems in the process of evolution are always influenced by the external environment, those influence may be random or the time delay. If the system add the random or the time delay terms, it makes up for the defects of some deterministic systems, and explains new evolutionary rules. Many researchers have discussed broadly the deterministic models in [6–8]. Stochastic lattice equations, driven by additive independent white noise, were discussed for the first time in

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[9], and then intensive researched in [10–20]. Furthermore, a kind of stochastic delay lattice systems were considered in [21–24], and these interesting results have attracted wide attention of scholars.

However, to the best of our knowledge, there is little literature about the existence of random attractors for a second-order stochastic delay lattice sine-Gordon equation on $\mathbb{Z}^k$. To this end, this paper is devoted to study this problem. The main ideas and methods used in the proofs are motivated by [21, 22, 24–27].

This paper is organized as follows. In Sect. 2, we introduce some basic concepts and propositions. In Sect. 3, we present some sufficient conditions for the existence of continuous stochastic dynamical systems. Section 4 is devoted to proving the existence of a random $\mathcal{D}$-attractor for a stochastic lattice sine-Gordon equation. In Sect. 5, we study the upper bound of the Kolmogorov ε-entropy for the random $\mathcal{D}$-attractor.

2 Preliminaries

In this section, we recall some basic concepts and propositions related to random attractors for stochastic dynamical systems (SDS), more details can be found in the literature [9, 28, 29].

Let $(X, \| \cdot \|_X)$ be a separable Hilbert space, and $(\Omega, \mathcal{F}, P)$ a probability space.

Definition 2.1 Let $\mathcal{D}$ be a collection of random subsets of X, stochastic process $S\{(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ a continuous random dynamical system and $\{B_0(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$. Then $\{B_0(\omega)\}_{\omega \in \Omega}$ is said to be a random absorbing set for S in $\mathcal{D}$ if for every $B \in \mathcal{D}$ for P-a.e. $\omega \in \Omega$, there exists some $t_B(\omega) > 0$ such that

$$S(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset B_0(\omega), \quad \forall t \geq t_B(\omega).$$

Definition 2.2 Let $\mathcal{D}$ be a collection of random subsets of X. Then a random set $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ in X is said to be a $\mathcal{D}$-random attractor for S, if the following conditions are satisfied, for P-a.e. $\omega \in \Omega$:

(1) $\{\mathcal{A}(\omega)\}$ is compact, and $\omega \rightarrow d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$;
(2) $\{\mathcal{A}(\omega)\}$ is invariant;
(3) $\{\mathcal{A}(\omega)\}$ attracts every set in $\mathcal{D}$, i.e., for all $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$,

$$\lim_{t \to \infty} d\big(S(t, \theta_{-t}\omega)B(\theta_{-t}\omega), \mathcal{A}(\omega)\big) = 0,$$

where d denotes the Hausdorff semi-metric.

Definition 2.3 Let $B(\omega) \subset X$ be a random set. For any $\varepsilon > 0$, $\omega \in \Omega$, let $\mathcal{N}_{\varepsilon, \omega}(B(\omega), X) = \mathcal{N}_{\varepsilon, \omega}(B(\omega))$ be the minimal number of deterministic open balls in X with radii ε that is necessary to cover $B(\omega)$. The number $\mathcal{K}_{\varepsilon}(B(\omega)) = \mathcal{K}_{\varepsilon}(B(\omega), X) = \ln \mathcal{N}_{\varepsilon, \omega}(B(\omega))$ is called the Kolmogorov ε-entropy of $B(\omega)$ in X.

Proposition 2.1 (See [30]) *Let $n \in \mathbb{N}$ and $\Upsilon = \{x = (x_i)_{|i| \leq n} : x_i \in \mathbb{R}, |x_i| \leq r\} \subset \mathbb{R}^{2n+1}$ be a regular polyhedron. Then Υ can be covered by $\mathcal{N}_{\varepsilon}(\Upsilon) = ([2r \cdot \frac{1}{\varepsilon}\sqrt{2n+1}] + 1)^{2n+1}$ balls in $\mathbb{R}^{2n+1}$ with radii $\frac{\varepsilon}{2}$, where $[\cdot]$ denotes the integer-valued function.*

Proposition 2.2 (See [29]) *If $r(\omega) > 0$ is tempered and $r(\theta_t\omega)$ is continuous in t for P-a.e.*
$\omega \in \Omega$, *then*

(1) *for any $t \in \mathbb{R}$, $r(\theta_t\omega)$ is tempered. Moreover, for any $h > 0$, $\max_{\tau \in [-h,0]} r(\theta_\tau\omega)$ is also tempered;*

(2) *for any $\beta > 0$ and P-a.e. $\omega \in \Omega$, $R(\omega) = \int_{-\infty}^{0} e^{\beta s} r(\theta_s\omega)\, ds < \infty$ is tempered, and $R(\theta_t\omega)$ is also continuous in t.*

Proposition 2.3 (See [9]) *Suppose that $B_0(\omega) \in \mathcal{D}(X)$ is a closed random absorbing set for the continuous $SDS\{S(t,\omega)\}_{t \geq 0,\omega \in \Omega}$, and, for a.e. $\omega \in \Omega$, each sequence $x_n \in S(t_n, \theta_{-t_n}\omega)B_0(\theta_{-t_n}\omega)$ with $t_n \to \infty$ has a convergent subsequence in X. Then the $SDS\{S(t,\omega)\}_{t \geq 0,\omega \in \Omega}$ has a unique $\mathcal{D}$-random attractor $A(\omega)$, which is given by*

$$A(\omega) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq s} S(t, \theta_{-t}\omega)B_0(\theta_{-t}\omega)}.$$

3 Second-order stochastic delay lattice sine-Gordon equation

Denote l^p $(p \geq 1)$ defined by

$$l^p = \left\{ u \,\Big|\, u = (u_i)_{i \in \mathbb{Z}^k}, i = (i_1, i_1, \ldots, i_k) \in \mathbb{Z}^k, u_i \in \mathbb{R}, \sum_{i \in \mathbb{Z}^k} |u_i|^p < +\infty \right\},$$

with the norm

$$\|u\|_p^p = \sum_{i \in \mathbb{Z}^k} |u_i|^p.$$

In particular, l^2 is a Hilbert space with the inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ given by

$$(u, v) = \sum_{i \in \mathbb{Z}^k} u_i v_i, \qquad \|u\|^2 = \sum_{i \in \mathbb{Z}^k} |u_i|^2,$$

for any $u = (u_i)_{i \in \mathbb{Z}^k}, v = (v_i)_{i \in \mathbb{Z}^k} \in l^2$.

Define a linear operator in the following way:

$$
\begin{aligned}
(Au_i)_{i \in \mathbb{Z}^k} &= 2k u_{(i_1,\ldots,i_j,\ldots,i_k)} - u_{(i_1-1,\ldots,i_j,\ldots,i_k)} - \cdots - u_{(i_1,\ldots,i_j-1,\ldots,i_k)} - \cdots - u_{(i_1,\ldots,i_j,\ldots,i_k-1)} \\
&\quad - u_{(i_1+1,\ldots,i_j,\ldots,i_k)} - \cdots - u_{(i_1,\ldots,i_j+1,\ldots,i_k)} - \cdots - u_{(i_1,\ldots,i_j,\ldots,i_k+1)},
\end{aligned}
\tag{3.1}
$$

$$(B_j u)_i = u_{(i_1,\ldots,i_j+1,\ldots,i_k)} - u_{(i_1,\ldots,i_j,\ldots,i_k)}, \qquad \left(B_j^* u\right)_i = u_{(i_1,\ldots,i_j,\ldots,i_k)} - u_{(i_1,\ldots,i_j-1,\ldots,i_k)},$$

$$(A_j u)_i = 2u_{(i_1,\ldots,i_j,\ldots,i_k)} - u_{(i_1,\ldots,i_j+1,\ldots,i_k)} - u_{(i_1,\ldots,i_j-1,\ldots,i_k)}.$$

Then B_j^* is the adjoint operator of B_j, and

$$A = A_1 + A_2 + \cdots + A_k, \qquad A_j = B_j B_j^* = B_j^* B_j, \quad \text{for } j = 1, 2, \ldots, k.$$

By using the above equalities, we have

$$(Au, v) = \sum_{j=1}^{k} (B_j u, B_j v) = \sum_{j=1}^{k} \left(B_j^* u, B_j^* v\right).$$

For any $u = (u_i)_{i \in \mathbb{Z}^k}$, $v = (v_i)_{i \in \mathbb{Z}^k} \in l^2$, we define a new inner product and norm on l^2 by

$$(u, v)_\lambda = \sum_{j=1}^{k} (B_j u, B_j v) + \lambda(u, v), \qquad \|u\|_\lambda^2 = (u, u)_\lambda = \sum_{j=1}^{k} \|B_j u\|^2 + \lambda \|u\|^2.$$

It is obvious that

$$\|B_j u\|^2 \le 4\|u\|^2, \qquad \lambda \|u\|^2 \le \|u\|_\lambda^2 \le (4k + \lambda)\|u\|^2.$$

Denote

$$l^2 = \big(l^2, (\cdot, \cdot), \| \cdot \|\big), \qquad l_\lambda^2 = \big(l^2, (\cdot, \cdot)_\lambda, \| \cdot \|_\lambda\big).$$

Then the norms $\| \cdot \|$ and $\| \cdot \|_\lambda$ are equivalent.

