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Existence of solution for a resonant p-Laplacian second-order m-point boundary value problem on the half-line with two dimensional kernel

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Abstract

The existence of a solution for a second-order p-Laplacian boundary value problem at resonance with two dimensional kernel will be considered in this paper. A semi-projector, the Ge and Ren extension of Mawhin's coincidence degree theory, and algebraic processes will be used to establish existence results, while an example will be given to validate our result.

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1 Introduction

The following second-order p-Laplacian boundary value problem will be considered in this work:

$$\begin{cases} (\varphi_p(u'(t)))' + g(t,u(t),u'(t)) = 0, & t \in (0,+\infty), \\ \varphi_p(u'(0)) = \int_0^{+\infty} \nu(t)\varphi_p(u'(t))\,dt, & \varphi_p(u'(+\infty)) = \sum_{j=1}^{m} \beta_j \int_0^{\eta_j} \varphi_p(u'(t))\,dt, \end{cases} \tag{1.1}$$

where $g : [0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is an L^1-Carathéodory function, $0 < \eta_1 < \eta_2 < \cdots \le \eta_m < +\infty$, $\beta_j \in \mathbb{R}$, $j = 1,2,\ldots,m$, $\nu \in L^1[0,+\infty)$, $\nu(t) > 0$ on $[0,+\infty)$, and

$$\varphi_p(s) = |s|^{p-2}s, \quad p \ge 2.$$

There are many real life applications of boundary value problems with integral and multi-point boundary conditions on an unbounded domain, for instance, in the study of physical phenomena such as the study of an unsteady flow of fluid through a semi-infinite porous medium and radially symmetric solutions of nonlinear elliptic equations. They also arise in plasma physics and in the study of drain flows; see [1–3].

Boundary value problems are said to be at resonance if the solution of the corresponding homogeneous boundary value problem is non-trivial. Many authors in the literature have considered resonant problems. López-Somoza and Minhós [4] obtained existence results for a resonant multi-point second-order boundary value problem on the half-line, Capitanelli, Fragapane and vivaldi [5] addressed regularity results for p-Laplacians in pre-fractal domains, while Jiang and Kosmatov [6] considered resonant p-Laplacian problems with functional boundary conditions. For other work on resonant problems without p-Laplacian operator, see [7–10], while for problems with the p-Laplacian operator, see [11–16]. In [17], Jiang considered the following p-Laplacian operator:

$$\begin{cases} (\varphi_p(u'))' + f(t,u,u') = 0, & 0 < t < +\infty, \\ u(0) = 0, & \varphi_p(u(+\infty)) = \sum_{i=1}^{n} \alpha_i \varphi_p(u'(\xi_i)), \end{cases}$$

where $\alpha_i > 0$, $i = 1, 2, \dots, n$, $\sum_{i=1}^{n} \alpha_i = 1$.

To the best of our knowledge p-Laplacian problems with two dimensional kernel on the half-line have not received much attention in the literature.

We will give the required lemmas, theorem and definitions in Sect. 2, Sect. 3 will be dedicated to stating and proving condition for existence of solutions, while an example will be given in Sect. 4 to validate the result obtained.

2 Preliminaries

In this section, we will give some definitions and lemmas that will be used in this work.

Definition 2.1 ([11]) A map $w : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is $L^1[0, +\infty)$-Carathéodory, if the following conditions are satisfied:

(i) for each $(d, e) \in \mathbb{R}^2$, the mapping $t \to w(t, d, e)$ is Lebesgue measurable;
(ii) for a.e. $t \in [0, \infty)$, the mapping $(d, e) \to w(t, d, e)$ is continuous on $\mathbb{R}^2$;
(iii) for each $k > 0$, there exists $\varphi_k(t) \in L_1[0, +\infty)$ such that, for a.e. $t \in [0, \infty)$ and every $(d, e) \in [-k, k]$, we have

$$\left| w(t, d, e) \right| \le \varphi_k(t).$$

Definition 2.2 ([18]) Let $(U, \|\cdot\|_U)$ and $(Z, \|\cdot\|_Z)$ be two Banach spaces. The continuous operator $M : U \cap \operatorname{dom} M \to Z$, is quasi-linear if the following hold:

(i) $\operatorname{Im} M = M(U \cap \operatorname{dom} M)$ is a closed subset of Z;
(ii) $\ker M = \{u \in U \cap \operatorname{dom} M : Mu = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < +\infty$.