Let $H = l_\lambda^2 \times l^2$ be endowed with the inner product and norm

$$(\psi_1, \psi_2)_H = \big(u^{(1)}, u^{(2)}\big)_\lambda + \big(v^{(1)}, v^{(2)}\big), \qquad \|\psi\|_H^2 = \|u\|_\lambda^2 + \|v\|^2,$$

for $\psi_j = (u^{(j)}, v^{(j)})^T = ((u_i^{(j)}, v_i^{(j)}))_{i \in \mathbb{Z}^k}^T \in H, j = 1, 2, \psi = (u, v)^T = ((u_i, v_i))_{i \in \mathbb{Z}^k}^T \in H$. In addition, the space $H_0 = C([-h, 0], H)$ is endowed with $\|\psi\|_{H_0} = \max_{\tau \in [-h, 0]} \|\psi(\tau)\|_H$.

In the following, we consider the probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \big\{\omega \in C\big(\mathbb{R}, l^2\big) : \omega(0) = 0\big\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P the corresponding Wiener measure on $(\Omega, \mathcal{F})$. We will identify ω with

$$\omega(t) = w(t), \quad t \in \mathbb{R}.$$

Define the time shift by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system with the filtration

$$\mathcal{F} = \bigvee_{s \le t} \mathcal{F}_s^t, \quad t \in \mathbb{R},$$

where $\mathcal{F}_s^t = \sigma\{w(t_2) - w(t_1) : s \le t_1 \le t_2 \le t\}$ is the smallest σ-algebra generated by $w(t_2) - w(t_1)$ for all $s \le t_1 \le t_2 \le t$.

For convenience, we rewrite Eq. (1.1) as

$$\begin{cases} \ddot{u} + \alpha \dot{u} + Au + \lambda u + \beta \sin u = f(u_t) + g + \dot{w}, & t > 0, \\ u_0 = u(\tau), \qquad \dot{u}_0 = \dot{u}(\tau), \quad \tau \in [-h, 0], \end{cases} \tag{3.2}$$

where $u = (u_i)_{i \in \mathbb{Z}^k}$, $\dot{u} = (\dot{u}_i)_{i \in \mathbb{Z}^k}$, $\ddot{u} = (\ddot{u}_i)_{i \in \mathbb{Z}^k}$, $u_0 = (u_{0i})_{i \in \mathbb{Z}^k}$, $\dot{u}_0 = (\dot{u}_{0i})_{i \in \mathbb{Z}^k}$, $Au = (Au_i)_{i \in \mathbb{Z}^k}$, $\lambda u = (\lambda u_i)_{i \in \mathbb{Z}^k}$, $\beta \sin u = \beta (\sin u_i)_{i \in \mathbb{Z}^k}$, $f(u_t) = (f_i(u_{it}))_{i \in \mathbb{Z}^k}$, $g = (g_i)_{i \in \mathbb{Z}^k}$ and $\dot{w} = (\epsilon_i \dot{w}_i)_{i \in \mathbb{Z}^k}$. Let $\bar{v} = \dot{u} + \delta u$, where δ is a positive constant and satisfies

$$4\delta + \frac{2\delta\alpha^2}{\lambda} + \frac{4\beta^2}{\lambda\delta} - 2\alpha < 0, \tag{3.3}$$

then Eq. (3.2) can be rewritten as

$$\begin{cases} \dot{\varphi}(t) + D\varphi(t) = F(\varphi(t)) + G(t), & t > 0, \\ \varphi_0 = (u_0, \bar{v}_0)^T = (u(\tau), \dot{u}(\tau) + \delta u(\tau))^T, & \tau \in [-h, 0], \end{cases} \tag{3.4}$$

where $\varphi = (u, \bar{v})^T$, $\varphi_t = (u_t, \bar{v}_t)^T$, $F(\varphi(t)) = (0, f(u_t) + g - \beta \sin u)^T$, $G = (0, \dot{w})^T$ and

$$D\varphi(t) = \begin{pmatrix} \delta u - \bar{v} \\ Au + \lambda u + (\delta - \alpha)(\delta u - \bar{v}) \end{pmatrix}.$$

Also, we make the following assumptions:

(H_1) $f_i : C([-h, 0]; \mathbb{R}) \to \mathbb{R}$ is continuous and $f_i(0) = 0$;

(H_2) $|f_i(\xi)| \leq M_{0,i} + M_{1,i} \max_{\tau \in [-h,0]} |\xi(\tau)|$ for all $\xi \in C([-h, 0]; \mathbb{R})$, where $M_{r,i} \geq 0$, $(M_{r,i})_{i \in \mathbb{Z}^k} \in l^2$, $M_r^2 := \sum_{i \in \mathbb{Z}^k} M_{r,i}^2$ $(r = 0, 1)$;

(H_3) for any bounded set $Y \subset l^2$, there exists a constant $L_f > 0$, such that

$$\|f(u) - f(v)\| \leq L_f \|u - v\|, \quad \forall u, v \in Y.$$

Lemma 3.1 *Suppose (H_1)–(H_3) hold. For any $T > 0$ and an initial data $\varphi_0 \in H_0$, there exists a unique solution $\varphi_t \in L^2(\Omega, C([0, T]); H)$ of Eq. (3.4) with $\varphi_t(\cdot, \varphi_0) \in H_0$ for $t \in [0, T]$ and $\varphi_0(\cdot, \varphi_0) = \varphi_0$. Moreover, $\varphi_t(\cdot, \varphi_0)$ depends continuously on the initial data φ_0 for each $\omega \in \Omega$.*

Proof Rewriting (3.4) as

$$\varphi(t) = \varphi(0) + \int_0^t \left(-D\varphi(s) + F(\varphi(s)) + G(s)\right) ds, \quad t > 0.$$

By (H_1) and (H_3), we know that

$$\begin{aligned} \left\| F(\varphi(t)) \right\|_H^2 &= \left\| f(u_t) + g - \beta \sin u \right\|^2 \leq 3L_f^2 \|u_t\|^2 + 3\|g\|^2 + 3\beta^2 \|u\|^2 \\ &\leq \frac{3L_f^2 + 3\beta^2}{\lambda} \|\varphi_t\|_{H_0}^2 + 3\|g\|^2 \end{aligned}$$

and

$$\begin{aligned} \left\| D\varphi(t) \right\|_H^2 &= \|\delta u - \bar{v}\|_\lambda^2 + \left\| Au + \lambda u + (\delta - \alpha)(\delta u - \bar{v}) \right\|^2 \\ &\leq 2\delta^2 \|u\|_\lambda^2 + 2\|\bar{v}\|_\lambda^2 + 3\|Au\|^2 + 3\lambda^2 \|u\|^2 + 3(\delta - \alpha)^2 \|\delta u - \bar{v})\|^2 \\ &\leq \left(2\delta^2 + \frac{54k^2}{\lambda} + 3\lambda + \frac{6\delta^2(\delta - \alpha)^2}{\lambda}\right) \|u\|_\lambda^2 + \left(6(\delta - \alpha)^2 + 2(4k + \lambda)\right) \|\bar{v}\|^2 \\ &\leq d_1 \left\| \varphi(t) \right\|_H^2 \leq d_1 \|\varphi_t\|_{H_0}^2, \end{aligned}$$

where $d_1 = \max\{2\delta^2 + \frac{54k^2}{\lambda} + 3\lambda + \frac{6\delta^2(\delta-\alpha)^2}{\lambda}, 6(\delta - \alpha)^2 + 2(4k + \lambda)\}$. Thus, F and D map the bounded sets into bounded sets. In this way, by the standard theory of differential equations, we find that there exists a unique local solution. Then calculations in blow shows that this local solution is actually global. Indeed, suppose the solutions $\varphi^{(1)}(t)$, $\varphi^{(2)}(t)$ of Eq. (3.4) with the initial data $\varphi_0^{(1)}$, $\varphi_0^{(2)} \in H_0$, respectively, we have

$$
\begin{aligned}
\frac{d}{dt} &\left\| \varphi^{(1)}(t) - \varphi^{(2)}(t) \right\|_H^2 \\
&= -2\big(D\big(\varphi^{(1)}(t) - \varphi^{(2)}(t)\big), \varphi^{(1)}(t) - \varphi^{(2)}(t)\big)_H \\
&\quad + 2\big(F\big(\varphi^{(1)}(t)\big) - F\big(\varphi^{(2)}(t)\big), \varphi^{(1)}(t) - \varphi^{(2)}(t)\big)_H \\
&\leq \left\| D\big(\varphi^{(1)}(t) - \varphi^{(2)}(t)\big) \right\|_H^2 + 2\left\| \varphi^{(1)}(t) - \varphi^{(2)}(t) \right\|_H^2 + \left\| F\big(\varphi^{(1)}(t)\big) - F\big(\varphi^{(2)}(t)\big) \right\|_H^2 \\
&\leq (d_1 + 2)\left\| \varphi^{(1)}(t) - \varphi^{(2)}(t) \right\|_H^2 + \left\| F\big(\varphi^{(1)}(t)\big) - F\big(\varphi^{(2)}(t)\big) \right\|_H^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\left\| \varphi^{(1)}(t) - \varphi^{(2)}(t) \right\|_H^2 \\
&\leq \left\| \varphi^{(1)}(0) - \varphi^{(2)}(0) \right\|_H^2 + (d_1 + 2)\int_0^t \left\| \varphi^{(1)}(s) - \varphi^{(2)}(s) \right\|_H^2 \\
&\quad + \int_0^t \left\| F\big(\varphi^{(1)}(s)\big) - F\big(\varphi^{(2)}(s)\big) \right\|_H^2 ds \\
&\leq \left\| \varphi^{(1)}(0) - \varphi^{(2)}(0) \right\|_H^2 + \left(d_1 + 2 + \frac{2\beta^2}{\lambda} \right)\int_0^t \left\| \varphi^{(1)}(s) - \varphi^{(2)}(s) \right\|_H^2 \\
&\quad + \frac{2L_f^2}{\lambda}\int_0^t \left\| \varphi_s^{(1)} - \varphi_s^{(2)} \right\|_H^2 ds \\
&= \left(1 + \frac{2hL_f^2}{\lambda} \right)\left\| \varphi^{(1)}(0) - \varphi^{(2)}(0) \right\|_{H_0}^2 + \left(d_1 + 2 + \frac{2\beta^2 + 2L_f^2}{\lambda} \right)\int_0^t \left\| \varphi^{(1)}(s) - \varphi^{(2)}(s) \right\|_H^2.
\end{aligned}
$$

Applying the Gronwall inequality, we have

$$
\left\| \varphi^{(1)}(t) - \varphi^{(2)}(t) \right\|_H^2 \leq \left(1 + \frac{2hL_f^2}{\lambda} \right) e^{(d_1 + 2 + \frac{2\beta^2 + 2L_f^2}{\lambda})t} \left\| \varphi^{(1)}(0) - \varphi^{(2)}(0) \right\|_{H_0}^2, \quad \forall t \in [0, T],
$$

from which we get

$$
\sup_{t \in [0,T]} \left\| \varphi^{(1)}(t) - \varphi^{(2)}(t) \right\|_H^2 \leq \left(1 + \frac{2hL_f^2}{\lambda} \right) e^{(d_1 + 2 + \frac{2\beta^2 + 2L_f^2}{\lambda})T} \left\| \varphi^{(1)}(0) - \varphi^{(2)}(0) \right\|_{H_0}^2.
$$

The proof is complete.   □

Lemma 3.2 *Suppose* (H_1)–(H_3) *hold, Eq.* (3.4) *generates a continuous random dynamical system* φ_t *over* $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in R})$.

Proof The proof is similar to that of Theorem 3.2 in [9], so here it is omitted.   □

4 Existence of random attractor

This section will be devoted to prove the existence of a $\mathcal{D}$-random attractor for $\{S(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ in H_0. Firstly, we introduce an Ornstein–Uhlenbeck process in l^2 on the metric dynamical system $(\Omega,\mathcal{F},P,(\theta_t)_{t\in R})$ given by the Wiener process:

$$z(\theta_t\omega) = -\alpha \int_{-\infty}^0 e^{\alpha s}\theta_t\omega(s)\,ds,$$

where $\alpha > 0$, the above integral solves the following Itô equation:

$$dz + \alpha z\,dt = dw.$$

In fact, there exists a θ_t-invariant set $\Omega' \subset \Omega$ such that
  (i) the mapping $t \to z(\theta_t\omega)$ is continuous for P-a.s. $\omega \in \Omega'$;
  (ii) the random variable $\|z(\theta_t\omega)\|$ is tempered.
  Denote

$$\psi(t) = \big(u(t),v(t)\big)^T = \varphi(t) - \big(0,z(\theta_t\omega)\big)^T,$$

where $\varphi(t)$ is the solution of Eq. (3.4). Then $\psi(t)$ satisfies

$$\begin{cases} \dot{\psi}(t) + D\psi(t) = C(\psi(t),t,\omega), \quad t > 0, \\ \psi_0 = \varphi_0 - (0,z(\theta_{0_-}\omega))^T, \end{cases} \tag{4.1}$$

where $z(\theta_{t_-}\omega) = z(\theta_{t+\tau}\omega)$, $\tau \in [-h,0]$, for any $t \geq 0$, $\psi_t = \varphi_t - (0,z(\theta_{t_-}\omega))^T$, $C(\psi(t),t,\omega) = (z(\theta_t\omega),f(u_t) + g - \beta\sin u + \delta z(\theta_t\omega))^T$, and

$$D\psi = \begin{pmatrix} \delta u - v \\ Au + \lambda u + (\delta - \alpha)(\delta u - v) \end{pmatrix}.$$

Lemma 4.1 *Suppose (H_1)–(H_3) hold, and $\delta > \frac{2M_1 e^{\delta h}}{\sqrt{\lambda}}$, Then there exists a random absorbing set $B_0(\omega) \in \mathcal{D}(H_0)$ for $\{S(t,\omega)\}_{t\geq 0,\omega\in\Omega}$.*

Proof Taking the inner product $(\cdot,\cdot)_H$ on both sides of (4.1) with $\psi(t)$, we get

$$\frac{d}{dt}\big\|\psi(t)\big\|_H^2 + 2\big(D\psi(t),\psi(t)\big)_H = 2\big(C,\psi(t)\big)_H. \tag{4.2}$$

Now, we estimate the terms of (4.2) one by one. We have

$$\begin{aligned} 2\big(D\psi(t),\psi(t)\big)_H &= 2\big(\delta\|u\|_\lambda^2 + (\alpha - \delta)\|v\|^2 - \delta(\alpha - \delta)(u,v)\big) \\ &\geq 2\left(\delta\|u\|_\lambda^2 + (\alpha - \delta)\|v\|^2 - \frac{\delta\alpha}{\sqrt{\lambda}}\|u\|_\lambda\|v\|\right) \\ &\geq 2\delta\|u\|_\lambda^2 + 2(\alpha - \delta)\|v\|^2 - \frac{\delta}{2}\|u\|_\lambda^2 - \frac{2\delta\alpha^2}{\lambda}\|v\|^2 \\ &= \frac{3\delta}{2}\|u\|_\lambda^2 + 2\left(\alpha - \delta - \frac{\delta\alpha^2}{\lambda}\right)\|v\|^2 \end{aligned} \tag{4.3}$$

and

$$2\big(C,\psi(t)\big)_H$$

$$= 2\big(u,z(\theta_t\omega)\big)_\lambda + 2\big(f(u_t),v\big) + 2(g,v) - 2(\beta\sin u,v) + 2\big(\delta z(\theta_t\omega),v\big)$$

$$\leq \frac{\delta}{4}\|u\|_\lambda^2 + \frac{4}{\delta}\big\|z(\theta_t\omega)\big\|_\lambda^2 + 4\delta\big\|z(\theta_t\omega)\big\|^2 + \frac{\delta}{4}\|v\|^2 + 2\sum_{i\in\mathbb{Z}^k}|g_i||v_i|$$

$$+ 2\sum_{i\in\mathbb{Z}^k}\Big(M_{0,i} + M_{1,i}\max_{\tau\in[-h,0]}|u_{it}|\Big)|v_i| + 2\beta\sum_{i\in\mathbb{Z}^k}|\sin u_i||v_i| \tag{4.4}$$

$$\leq \frac{\delta}{2}\|u\|_\lambda^2 + \Big(\frac{16k+4\lambda}{\delta} + 4\delta\Big)\big\|z(\theta_t\omega)\big\|^2 + \Big(\frac{3\delta}{4} + \frac{4\beta^2}{\delta\lambda}\Big)\|v\|^2$$

$$+ \frac{4}{\delta}\big(M_0^2 + \|g\|^2\big) + \frac{2M_1}{\sqrt{\lambda}}\|\psi_t\|_{H_0}^2.$$

By (4.2)–(4.4) and (3.3), we have

$$\frac{d}{dt}\big\|\psi(t)\big\|_H^2 + \delta\big\|\psi(t)\big\|_H^2$$

$$\leq \Big(4\delta + \frac{2\delta\alpha^2}{\lambda} + \frac{4\beta^2}{\delta\lambda} - 2\alpha\Big)\|v\|^2 + \Big(\frac{16k+4\lambda}{\delta} + 4\delta\Big)\big\|z(\theta_t\omega)\big\|^2$$

$$+ \frac{2M_1}{\sqrt{\lambda}}\|\psi_t\|_{H_0}^2 + \frac{4}{\delta}\big(M_0^2 + \|g\|^2\big) \tag{4.5}$$

$$\leq \Big(\frac{16k+4\lambda}{\delta} + 4\delta\Big)\big\|z(\theta_t\omega)\big\|^2 + \frac{2M_1}{\sqrt{\lambda}}\|\psi_t\|_{H_0}^2 + \frac{4}{\delta}\big(M_0^2 + \|g\|^2\big),$$

which gives

$$\frac{d}{dt}\big(e^{\delta t}\big\|\psi(t)\big\|_H^2\big) \leq \Big(\frac{16k+4\lambda}{\delta} + 4\delta\Big)e^{\delta t}\big\|z(\theta_t\omega)\big\|^2$$

$$+ \frac{2M_1 e^{\delta t}}{\sqrt{\lambda}}\|\psi_t\|_{H_0}^2 + \frac{4e^{\delta t}}{\delta}\big(M_0^2 + \|g\|^2\big). \tag{4.6}$$

One deduces by integrating the above inequality on [0,t] that

$$e^{\delta t}\big\|\psi(t)\big\|_H^2 \leq \big\|\psi(0)\big\|_H^2 + \Big(\frac{16k+4\lambda}{\delta} + 4\delta\Big)\int_0^t e^{\delta s}\big\|z(\theta_s\omega)\big\|^2\,ds$$

$$+ \frac{2M_1}{\sqrt{\lambda}}\int_0^t e^{\delta s}\|\psi_s\|_{H_0}^2\,ds + \frac{4e^{\delta t}}{\delta^2}\big(M_0^2 + \|g\|^2\big). \tag{4.7}$$

For fixed $\tau\in[-h,0]$, we have, for all $t\geq 0$,

$$e^{\delta t}\big\|\psi(t+\tau)\big\|_H^2 \leq e^{-\delta\tau}\big\|\psi(0)\big\|_H^2 + e^{-\delta\tau}\Big(\frac{16k+4\lambda}{\delta} + 4\delta\Big)\int_0^{t+\tau} e^{\delta s}\big\|z(\theta_s\omega)\big\|^2\,ds$$

$$+ e^{-\delta\tau}\frac{2M_1}{\sqrt{\lambda}}\int_0^{t+\tau} e^{\delta s}\|\psi_s\|_{H_0}^2\,ds + e^{-\delta\tau}\frac{4e^{\delta(t+\tau)}}{\delta^2}\big(M_0^2 + \|g\|^2\big) \tag{4.8}$$

$$\leq e^{\delta h}\|\psi(0)\|_H^2 + e^{\delta h}\left(\frac{16k + 4\lambda}{\delta} + 4\delta\right)\int_0^t e^{\delta s}\|z(\theta_s\omega)\|^2\,ds$$

$$+ e^{\delta h}\frac{2M_1}{\sqrt{\lambda}}\int_0^t e^{\delta s}\|\psi_s\|_{H_0}^2\,ds + \frac{4e^{\delta t}}{\delta^2}(M_0^2 + \|g\|^2).$$