Definition 2.3 ([19]) Let U be a Banach space and $U_1 \subset U$ a subspace. Let $P, Q : U \to U_1$ be operators, then P is a projector if

(i) $P^2 = P$;
(ii) $P(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 P u_1 + \lambda_2 P u_2$ where $u_1, u_2 \in U$, $\lambda_1, \lambda_2 \in \mathbb{R}$,

and Q is a semi-projector if

(i) $Q^2 = Q$;
(ii) $Q(\lambda u) = \lambda Q u$ where $u \in U$, $\lambda \in \mathbb{R}$.

Let $U_1 = \ker M$ and U_2 be the complement space of U_1 in U, then $U = U_1 \oplus U_2$. Similarly, if Z_1 is a subspace of Z and Z_2 is the complement space of Z_1 in Z, then $Z = Z_1 \oplus Z_2$. Let

$P : U \to U_1$ be a projector, $Q : Z \to Z_1$ be a semi-projector and $\Omega \subset U$ an open bounded set with $\theta \in \Omega$ the origin. Also, let N_1 be denoted by N, let $N_\lambda : \overline{\Omega} \to Z$, where $\lambda \in [0,1]$ is a continuous operator and $\Sigma_\lambda = \{u \in \overline{\Omega} : Mu = N_\lambda u\}$.

Definition 2.4 ([20]) Let U be the space of all continuous and bounded vector-valued functions on $[0,+\infty)$ and $X \subset U$. Then X is said to be relatively compact if the following statements hold:

(i) X is bounded in U;

(ii) all functions from X are equicontinuous on any compact subinterval of $[0,+\infty)$;

(iii) all functions from X are equiconvergent at ∞, i.e. $\forall\epsilon > 0$, $\exists$ a $T = T(\epsilon)$ such that $\|A(t) - A(+\infty)\|_{R^n} < \epsilon$ $\forall t > T$ and $A \in X$.

Definition 2.5 ([18]) Let $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ be a continuous operator. The operator N_λ is said to be M-compact in $\overline{\Omega}$ if there exist a vector subspace $Z_1 \in Z$ such that $\dim Z_1 = \dim U_1$ and a compact and continuous operator $R : \overline{\Omega} \times [0,1] \to U_2$ such that, for $\lambda \in [0,1]$, the following holds:

(i) $(I - Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I - B)Z$,

(ii) $QN_\lambda u = 0 \Leftrightarrow QNu = 0$, $\lambda \in (0,1)$,

(iii) $R(\cdot,u)$ is the zero operator and $R(\cdot,\lambda)|_{\Sigma_\lambda} = (I - P)|_{\Sigma_\lambda}$,

(iv) $M[P + R(\cdot,\lambda)] = (I - Q)N_\lambda$.

Lemma 2.1 ([19]) *The following are properties of the function* $\varphi_p : \mathbb{R} \to \mathbb{R}$:

(i) *It is continuous, monotonically increasing and invertible. Its inverse* $\varphi_p^{-1} = \varphi_q$, *where* $q > 1$ *and satisfies* $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) *For any* $x, y > 0$,

  (a) $\varphi_p(x + y) \le \varphi_p(x) + \varphi_p(y)$, *if* $1 < p < 2$,

  (b) $\varphi_p(x + y) \le 2^{p-2}(\varphi_p(x) + \varphi_p(y))$, *if* $p \ge 2$.

Theorem 2.1 ([18]) *Let* $(U, \|\cdot\|_U)$ *and* $(Z, \|\cdot\|_Z)$ *be two Banach spaces and* $\Omega \subset U$ *an open and bounded set. If the following holds*:

(A_1) *The operator* $M : U \cap \operatorname{dom} M \to Z$ *is a quasi-linear,*

(A_2) *the operator* $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ *is* M*-compact,*

(A_3) $Mu \ne N_\lambda u$, *for* $\lambda \in (0,1)$, $u \in \partial\Omega \cap \operatorname{dom} M$,

(A_4) $\deg\{JQN, \Omega \cap \ker M, 0\} \ne 0$, *where the operator* $J : Z_1 \to U_1$ *is a homeomorphism with* $J(\theta) = \theta$ *and* deg *is the Brouwer degree,*

then the equation $Mu = Nu$ *has at least one solution in* $\overline{\Omega}$.