This implies that

$$
\begin{aligned}
e^{\delta t}\|\psi_t\|_{H_0}^2 &\leq e^{\delta h}\|\psi(0)\|_{H_0}^2 + e^{\delta h}\left(\frac{16k + 4\lambda}{\delta} + 4\delta\right)\int_0^t e^{\delta s}\|z(\theta_s\omega)\|^2\,ds \\
&\quad + e^{\delta h}\frac{2M_1}{\sqrt{\lambda}}\int_0^t e^{\delta s}\|\psi_s\|_{H_0}^2\,ds + \frac{4e^{\delta t}}{\delta^2}(M_0^2 + \|g\|^2).
\end{aligned}
\tag{4.9}
$$

Let us introduce the following notations:

$$c_1 = e^{\delta h}\left(\frac{16k + 4\lambda}{\delta} + 4\delta\right), \qquad c_2 = \frac{4}{\delta^2}(M_0^2 + \|g\|^2), \qquad c_3 = \frac{2M_1 e^{\delta h}}{\sqrt{\lambda}}.$$

Thus, the inequality (4.9) can be rewritten as

$$e^{\delta t}\|\psi_t\|_{H_0}^2 \leq e^{\delta h}\|\psi(0)\|_{H_0}^2 + c_1\int_0^t e^{\delta s}\|z(\theta_s\omega)\|^2\,ds + c_2 e^{\delta t} + c_3\int_0^t e^{\delta s}\|\psi_s\|_{H_0}^2\,ds. \tag{4.10}$$

Applying the Gronwall inequality to the above inequality yields

$$
\begin{aligned}
e^{\delta t}\|\psi_t\|_{H_0}^2 &\leq e^{\delta h}\|\psi(0)\|_{H_0}^2 + c_1\int_0^t e^{\delta s}\|z(\theta_s\omega)\|^2\,ds + c_2 e^{\delta t} \\
&\quad + c_3\int_0^t\left(e^{\delta h}\|\psi(0)\|_{H_0}^2 + c_1\int_0^s e^{\delta\upsilon}\|z(\theta_\upsilon\omega)\|^2\,d\upsilon + c_2 e^{\delta s}\right)e^{c_3(t-s)}\,ds \\
&\leq e^{\delta h}\|\psi(0)\|_{H_0}^2 e^{c_3 t} + \frac{c_2\delta}{\delta - c_3}e^{\delta t} + c_1 e^{c_3 t}\int_0^t e^{(\delta - c_3)s}\|z(\theta_s\omega)\|^2\,ds,
\end{aligned}
\tag{4.11}
$$

and hence

$$
\begin{aligned}
\|\psi_t(\cdot,\omega,\psi_0(\omega))\|_{H_0}^2 &\leq e^{\delta h}\|\psi(0)\|_{H_0}^2 e^{-(\delta - c_3)t} \\
&\quad + \frac{c_2\delta}{\delta - c_3} + c_1\int_0^t e^{(\delta - c_3)(s-t)}\|z(\theta_s\omega)\|^2\,ds.
\end{aligned}
\tag{4.12}
$$

Replacing ω by $\theta_{-t}\omega$ in (4.12), we have

$$
\begin{aligned}
\|\psi_t&(\cdot,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega))\|_{H_0}^2 \\
&\leq e^{\delta h}\|\psi_0(\theta_{-t}\omega)\|_{H_0}^2 e^{-(\delta - c_3)t} + \frac{c_2\delta}{\delta - c_3} + c_1\int_0^t e^{(\delta - c_3)(s-t)}\|z(\theta_{s-t}\omega)\|^2\,ds \\
&\leq e^{\delta h}\|\psi_0(\theta_{-t}\omega)\|_{H_0}^2 e^{-(\delta - c_3)t} + \frac{c_2\delta}{\delta - c_3} + c_1\int_{-\infty}^0 e^{(\delta - c_3)s}\|z(\theta_s\omega)\|^2\,ds.
\end{aligned}
\tag{4.13}
$$

Since $\varphi_t(\cdot, \omega, \varphi_0(\omega)) = \psi_t(\cdot, \omega, \varphi_0(\omega) - (0, z(\theta_{0_-}\omega))^T) + (0, z(\theta_{t_-}\omega))^T$, it follows from (4.13) that

$$
\begin{aligned}
&\left\| \varphi_t\left(\cdot, \theta_{-t}\omega, \varphi_0(\theta_{-t}\omega)\right) \right\|_{H_0}^2 \\
&= \left\| \psi_t\left(\cdot, \theta_{-t}\omega, \varphi_0(\theta_{-t}\omega) - \left(0, z(\theta_{-t}\omega)\right)^T\right) + \left(0, z(\theta_{0_-}\omega)\right)^T \right\|_{H_0}^2 \\
&\le 2\left\| \psi_t\left(\cdot, \theta_{-t}\omega, \varphi_0(\theta_{-t}\omega) - \left(0, z(\theta_{-t}\omega)\right)^T\right) \right\|_{H_0}^2 + 2\left\| \left(0, z(\theta_{0_-}\omega)\right)^T \right\|_{H_0}^2 \\
&\le 4e^{\delta h}\left(\left\| \varphi_0(\theta_{-t}\omega) \right\|_{H_0}^2 + \max_{\tau \in [-h,0]} \left\| z(\theta_{-t+\tau}\omega) \right\|^2 \right)e^{-(\delta - c_3)t} + \frac{2c_2\delta}{\delta - c_3} \\
&\quad + 2c_1 \int_{-\infty}^0 e^{(\delta - c_3)s}\left\| z(\theta_s\omega) \right\|^2 ds + 2\max_{\tau \in [-h,0]} \left\| z(\theta_\tau\omega) \right\|^2.
\end{aligned}
\tag{4.14}
$$

By assumption, $B(\omega) \in \mathcal{D}(H_0)$ is tempered. On the other hand, by Proposition 2.2, we know that $\max_{\tau \in [-h,0]} \|z(\theta_\tau\omega)\|^2$, $\max_{\tau \in [-h,0]} \|z(\theta_{-t+\tau}\omega)\|^2$, and $\int_{-\infty}^0 e^{(\delta - c_3)s}\|z(\theta_s\omega)\|^2 ds$ are also tempered. Thus, if $\varphi_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, then there exists some $T_B(\omega) > 0$ such that, for all $t \ge T_B(\omega)$,

$$
\begin{aligned}
&\left\| \varphi_t\left(\cdot, \theta_{-t}\omega, \varphi_0(\theta_{-t}\omega)\right) \right\|_{H_0}^2 \\
&\le \frac{4c_2\delta}{\delta - c_3} + 2c_1 \int_{-\infty}^0 e^{(\delta - c_3)s}\left\| z(\theta_s\omega) \right\|^2 ds + 2\max_{\tau \in [-h,0]} \left\| z(\theta_\tau\omega) \right\|^2 \equiv R_0^2(\omega),
\end{aligned}
\tag{4.15}
$$

that is, $B_0(\omega) = \{\xi \in H_0 : \|\xi\|_{H_0} \le R_0(\omega)\}$ is a random absorbing set for $\{S(t,\omega)\}_{t \ge 0, \omega \in \Omega}$. The proof is complete. □

For convenience, we denote $\|\psi\|_H^2 = \sum_{i \in \mathbb{Z}^k} |\psi_i|_H^2$, where $|\psi_i|_H^2 = \sum_{j=1}^k (B_j u)_i^2 + \lambda u_i^2 + v_i^2$, for any $\psi = (\psi_i)_{i \in \mathbb{Z}^k} = ((u_i, v_i))_{i \in \mathbb{Z}^k}^T$.

Lemma 4.2 *Suppose the conditions of Lemma 4.1 hold, and $\varphi_0(\omega) \in B_0(\omega)$. Then, for any $\varepsilon > 0$, there exist $M(\varepsilon, \omega) \in N$ and $T(\varepsilon, \omega) > 0$ such that the solution $\varphi(t, \omega, \varphi_0(\omega))$ of Eq. (3.4) satisfies*

$$
\max_{\tau \in [-h,0]} \sum_{\|i\| > M(\varepsilon,\omega)} \left| \varphi_i\left(t + \tau, \theta_{-t}\omega, \varphi_0(\theta_{-t}\omega)\right) \right|_H^2 < \varepsilon, \quad \forall t \ge T(\varepsilon, \omega),
$$

where $i \in \mathbb{Z}^k$ and $\|i\| = \max_{1 \le j \le k} |i_j|$.