Let

$$U = \left\{u \in C^2[0,+\infty) : u, \varphi_p(u') \in AC[0,+\infty), \lim_{t\to+\infty} e^{-t}\left|u^{(i)}(t)\right| \text{ exist}, i = 0,1\right\},$$

with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$ defined on U where $\|u\|_\infty = \sup_{t\in[0,+\infty)} e^{-t}|u|$. The space $(U, \|\cdot\|)$ by a standard argument is a Banach Space.

Let $Z = L^1[0,+\infty)$ with the norm $\|w\|_{L^1} = \int_0^{+\infty} |w(v)|\,dv$. Define M as a continuous operator such that $M : \operatorname{dom} M \subset U \to Z$ where

$$\operatorname{dom} M = \biggl\{ u \in U : \bigl(\varphi_p(u')\bigr)' \in L^1[0,+\infty), \varphi_p\bigl(u'(0)\bigr) = \int_0^{+\infty} \nu(t)\varphi_p\bigl(u'(t)\bigr)\,dt, \\ \lim_{t\to+\infty}\bigl(\varphi_p\bigl(u'(t)\bigr)\bigr) = \sum_{j=1}^{m} \beta_j \int_0^{\eta_j} \varphi_p\bigl(u'(t)\bigr)\,dt \biggr\}$$

and $Mu = (\varphi_p(u'(t)))'$. We will define the operator $N_\lambda u : \overline{\Omega} \to Z$ by

$$N_\lambda u = -\lambda g\bigl(t, u(t), u'(t)\bigr), \quad \lambda \in [0,1], t \in [0,+\infty),$$

where $\Omega \subset U$ is an open and bounded set. Then the boundary value problem (1.1) in abstract form is $Mu = Nu$.

Throughout the paper we will assume the hypotheses:

(ϕ_1) $\sum_{j=1}^{m} \beta_j\eta_j = \int_0^{+\infty} \nu(t)\,dt = 1$;

(ϕ_2)

$$C = \begin{vmatrix} Q_1 e^{-t} & Q_2 e^{-t} \\ Q_1 t e^{-t} & Q_2 t e^{-t} \end{vmatrix} := \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = c_{11}\cdot c_{22} - c_{12}\cdot c_{21} \neq 0,$$

where

$$Q_1 w = \int_0^{+\infty} \nu(t) \int_0^t w(s)\,ds\,dt,$$

and

$$Q_2 w = \sum_{j=1}^{m} \beta_j \int_0^{\eta_j} \int_t^{+\infty} w(s)\,ds\,dt.$$

It is obvious that $\ker M = \{u \in \operatorname{dom} M : u = a + bt : a, b \in \mathbb{R}, t \in [0,+\infty)\}$ and $\operatorname{Im} M = \{w : w \in Z, Q_1 w = Q_2 w = 0\}$.

Clearly, $\ker M = 2$ is linearly homeomorphic to $\mathbb{R}^2$ and $\operatorname{Im} M \subset Z$ is closed, hence, the operator $M : \operatorname{dom} M \subset U \to Z$ is quasi-linear.

We next define the projector $P : U \to U_1$ as

$$Pu(t) = u(0) + u'(0)t, \quad u \in U, \tag{2.1}$$

and the operators $\Delta_1, \Delta_2 : Z \to Z_1$ as

$$\Delta_1 w = \frac{1}{C}(\delta_{11} Q_1 w + \delta_{12} Q_2 w)e^{-t},$$

and

$$\Delta_2 w = \frac{1}{C}(\delta_{21} Q_1 w + \delta_{22} Q_2 w)e^{-t},$$

where δ_{ij} is the co-factor of c_{ij}, $i,j = 1,2$. Then the operator $Q : Z \to Z_1$ will be defined as

$$Qw = (\Delta_1 w) + (\Delta_2 w) \cdot t \tag{2.2}$$

where Z_1 is the complement space of $\operatorname{Im} M$ in Z. Then the operator $Q : Z \to Z_1$ can easily be shown to be a semi-projector.

Let the operator $R : U \times [0,1] \to U_2$ be defined by

$$R(u,\lambda)(t) = \int_0^t \varphi_q\left(\varphi_p\big(u'(0)\big) - \int_0^\tau \lambda\big(g\big(s,u(s),u'(s)\big) - QNu(s)\big)\,ds\right)d\tau - u'(0)t,$$

where U_2 is the complement space of $\ker M$ in U.

Lemma 2.2 *If g is a $L^1[0,+\infty)$-Carathéodory function, then $R : U \times [0,1] \to U_2$ is M-compact.*

Proof Let the set $\Omega \subset U$ be nonempty, open and bounded, then, for $u \in \overline{\Omega}$, there exists a constant $k > 0$ such that $\|u\| < k$. Since g is an $L^1[0,+\infty)$-Carathéodory function, there exists $\psi_k \in L^1[0,+\infty)$ such that, for a.e. $t \in [0,+\infty)$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}\|N_\lambda u\|_{L^1} + \|QN_\lambda u\|_{L^1} &= \int_0^{+\infty} \big|N_\lambda u(v)\big|\,dv + \int_0^{+\infty} \big|QN_\lambda u(v)\big|\,dv \\ &\le \|\psi_k\|_{L^1} + \|QNu\|_{L^1}.\end{aligned}$$

Now for any $u \in \overline{\Omega}$, $\lambda \in [0,1]$, we have

$$\begin{aligned}\big\|R(u,\lambda)\big\|_\infty &= \sup_{t\in[0,+\infty)} e^{-t}\big|R(u,\lambda)(t)\big| \le \frac{1}{e}\varphi_q\big(\varphi_p(k) + \|Nu_\lambda\|_{L^1} + \|QN_\lambda u\|_{L^1}\big) + k \\ &\le \varphi_q\big(\varphi_p(k) + \|\psi_k\|_{L^1} + \|QNu\|_{L^1}\big) + k < +\infty\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\big\|R'(u,\lambda)\big\|_\infty &= \sup_{t\in[0,+\infty)} e^{-t}\big|R'(u,\lambda)(t)\big| \\ &\le \varphi_q\big(\varphi_p(k) + \|\psi_k\|_{L^1} + \|QNu\|_{L^1}\big) + k < +\infty.\end{aligned} \tag{2.4}$$

Therefore it follows from (2.3) and (2.4) that $R(u,\lambda)\overline{\Omega}$ is uniformly bounded.

Next we show that $R(u,\lambda)\overline{\Omega}$ is equicontinuous in a compact set. Let $u \in \overline{\Omega}$, $\lambda \in [0,1]$. For any $T \in [0,+\infty)$, with $t_1, t_2 \in [0,T]$ where $t_1 < t_2$, we have

$$\begin{aligned}&\big|e^{t_2}R(u,\lambda)(t_2) - e^{t_1}R(u,\lambda)(t_1)\big| \\ &\quad= \bigg|e^{t_2}\int_0^{t_2} \varphi_q\left(\varphi_p\big(u'(0)\big) - \int_0^\tau \lambda\big(g\big(s,u(s),u'(s)\big) - QNu(s)\big)\,ds\right)d\tau - u'(0)t_2e^{-t_2} \\ &\qquad - e^{-t_1}\int_0^{-t_1} \varphi_q\left(\varphi_p\big(u'(0)\big) - \int_0^\tau \lambda\big(g\big(s,u(s),u'(s)\big) - QNu(s)\big)\,ds\right)d\tau + u'(0)t_1e^{t_1}\bigg| \\ &\quad\le \big|e^{t_2} - e^{-t_1}\big| \int_0^{t_1} \varphi_q\left(\varphi_p\big(\big|u'(0)\big|\big) + \int_0^\tau \lambda\big|g\big(s,u(s),u'(s)\big) - QNu(s)\big|\,ds\right)d\tau\end{aligned}$$

$$
\begin{aligned}
&+ e^{-t_2}\int_{t_1}^{t_2} \varphi_q\left(\varphi_p\left(\left|u'(0)\right|\right) + \int_0^{\tau} \lambda\left|g\left(s,u(s),u'(s)\right) - QNu(s)\right| ds\right) d\tau \\
&+ \left|t_1 e^{-t_1} - t_2 e^{-t_2}\right|\left|u'(0)\right| \\
&\le \left(e^{t_2} - e^{-t_1}\right)\varphi_q\left(\varphi_p(k) + \|\psi_k\|_{L^1} + \|QNu\|_{L^1}\right)t_1 \\
&+ e^{-t_2}\varphi_q\left(\varphi_p(k) + \|\psi_k\|_{L^1} + \|QNu\|_{L^1}\right)(t_2 - t_1) + \left|t_1 e^{-t_1} - t_2 e^{-t_2}\right| r \\
&\to 0, \quad \text{as } t_1 \to t_2,
\end{aligned} \tag{2.5}
$$