Proof Define a smooth increasing function $\eta(x) \in C(R_+, [0, 1]) \bigcap C^1(R_+, R_+)$ such that

$$
\eta(x) = \begin{cases} 0, & 0 \le x \le 1; \\ 1, & x \ge 2, \end{cases}
\tag{4.16}
$$

and $|\eta(x)'| \le \eta_0$ (constant) for all $x \in R_+$. Set $\xi = (\xi_i)_{i \in \mathbb{Z}^k} = (p, q)^T$ with $\xi_i = (p_i, q_i)^T = (\eta(\frac{\|i\|}{M})u_i, \eta(\frac{\|i\|}{M})v_i)^T$, where M is a fixed positive integer. Taking the inner product $(\cdot, \cdot)_H$ of Eq. (4.1) with ξ, we have

$$
(\dot{\psi}, \xi)_H + (D\psi, \xi)_H = \left(C(\psi, t, \omega), \xi \right)_H.
\tag{4.17}
$$

Next, we estimate the terms of (4.17) one by one. Firstly,

$$
\begin{aligned}
(\dot{\psi}, \xi)_H &= \sum_{i \in \mathbb{Z}^k} \sum_{j=1}^{k} (B_j \dot{u})_i (B_j p)_i + \sum_{i \in \mathbb{Z}^k} \lambda \dot{u}_i p_i + \sum_{i \in \mathbb{Z}^k} \dot{v}_i q_i \\
&= \sum_{i \in \mathbb{Z}^k} \sum_{j=1}^{k} (B_j \dot{u})_i \left[\eta\left(\frac{\|i\|}{M} \right) (B_j u)_i + (B_j p)_i - \eta\left(\frac{\|i\|}{M} \right) (B_j u)_i \right] \\
&\quad + \sum_{i \in \mathbb{Z}^k} \lambda \dot{u}_i p_i + \sum_{i \in \mathbb{Z}^k} \dot{v}_i q_i \\
&= \frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M} \right) \left[\sum_{j=1}^{k} (B_j u)_i^2 + \lambda u_i^2 + v_i^2 \right] \\
&\quad + \sum_{i \in \mathbb{Z}^k} \sum_{j=1}^{k} (B_j \dot{u})_i \left[(B_j p)_i - \eta\left(\frac{\|i\|}{M} \right) (B_j u)_i \right]
\end{aligned}
\tag{4.18}
$$

and

$$
\begin{aligned}
&\sum_{i \in \mathbb{Z}^k} \sum_{j=1}^{k} (B_j \dot{u})_i \left[(B_j p)_i - \eta\left(\frac{\|i\|}{M} \right) (B_j u)_i \right] \\
&\leq \frac{2k\eta_0}{M} \left((1+\delta) \|u\|^2 + \|v\|^2 + \|z(\theta_t \omega)\|^2 \right),
\end{aligned}
\tag{4.19}
$$

by (4.18) and (4.19) we have

$$
(\dot{\psi}, \xi)_H \geq \frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M} \right) |\psi_i|_H^2 - \frac{2k\eta_0}{M} \left((1+\delta) \|u\|^2 + \|v\|^2 + \|z(\theta_t \omega)\|^2 \right).
\tag{4.20}
$$

Secondly,

$$
\begin{aligned}
(D\psi, \xi)_H &= \delta(u, p)_\lambda - (v, p)_\lambda + (Au, q) + \lambda(u, q) + (\delta - \alpha)\delta(u, q) + (\alpha - \delta)(v, q) \\
&= \delta \sum_{j=1}^{k} (B_j u, B_j p) + \lambda\delta(u, p) - \sum_{j=1}^{k} (B_j v, B_j p) + \sum_{j=1}^{k} (B_j u, B_j q) \\
&\quad + (\delta - \alpha)\delta(u, q) + (\alpha - \delta)(v, q)
\end{aligned}
\tag{4.21}
$$

and

$$
\delta \sum_{j=1}^{k} (B_j u, B_j p) = \delta \sum_{j=1}^{k} \sum_{i \in \mathbb{Z}^k} (B_j u)_i (B_j p)_i \geq \delta \sum_{j=1}^{k} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M} \right) (B_j u)_i^2 - \frac{2k\delta\eta_0}{M} \|u\|^2,
\tag{4.22}
$$

$$
\sum_{j=1}^{k} (B_j u, B_j q) - \sum_{j=1}^{k} (B_j v, B_j p) \geq -\frac{k\eta_0}{M} \left(\sqrt{\lambda} \|u\|^2 + \frac{1}{\sqrt{\lambda}} \|v\|^2 \right) \geq -\frac{k\eta_0}{\sqrt{\lambda} M} \|\psi\|_H^2,
\tag{4.23}
$$

$$
\delta\lambda(u, p) = \delta\lambda \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M} \right) u_i^2,
\tag{4.24}
$$

$$
(\alpha - \delta)(v, q) = (\alpha - \delta) \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M} \right) v_i^2,
\tag{4.25}
$$

$$(\delta - \alpha)\delta(u, q) = (\delta - \alpha)\delta \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) u_i v_i$$

$$\geq -\frac{\lambda\delta}{4} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) u_i^2 - \frac{\alpha^2 \delta}{\lambda} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) v_i^2. \tag{4.26}$$

By (4.21)–(4.26) we obtain

$$2(D\psi, \xi)_H \geq 2\delta \sum_{i \in \mathbb{Z}^k} \sum_{j=1}^{k} \eta\left(\frac{\|i\|}{M}\right) (B_j u)_i^2 - \frac{4k\delta\eta_0}{M} \|u\|^2 - \frac{2k\eta_0}{\sqrt{\lambda}M} \|\psi\|_H^2$$

$$+ 2\delta\lambda \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) u_i^2 + 2(\alpha - \delta) \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) v_i^2$$

$$- \frac{\lambda\delta}{2} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) u_i^2 - \frac{2\alpha^2\delta}{\lambda} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) v_i^2 \tag{4.27}$$

$$= 2\delta \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) |\psi_i|_H^2 - \frac{4k\delta\eta_0}{M} \|u\|^2 - \frac{2k\eta_0}{\sqrt{\lambda}M} \|\psi\|_H^2$$

$$- \frac{\lambda\delta}{2} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) u_i^2 + \left(2\alpha - 4\delta - \frac{2\alpha^2\delta}{\lambda}\right) \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) v_i^2.$$

Thirdly,

$$\big(C(\psi, t, \omega), \xi\big)_H = (z, p)_\lambda + \big(f(u_t), q\big) + (g, q) + (-\beta \sin u, q) + (\delta z, q) \tag{4.28}$$

and

$$(z, p)_\lambda = \sum_{j=1}^{k} (B_j z, B_j p) + \lambda(z, p)$$

$$\leq \frac{\delta}{2} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) \sum_{j=1}^{k} (B_j u)_i^2 + \frac{\lambda\delta}{8} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) u_i^2 + \frac{k\eta_0}{2M} \big(\|u\|^2 + 4\|z\|^2\big) \tag{4.29}$$

$$+ \frac{2k + 2\lambda}{\delta} \sum_{\|i\| \geq M-1} z_i^2,$$

$$\delta(z, q) = \delta \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) z_i v_i \leq \frac{\delta}{6} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) v_i^2 + \frac{3\delta}{2} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) z_i^2, \tag{4.30}$$

$$(-\beta \sin u, q) = -\beta \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) \sin u_i v_i$$

$$\leq \frac{\lambda\delta}{8} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) u_i^2 + \frac{2\beta^2}{\lambda\delta} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) v_i^2, \tag{4.31}$$

$$(g, q) = \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) g_i v_i \leq \frac{\delta}{6} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) v_i^2 + \frac{3}{2\delta} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) g_i^2, \tag{4.32}$$

$$\big(f(u_t), q\big) = \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) f_i(u_{it}) v_i \le \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)\Big(M_{0,i} + M_{1,i} \max_{\tau\in[-h,0]} |u_{it}|\Big)|v_i|$$

$$= \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) M_{0,i}|v_i| + \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) M_{1,i} \max_{\tau\in[-h,0]} |u_{it}||v_i| \tag{4.33}$$

$$\le \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)\left(\frac{3}{2\delta}M_{0,i}^2 + \frac{\delta}{6}v_i^2\right) + \frac{M_1}{\sqrt{\lambda}}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_{it}|_{H_0}^2.$$

Combining (4.28)–(4.33), we have

$$2\big(C(\psi, t\omega), \xi\big)_H$$

$$\le \delta \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)\left(\sum_{j=1}^{k}(B_j u)_i^2\right) + \frac{\lambda\delta}{2}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)u_i^2 + \delta \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)v_i^2$$

$$+ \frac{k\eta_0}{M}\big(\|u\|^2 + 4\|z\|^2\big) + \frac{4k+4\lambda}{\delta}\sum_{\|i\|\ge M-1} z_i^2 + 3\delta \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)z_i^2 \tag{4.34}$$

$$+ \frac{4\beta^2}{\lambda\delta}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)v_i^2 + \frac{3}{\delta}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)g_i^2$$

$$+ \frac{3}{\delta}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)M_{0,i}^2 + \frac{2M_1}{\sqrt{\lambda}}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_{it}|_{H_0}^2.$$

By (4.17), (4.20), (4.27) and (4.34), we get

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_i|_H^2 + \delta \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_i|_H^2$$

$$\le \frac{k\eta_0}{M}\big((5+8\delta)\|u\|^2 + 4\|v\|^2 + 8\|z\|^2\big) + \frac{3}{\delta}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)g_i^2$$

$$+ \left(\frac{4k+4\lambda}{\delta} + 3\delta\right)\sum_{\|i\|\ge M-1} z_i^2 + \left(\frac{4\beta^2}{\lambda\delta} + 4\delta + \frac{2\alpha^2\delta}{\lambda} - 2\alpha\right)\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)v_i^2 \tag{4.35}$$

$$+ \frac{3}{\delta}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)M_{0i}^2 + \frac{2M_1}{\sqrt{\lambda}}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_{it}|_{H_0}^2 + \frac{2k\eta_0}{\sqrt{\lambda}M}\|\psi\|_H^2.$$