and

$$
\begin{aligned}
&\left|e^{-t_2}R'(u,\lambda)(t_2) - e^{-t_1}R'(u,\lambda)(t_1)\right| \\
&= \left|e^{t_2}\varphi_q\left(\varphi_p\left(u'(0)\right) - \int_0^{t_2} \lambda\left(g\left(s,u(s),u'(s)\right) - QNu(s)\right) ds\right) - u'(0)e^{-t_2}\right. \\
&\quad \left. - e^{-t_1}\varphi_q\left(\varphi_p\left(u'(0)\right) - \int_0^{t_1} \lambda\left(g\left(s,u(s),u'(s)\right) - QNu(s)\right) ds\right) + u'(0)e^{-t_1}\right| \\
&\le \left(e^{t_2} - e^{-t_1}\right)\varphi_q\left(\varphi_p(k) + \|\psi_k\|_{L^1} + \|QNu\|_{L^1}\right) + \left(e^{-t_1} - e^{-t_2}\right)k \\
&\to 0, \quad \text{as } t_1 \to t_2.
\end{aligned} \tag{2.6}
$$

Thus, (2.5) and (2.6) show that $R(u,\lambda)\overline{\Omega}$ is equicontinuous on $[0,T]$.

We will now prove that $R(u,\lambda)\overline{\Omega}$ is equiconvergent at ∞. Since $\lim_{t\to+\infty} e^{-t} = 0$,

$$
\lim_{t\to+\infty} e^{-t}R(u,\lambda)(t) = \lim_{t\to+\infty} e^{-t}R'(u,\lambda)(t) = 0.
$$

Hence,

$$
\begin{aligned}
&\left|e^{-t}R(u,\lambda)(t) - \lim_{t\to+\infty} e^{-t}R(u,\lambda)(t)\right| \\
&= \left|e^{-t}\int_0^{t} \varphi_q\left(\varphi_p\left(u'(0)\right) - \int_0^{\tau} \lambda\left(g\left(s,u(s),u'(s)\right) - QNu(s)\right) ds\right) d\tau - te^{-t}u'(0) - 0\right| \\
&\le te^{-t}\varphi_q\left(\varphi_p(k) + \|\psi_k\|_{L^1} + \|QNu\|_{L^1}\right) + kte^{-t} \\
&\to 0, \quad \text{uniformly as } t \to +\infty,
\end{aligned} \tag{2.7}
$$

and

$$
\begin{aligned}
&\left|e^{-t}R'(u,\lambda)(t) - \lim_{t\to+\infty} e^{-t}R'(u,\lambda)(t)\right| \\
&= \left|e^{-t}\varphi_q\left(\varphi_p\left(u'(0)\right) - \int_0^{t} \lambda\left(g\left(s,u(s),u'(s)\right) - QNu(s)\right) ds\right) - e^{-t}u'(0) - 0\right| \\
&\le e^{-t}\varphi_q\left(\varphi_p(k) + \|\psi_k\|_{L^1} + \|QNu\|_{L^1}\right) + ke^{-t} \\
&\to 0, \quad \text{uniformly as } t \to +\infty.
\end{aligned} \tag{2.8}
$$

Therefore $R(u,\lambda)\overline{\Omega}$ is equiconvergent at $+\infty$. It then follows from Definition 2.4 that $R(u,\lambda)$ is compact. □

Lemma 2.3 *The operator N_λ is M-compact.*

Proof Since Q is a semi-projector, $Q(I-Q)N_\lambda(\overline{\Omega}) = 0$. Hence, $(I-Q)N_\lambda(\overline{\Omega}) \subset \ker Q = \operatorname{Im} M$. Conversely, let $w \in \operatorname{Im} M$, then $w = w - Qw = (I-Q)w \in (I-Q)Z$. Hence, condition (i) of definition (2.5) is satisfied. It can easily be shown that condition (ii) of Definition 2.5 holds.