It follows from (3.3) that

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_i|_H^2 + \delta \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_i|_H^2$$

$$\le \frac{k\eta_0}{M}\big((5+8\delta)\|u\|^2 + 4\|v\|^2 + 8\|z\|^2\big) + \frac{3}{\delta}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)\big(g_i^2 + M_{0i}^2\big)$$

$$+ \left(\frac{4k+4\lambda}{\delta} + 3\delta\right)\sum_{\|i\|\ge M-1} z_i^2 + \frac{2M_1}{\sqrt{\lambda}}\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_{it}|_{H_0}^2 + \frac{2k\eta_0}{\sqrt{\lambda}M}\|\psi\|_H^2 \tag{4.36}$$

$$= h_1 \|\psi\|_H^2 + \frac{8k\eta_0}{M}\|z\|^2 + \frac{3}{\delta} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)\left(g_i^2 + M_{0i}^2\right)$$

$$+ h_2 \sum_{\|i\|\geq M-1} z_i^2 + \frac{2M_1}{\sqrt{\lambda}} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_{it}|_{H_0}^2,$$

where $h_1 = \frac{k\eta_0}{M}\max\{\frac{5+8\delta}{\lambda}, 4\} + \frac{2k\eta_0}{\sqrt{\lambda}M}$, $h_2 = \frac{4k+4\lambda}{\delta} + 3\delta$. One deduces by integrating the above inequality on [0,t] that

$$e^{\delta t} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_i|_H^2$$

$$\leq \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_i(0)|_H^2 + h_1 \int_0^t e^{\delta s}\|\psi\|_H^2\,ds + \frac{8k\eta_0}{M}\int_0^t e^{\delta s}\|z\|^2\,ds$$

$$+ \frac{3}{\delta^2} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)(g_i^2 + M_{0i}^2)e^{\delta t} + h_2 \int_0^t e^{\delta s} \sum_{\|i\|\geq M-1} z_i^2\,ds \qquad (4.37)$$

$$+ \frac{2M_1}{\sqrt{\lambda}} \int_0^t \sum_{i\in\mathbb{Z}^k} e^{\delta s} \eta\left(\frac{\|i\|}{M}\right)|\psi_{it}|_{H_0}^2\,ds.$$

Note that $\|\psi(\tau)\|_H \leq \|\psi_0\|_{H_0}$ for fixed $\tau \in [-h, 0]$, hence we get, for all $t \geq 0$,

$$e^{\delta t} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_i(t+\tau)|_H^2$$

$$\leq e^{-\delta\tau} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_{i0}|_H^2 + h_1 e^{-\delta\tau}\int_0^t e^{\delta s}\|\psi\|_H^2\,ds + \frac{8k\eta_0}{M}e^{-\delta\tau}\int_0^t e^{\delta s}\|z\|^2\,ds$$

$$+ h_2 e^{-\delta\tau}\int_0^t e^{\delta s} \sum_{\|i\|\geq M-1} z_i^2\,ds + \frac{2M_1}{\sqrt{\lambda}}e^{-\delta\tau}\int_0^t \sum_{i\in\mathbb{Z}^k} e^{\delta s}\eta\left(\frac{\|i\|}{M}\right)|\psi_{it}|_{H_0}^2\,ds \qquad (4.38)$$

$$+ \frac{3}{\delta^2}e^{\delta t} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)(M_{0i}^2 + g_i^2),$$

from which we get

$$e^{\delta t} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_{it}|_{H_0}^2$$

$$\leq e^{\delta h} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)|\psi_{i0}|_{H_0}^2 + k_1 e^{\delta t} + k_2 \int_0^t e^{\delta s}\|\psi\|_H^2\,ds + k_3 \int_0^t e^{\delta s}\|z\|^2\,ds \qquad (4.39)$$

$$+ k_4 \int_0^t e^{\delta s} \sum_{\|i\|\geq M-1} z_i^2\,ds + c_3 \int_0^t \sum_{i\in\mathbb{Z}^k} e^{\delta s}\eta\left(\frac{\|i\|}{M}\right)|\psi_{it}|_{H_0}^2\,ds,$$

where $k_1 = \frac{3}{\delta^2} \sum_{i \in \mathbb{Z}^k} \eta(\frac{\|i\|}{M})(M_{0i}^2 + g_i^2)$, $k_2 = h_1 e^{\delta h}$, $k_3 = \frac{8k\eta_0}{M} e^{\delta h}$, $k_4 = h_2 e^{\delta h}$. Using the Gronwall inequality yields

$$
\begin{aligned}
e^{\delta t} \sum_{i \in \mathbb{Z}^k} &\eta\left(\frac{\|i\|}{M}\right) |\psi_{it}|_{H_0}^2 \\
&\leq e^{\delta h} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) |\psi_{i0}|_{H_0}^2 e^{c_3 t} + \frac{k_1 \delta}{\delta - c_3} e^{\delta t} + k_2 e^{c_3 t} \int_0^t e^{(\delta - c_3)s} \|\psi\|_H^2 \, ds \\
&\quad + k_3 e^{c_3 t} \int_0^t e^{(\delta - c_3)s} \|z\|^2 \, ds + k_4 e^{c_3 t} \int_0^t e^{(\delta - c_3)s} \sum_{\|i\| \geq M-1} z_i^2 \, ds.
\end{aligned}
\tag{4.40}
$$

This implies that

$$
\begin{aligned}
\sum_{i \in \mathbb{Z}^k} &\eta\left(\frac{\|i\|}{M}\right) |\psi_{it}|_{H_0}^2 \\
&\leq e^{\delta h} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) |\psi_{i0}|_{H_0}^2 e^{-(\delta - c_3)t} + \frac{k_1 \delta}{\delta - c_3} + k_2 e^{-(\delta - c_3)t} \int_0^t e^{(\delta - c_3)s} \|\psi\|_H^2 \, ds \\
&\quad + k_3 e^{-(\delta - c_3)t} \int_0^t e^{(\delta - c_3)s} \|z\|^2 \, ds + k_4 e^{-(\delta - c_3)t} \int_0^t e^{(\delta - c_3)s} \sum_{\|i\| \geq M-1} z_i^2 \, ds.
\end{aligned}
\tag{4.41}
$$

Replacing ω with $\theta_{-t}\omega$, we have

$$
\begin{aligned}
\sum_{i \in \mathbb{Z}^k} &\eta\left(\frac{\|i\|}{M}\right) \left|\psi_{it}\left(\cdot, \theta_{-t}\omega, \psi_0(\theta_{-t}\omega)\right)\right|_{H_0}^2 \\
&\leq e^{\delta h} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) \left|\psi_{i0}(\theta_{-t}\omega)\right|_{H_0}^2 e^{-(\delta - c_3)t} + \frac{k_1 \delta}{\delta - c_3} \\
&\quad + k_2 e^{-(\delta - c_3)t} \int_0^t e^{(\delta - c_3)s} \left\|\psi\left(s, \theta_{-t}\omega, \psi_0(\theta_{-t}\omega)\right)\right\|_H^2 \, ds \\
&\quad + k_3 \int_{-\infty}^0 e^{(\delta - c_3)s} \left\|z(\theta_s\omega)\right\|^2 \, ds + k_4 \int_{-\infty}^0 e^{(\delta - c_3)s} \sum_{\|i\| \geq M-1} z_i^2(\theta_s\omega) \, ds.
\end{aligned}
\tag{4.42}
$$

Next, we estimate the terms of (4.42). Note that $\|\varphi_0(\theta_{-t}\omega)\|^2$, $\max_{\tau \in [-h,0]} \|z(\theta_{-t+\tau}\omega)\|^2$ are tempered, we find that, for every $\varepsilon > 0$, there exists some $T_1(\varepsilon, \omega) > 0$ such that, for all $t \geq T_1(\varepsilon, \omega)$,

$$
\begin{aligned}
e^{\delta h} \sum_{i \in \mathbb{Z}^k} &\eta\left(\frac{\|i\|}{M}\right) \left|\psi_{i0}(\theta_{-t}\omega)\right|_{H_0}^2 e^{-(\delta - c_3)t} \\
&\leq e^{\delta h} \left\|\psi_0(\theta_{-t}\omega)\right\|_{H_0}^2 e^{-(\delta - c_3)t} \\
&\leq 2e^{\delta h} \left(\left\|\varphi_0(\theta_{-t}\omega)\right\|_{H_0}^2 + \max_{\tau \in [-h,0]} \left\|z(\theta_{-t+\tau}\omega)\right\|^2\right) e^{-(\delta - c_3)t} < \frac{\varepsilon}{4}.
\end{aligned}
\tag{4.43}
$$

Since $M_0, g \in l^2$, there exists $M_1(\varepsilon, \omega) > 0$ such that

$$
\frac{k_1 \delta}{\delta - c_3} < \frac{\varepsilon}{4}, \quad \forall M \geq M_1(\varepsilon, \omega).
\tag{4.44}
$$

In the light of $\int_{-\infty}^{0} e^{(\delta-c_3)s}\|z(\theta_s\omega)\|^2\,ds < \infty$ and the Lebesgue theorem of dominated convergence, there exists some $M_2(\varepsilon,\omega) > 0$ such that, for all $M \geq M_2(\varepsilon,\omega)$,

$$k_3 \int_{-\infty}^{0} e^{(\delta-c_3)s}\big\|z(\theta_s\omega)\big\|^2\,ds + k_4 \int_{-\infty}^{0} e^{(\delta-c_3)s} \sum_{\|i\|\geq M-1} z_i^2(\theta_s\omega)\,ds < \frac{\varepsilon}{4}. \tag{4.45}$$

Moreover, it follows from (4.13) that

$$
\begin{aligned}
&k_2 e^{-(\delta-c_3)t} \int_0^t e^{(\delta-c_3)s}\big\|\psi\big(s,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega)\big)\big\|_H^2\,ds \\
&\quad \leq k_2 e^{-(\delta-c_3)t} \int_0^t e^{(\delta-c_3)s}\left(e^{\delta h}\big\|\psi_0(\theta_{-t}\omega)\big\|_{H_0}^2 e^{-(\delta-c_3)s} + \frac{c_2\delta}{\delta-c_3}\right)ds \\
&\qquad + k_2 e^{-(\delta-c_3)t} \int_0^t e^{(\delta-c_3)s} c_1 \int_{-\infty}^{0} e^{(\delta-c_3)\upsilon}\big\|z(\theta_\upsilon\omega)\big\|^2\,d\upsilon\,ds \\
&\quad \leq k_2 e^{\delta h}\big\|\psi_0(\theta_{-t}\omega)\big\|_{H_0}^2 te^{-(\delta-c_3)t} + \frac{c_2 k_2\delta}{(\delta-c_3)^2} + \frac{c_1 k_2}{\delta-c_3}\int_{-\infty}^{0} e^{(\delta-c_3)\upsilon}\big\|z(\theta_\upsilon\omega)\big\|^2\,d\upsilon,
\end{aligned}
\tag{4.46}
$$

which shows that there exist $T_2(\varepsilon,\omega) > 0$ and $M_3(\varepsilon,\omega) > 0$ such that

$$k_2 e^{-(\delta-c_3)t}\int_0^t e^{(\delta-c_3)s}\big\|\psi\big(s,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega)\big)\big\|_H^2\,ds < \frac{\varepsilon}{4},$$
$$\forall t \geq T_2(\varepsilon,\omega), M \geq M_3(\varepsilon,\omega). \tag{4.47}$$

We set

$$
\begin{aligned}
T(\varepsilon,\omega) &= \max\big\{T_1(\varepsilon,\omega), T_2(\varepsilon,\omega)\big\}, \\
M_4(\varepsilon,\omega) &= \max\big\{M_1(\varepsilon,\omega), M_2(\varepsilon,\omega), M_3(\varepsilon,\omega)\big\},
\end{aligned}
\tag{4.48}
$$

we have

$$\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)\big|\psi_{it}\big(\cdot,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega)\big)\big|_{H_0}^2 < \varepsilon, \quad \forall t \geq T(\varepsilon,\omega), M \geq M_4(\varepsilon,\omega), \tag{4.49}$$

from which we get

$$
\begin{aligned}
&\max_{\tau\in[-h,0]} \sum_{\|i\|\geq 2K} \big|\varphi_i\big(t+\tau,\theta_{-t}\omega,\varphi_0(\theta_{-t}\omega)\big)\big|_H^2 \\
&\quad \leq 2\max_{\tau\in[-h,0]} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)\big|\psi_i\big(t+\tau,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega)\big)\big|_H^2 \\
&\qquad + 2\max_{\tau\in[-h,0]} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)\big|z_i(\theta_\tau\omega)\big|^2 \\
&\quad = 2\sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)\big|\psi_{it}\big(\cdot,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega)\big)\big|_{H_0}^2 \\
&\qquad + 2\max_{\tau\in[-h,0]} \sum_{i\in\mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right)\big|z_i(\theta_\tau\omega)\big|^2 < 4\varepsilon,
\end{aligned}
\tag{4.50}
$$

where $M(\varepsilon, \omega) = \max\{M_4(\varepsilon, \omega), M_5(\varepsilon, \omega)\}$, and $M_5(\varepsilon, \omega) \in N$ leads to

$$\max_{\tau \in [-h, 0]} \sum_{i \in \mathbb{Z}^k} \eta\left(\frac{\|i\|}{M}\right) |z_i(\theta_\tau \omega)|^2 < \varepsilon, \quad M > M_5(\varepsilon, \omega). \tag{4.51}$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 4.1 *Suppose that Lemma* 4.1 *holds, then the* $SDS\{S(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ *over* $(\Omega, F, P, (\theta_t)_{t \in R})$ *defined by Eq.* (3.4) *has a unique $\mathcal{D}$-random attractor $\mathcal{A}(\omega)$.*

Proof In the light of Proposition 2.3 and Lemma 4.1, it suffices to prove that, for a.e. $\omega \in \Omega$, each sequence $\varphi_{t_n}(\cdot, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)) = S(t_n, \theta_{-t_n}\omega)\varphi_0(\theta_{-t_n}\omega)$ has a convergent subsequence in H_0 as $t_n \to \infty$ and $\varphi_0(\theta_{-t_n}\omega) \in B_0(\theta_{-t_n}\omega)$. By (4.14) we have

$$\left\| \varphi\big(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) \right\|_H \leq \left\| \varphi_{t_n}\big(\cdot, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) \right\|_{H_0} \leq C, \quad \forall \tau \in [-h, 0],$$

where $C > 0$ is a given constant. For a fixed $\tau \in [-h, 0]$, we can find a subsequence $\{\varphi(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega))\}$ and $\mu(\tau) \in H$ such that

$$\varphi\big(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) \to \mu(\tau) \quad \text{weakly in } H, \text{as } n \to \infty.$$

Next, we prove the above convergence is also strong. We know, for any $\varepsilon > 0$, there exist $N(\varepsilon, \omega)$ and $\hat{M}(\varepsilon, \omega)$ such that

$$\sum_{\|i\| > \hat{M}(\varepsilon, \omega)} \left| \varphi_i\big(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) \right|_H^2 < \varepsilon, \qquad \sum_{\|i\| > \hat{M}(\varepsilon, \omega)} \left| \mu_i(\tau) \right|_H^2 < \varepsilon, \tag{4.52}$$

and for $n \geq N(\varepsilon, \omega)$ we have

$$\sum_{\|i\| \leq \hat{M}(\varepsilon, \omega)} \left| \varphi_i\big(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) - \mu_i(\tau) \right|_H^2 < \varepsilon. \tag{4.53}$$

By (4.52) and (4.53), we have

$$\begin{aligned}
&\left\| \varphi\big(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) - \mu(\tau) \right\|_H^2 \\
&\quad \leq \sum_{\|i\| > \hat{M}(\varepsilon, \omega)} \left| \varphi_i\big(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) - \mu_i(\tau) \right|_H^2 \\
&\qquad + \sum_{\|i\| \leq \hat{M}(\varepsilon, \omega)} \left| \varphi_i\big(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) - \mu_i(\tau) \right|_H^2 \leq 5\varepsilon.
\end{aligned} \tag{4.54}$$

This shows that, for any $\tau \in [-h, 0]$, $\varphi(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)) \to \mu(\tau)$ is strong in H as $n \to \infty$. In addition, making use of the integral representation of solutions, we obtain for any $\tau_1, \tau_2 \in [-h, 0]$

$$\begin{aligned}
&\left\| \varphi\big(t_n + \tau_1, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) - \varphi\big(t_n + \tau_2, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) \right\|_H \\
&\quad \leq \left\| \psi\big(t_n + \tau_1, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) - \psi\big(t_n + \tau_2, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\big) \right\|_H \\
&\qquad + \left\| z(\theta_{\tau_1}\omega) - z(\theta_{\tau_2}\omega) \right\|
\end{aligned} \tag{4.55}$$

$$\leq \int_{\tau_2}^{\tau_1} \left(\left\| D\psi \left(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\right) \right\|_H + \left\| C\psi \left(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\right) \right\|_H \right) ds$$
$$+ \left\| z(\theta_{\tau_1}\omega) - z(\theta_{\tau_2}\omega) \right\|.$$

It is obvious that D is a linear operator from H into itself and the operator C is bounded. Hence we have

$$\lim_{|\tau_1-\tau_2|\to 0} \int_{\tau_2}^{\tau_1} \left(\left\| D\psi \left(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\right) \right\|_H + \left\| C\psi \left(t_n + \tau, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\right) \right\|_H \right) ds$$
$$= 0 \tag{4.56}$$

and

$$\lim_{|\tau_1-\tau_2|\to 0} \left\| z(\theta_{\tau_1}\omega) - z(\theta_{\tau_2}\omega) \right\| = 0. \tag{4.57}$$

Hence

$$\lim_{|\tau_1-\tau_2|\to 0} \left\| \varphi \left(t_n + \tau_1, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\right) - \varphi \left(t_n + \tau_2, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega)\right) \right\|_H = 0, \tag{4.58}$$

which is the required equicontinuity. In view of the Ascoli–Arzelá theorem, we conclude that there exists a subsequence $\{\varphi_{t_{n_k}}(\cdot, \theta_{-t_{n_k}}\omega, \varphi_0(\theta_{-t_{n_k}}\omega))\}$ of $\{\varphi_{t_n}(\cdot, \theta_{-t_n}\omega, \varphi_0(\theta_{-t_n}\omega))\}$ such that

$$\varphi_{t_{n_k}}\left(\cdot, \theta_{-t_{n_k}}\omega, \varphi_0(\theta_{-t_{n_k}}\omega)\right) \to \mu(\cdot) \quad \text{strongly in } H_0.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

5 An upper bound of the Kolmogorov ε-entropy

In this section, we study the upper bound of the Kolmogorov ε-entropy of the global random $\mathcal{D}$-attractor $\mathcal{A}(\omega)$ given by Theorem 4.1.

Theorem 5.1 *Under the same conditions of Theorem* 4.1, *for a.e.* $\omega \in \Omega$,

$$K_\varepsilon \left(\mathcal{A}(\omega)\right) \leq \left(2\hat{M}(\varepsilon, \omega) + 1\right) \ln\left(\left[\frac{R_0(\omega)}{\sqrt{\lambda}} \cdot \frac{4k + \lambda}{\varepsilon} \cdot \sqrt{2\hat{M}(\varepsilon, \omega) + 1}\right] + 1\right)$$
$$+ \left(2\hat{M}(\varepsilon, \omega) + 1\right) \ln\left(\left[R_0(\omega) \cdot \frac{4k + \lambda}{\varepsilon} \cdot \sqrt{2\hat{M}(\varepsilon, \omega) + 1}\right] + 1\right), \tag{5.1}$$

where $\hat{M}(\varepsilon, \omega) \doteq \hat{M}(\frac{\sqrt{4k+\lambda}-\sqrt{2}}{\sqrt{4k+\lambda}}\varepsilon, \omega, B_0)$ *is the minimal positive integer such that*

$$\sup_{(\varphi_{it})_{i\in\mathbb{Z}^k}=\varphi_t\in B_0} \left(\sum_{\|i\|>\hat{M}(\varepsilon, \omega)} |\varphi_{it}|_H^2 \right)^{\frac{1}{2}} \leq \frac{\sqrt{4k+\lambda}-\sqrt{2}}{\sqrt{4k+\lambda}}\varepsilon. \tag{5.2}$$

Proof By Lemma 4.1, we obtain $\mathcal{A}(\omega) = \varphi_t(t, \theta_{-t}\omega)\mathcal{A}(\theta_{-t}\omega) \subset B_0(\omega)$ for $t > T(\omega, \mathcal{A})$. Thus, for any $\varepsilon > 0$ and $\varphi_t = (\varphi_{it})_{i\in\mathbb{Z}^k} = ((u_{it}, v_{it})_{i\in\mathbb{Z}^k})^T = \varphi_t(t, \theta_{-t}\omega)\varphi_0(\theta_{-t}\omega) \in \mathcal{A}$, where $\varphi_0(\omega) \in$

$\mathcal{A}(\omega) \subset B_0(\omega)$, and by Lemma 4.2 there exists some $\hat{M}(\varepsilon, \omega) \doteq \hat{M}(\frac{\sqrt{4k+\lambda}-\sqrt{2}}{\sqrt{4k+\lambda}}\varepsilon, \omega, B_0) \in \mathbb{N}$ such that (5.2) holds. Next, we decompose φ into two parts as

$$\varphi_t = \varphi_t^{(1)} + \varphi_t^{(2)} = \left(\varphi_{it}^{(1)}\right)_{i \in \mathbb{Z}^k} + \left(\varphi_{it}^{(2)}\right)_{i \in \mathbb{Z}^k}, \tag{5.3}$$

where

$$\varphi_{it}^{(1)} = (\varsigma_{it}, \mu_{it})^T = \begin{cases} \varphi_{it}, & \|i\| \le \hat{M}(\varepsilon, \omega); \\ 0, & \|i\| > \hat{M}(\varepsilon, \omega); \end{cases} \tag{5.4}$$

and

$$\varphi_{it}^{(2)} = (\varsigma_{it}, \mu_{it})^T = \begin{cases} 0, & \|i\| \le \hat{M}(\varepsilon, \omega); \\ \varphi_{it}, & \|i\| > \hat{M}(\varepsilon, \omega). \end{cases} \tag{5.5}$$

By (5.2)–(5.5) we get

$$\left\|\varphi_t^{(2)}\right\|_H = \left(\sum_{\|i\|>\hat{M}(\varepsilon,\omega)} |\varphi_{it}|_H^2\right)^{\frac{1}{2}} \le \frac{\sqrt{4k+\lambda}-\sqrt{2}}{\sqrt{4k+\lambda}}\varepsilon,$$

$$\left\|\varphi_t^{(1)}\right\|_H^2 = \sum_{\|i\|\le\hat{M}(\varepsilon,\omega)} \left|\varphi_{it}^{(1)}\right|_H^2 = \sum_{\|i\|\le\hat{M}(\varepsilon,\omega)} \left(\sum_{j=1}^k (B_j\varsigma_t)_i^2 + \lambda\varsigma_{it}^2 + \mu_{it}^2\right) \le \|\varphi_t\|_{H_0}^2 \le R_0^2(\omega),$$

this implies that

$$|\varsigma_{it}| \le \frac{R_0(\omega)}{\sqrt{\lambda}}, \qquad |\mu_{it}| \le R_0(\omega), \quad \forall \|i\| \le \hat{M}(\varepsilon, \omega).$$

Consider the regular polyhedron

$$\Upsilon_1 = \left\{\varsigma_t = (\varsigma_{it})_{\|i\|\le\hat{M}(\varepsilon,\omega)} : \varsigma_{it} \in \mathbb{R}, |\varsigma_{it}| \le \frac{R_0(\omega)}{\sqrt{\lambda}}\right\} \subset \mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1},$$

by Lemma 2.1 we see that Υ_1 can be covered by

$$\mathcal{N}_{\varepsilon,\omega}^{(1)}(\Upsilon_1) = \left(\left[\frac{R_0(\omega)}{\sqrt{\lambda}} \cdot \frac{4k+\lambda}{\varepsilon} \cdot \sqrt{2\hat{M}(\varepsilon,\omega)+1}\right] + 1\right)^{2\hat{M}(\varepsilon,\omega)+1}$$

balls in $\mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1}$ with radii $\frac{\varepsilon}{4k+\lambda}$. Next, we study the other regular polyhedron

$$\Upsilon_2 = \left\{\mu_t = (\mu_{it})_{\|i\|\le\hat{M}(\varepsilon,\omega)} : \mu_{it} \in \mathbb{R}, |\mu_{it}| \le R_0(\omega)\right\} \subset \mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1},$$

it can be covered by

$$\mathcal{N}_{\varepsilon,\omega}^{(2)}(\Upsilon_2) = \left(\left[R_0(\omega) \cdot \frac{4k+\lambda}{\varepsilon} \cdot \sqrt{2\hat{M}(\varepsilon,\omega)+1}\right] + 1\right)^{2\hat{M}(\varepsilon,\omega)+1}$$

balls in $\mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1}$ with radii $\frac{\varepsilon}{4k+\lambda}$. Therefore, the polyhedron

$$\Upsilon = \Upsilon_1 \times \Upsilon_2 = \left\{ \varphi_t^{(1)} = (\varsigma_{it}, \mu_{jt})_{\|i\|,\|j\| \le \hat{M}(\varepsilon,\omega)} : |\varsigma_{it}| \le \frac{R_0(\omega)}{\sqrt{\lambda}}, |\mu_{it}| \le R_0(\omega) \right\}$$

$$\subset \mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1} \times \mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1}$$

can be covered by

$$\mathcal{N}_{\varepsilon,\omega}(\Upsilon) = \mathcal{N}_{\varepsilon,\omega}^{(1)}(\Upsilon_1) \times \mathcal{N}_{\varepsilon,\omega}^{(2)}(\Upsilon_2)$$

$$= \left(\left[\frac{R_0(\omega)}{\sqrt{\lambda}} \cdot \frac{4k+\lambda}{\varepsilon} \cdot \sqrt{2\hat{M}(\varepsilon,\omega)+1} \right] + 1 \right)^{2\hat{M}(\varepsilon,\omega)+1}$$

$$\times \left(\left[R_0(\omega) \cdot \frac{4k+\lambda}{\varepsilon} \cdot \sqrt{2\hat{M}(\varepsilon,\omega)+1} \right] + 1 \right)^{2\hat{M}(\varepsilon,\omega)+1}$$

balls in $\mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1} \times \mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1}$ with radii $\frac{\sqrt{2}\varepsilon}{4k+\lambda}$. Let the centers of those balls be

$$\varphi_{\iota t}^* = \left(\varsigma_{\iota it}^*, \mu_{\iota jt}^* \right)_{\|i\|,\|j\| \le \hat{M}(\varepsilon,\omega)} \subset \mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1} \times \mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1},$$

where $\iota = 1, 2, \dots, \mathcal{N}_{\varepsilon,\omega}(\Upsilon)$. We choose

$$\hat{\varphi}_{\iota t} = (\hat{\varphi}_{\iota ijt})_{i,j \in \mathbb{Z}^k} = \begin{cases} \varphi_{\iota t}^*, & \max\{\|i\|, \|j\|\} \le \hat{M}(\varepsilon,\omega); \\ 0, & \max\{\|i\|, \|j\|\} > \hat{M}(\varepsilon,\omega). \end{cases}$$

Then there exists some ι $(1 \le \iota \le \mathcal{N}_{\varepsilon,\omega}(\Upsilon))$ such that

$$\left\| \varphi_t^{(1)} - \hat{\varphi}_{\iota t} \right\|_H \le \sqrt{4k+\lambda} \left\| (\varphi_{it})_{\|i\| \le \hat{M}(\varepsilon,\omega)} - \varphi_{\iota t}^* \right\|_{\mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1} \times \mathbb{R}^{2\hat{M}(\varepsilon,\omega)+1}}$$

$$\le \sqrt{4k+\lambda} \cdot \frac{\sqrt{2}}{4k+\lambda} \varepsilon = \frac{\sqrt{2}}{\sqrt{4k+\lambda}} \varepsilon.$$

Thus, for any $\varphi \in \mathcal{A}(\omega) \subset \mathcal{B}_0(\omega)$, there exists some ι $(1 \le \iota \le \mathcal{N}_{\varepsilon,\omega}(\Upsilon))$ such that

$$\|\varphi_t - \hat{\varphi}_{\iota t}\|_H = \left\| \varphi_t^{(1)} + \varphi_t^{(2)} - \hat{\varphi}_{\iota t} \right\|_H \le \left\| \varphi_t^{(2)} \right\|_H + \left\| \varphi_t^{(1)} - \hat{\varphi}_{\iota t} \right\|_H$$

$$\le \frac{\sqrt{2}}{\sqrt{4k+\lambda}} \varepsilon + \frac{\sqrt{4k+\lambda} - \sqrt{2}}{\sqrt{4k+\lambda}} \varepsilon = \varepsilon,$$

which means that the global random $\mathcal{D}$-attractor $\mathcal{A}(\omega) \subset H$ can be covered by $\mathcal{N}_{\varepsilon,\omega}(\Upsilon)$ balls centered at $\hat{\varphi}_{\iota t}$, $\iota = 1, 2, \dots, \mathcal{N}_{\varepsilon,\omega}(\Upsilon)$, with radii ε. The proof is complete. □

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