Let $u \in \Sigma_\lambda = \{u \in \overline{\Omega} : Mu = N_\lambda u\}$, then $N_\lambda u \in \operatorname{Im} M$. Hence, $QN_\lambda u = 0$ and $R(u,0)(t) = 0$. From $(\varphi_p(u'(t)))' + g(t,u(t),u'(t)) = 0$, $t \in (0,+\infty)$, we have

$$\begin{aligned} R(u,\lambda)(t) &= \int_0^t \varphi_q\biggl(\varphi_p\bigl(u'(0)\bigr) - \int_0^\tau \lambda g\bigl(s,u(s),u'(s)\bigr)\,ds\biggr)\,d\tau - u'(0)t \\ &= \int_0^t \varphi_q\bigl(\varphi_p\bigl(u'(0)\bigr) + \varphi_p\bigl(u'(\tau)\bigr) - \varphi_p\bigl(u'(0)\bigr)\bigr)\,d\tau - u'(0)t \\ &= u(t) - u(0) - u'(0)t = u(t) - Pu(t) = \bigl[(I-P)u\bigr](t). \end{aligned}$$

Therefore, condition (iii) of definition (2.5) holds.

Let $u \in \overline{\Omega}$. Since $Mu = (\varphi_p(u'(t)))'$ we have

$$\begin{aligned} M\bigl[Pu + R(u,\lambda)\bigr](t) &= \bigl(\varphi_p\bigl(\bigl[Pu + R(u,\lambda)\bigr]\bigr)'(t)\bigr)' \\ &= \biggl(\varphi_p\biggl[u(0) + u'(0)t + \int_0^t \varphi_q\biggl(\varphi_p\bigl(u'(0)\bigr) - \int_0^\tau \lambda\bigl(g\bigl(s,u(s),u'(s)\bigr) \\ &\quad - QN(s)\bigr)\,ds\biggr)\,d\tau - u'(0)t\biggr]'\biggr)' \\ &= \biggl(\varphi_p\bigl(u'(0)\bigr) - \int_0^\tau \lambda\bigl(g\bigl(s,u(s),u'(s)\bigr) - QN(s)\bigr)\,ds\biggr)' = (I-Q)N_\lambda(t), \end{aligned}$$

that is, condition (iv) of definition (2.5) holds. Hence, N_λ is M-compact in $\overline{\Omega}$. □

3 Existence result

In this section, the conditions for existence of solutions for boundary value problem (1.1) will be stated and proved.

Theorem 3.1 *Assume g is a $L^{[}0,+\infty)$-Carathéodory function and the following hypotheses hold*:

(H_1) *there exist functions $x_1(t), x_2(t), x_3(t) \in L^1[0,+\infty)$ such that, for a.e. $t \in [0,+\infty)$,*

$$\bigl|g\bigl(t,u,u'\bigr)\bigr| \le e^{-t}\bigl(x_1(t)|u|^{p-1} + x_2(t)\bigl|u'\bigr|^{p-1}\bigr) + x_3(t), \tag{3.1}$$

(H_2) *for $u \in \operatorname{dom} M$ there exists a constant $A_0 > 0$, such that, if $|u(t)| > A_0$ for $t \in [0,+\infty)$ or $|u'(t)| > A_0$ for $t \in [0,+\infty]$, then either*

$$Q_1Nu(t) \neq 0 \quad \text{or} \quad Q_2Nu(t) \neq 0, \quad t \in [0,+\infty), \tag{3.2}$$

(H_3) *there exists a constant $l > 0$ such that, for $|a| > l$ or $|b| > l$ either*

$$Q_1N(a+bt) + Q_2N(a+bt) < 0, \quad t \in [0,+\infty), \tag{3.3}$$

or

$$Q_1N(a+bt)+Q_2N(a+bt)>0, \quad t\in[0,+\infty), \tag{3.4}$$

where $a,b\in\mathbb{R}$, $|a|+|b|>l$ *and* $t\in[0,+\infty)$.

Then the boundary value problem (1.1) *has at least one solution, provided*

$$2^{2q-4}\big(\|x_2\|_{L^1}+2^{q-2}\|x_1\|_{L^1}\big)<1, \quad \text{for } 1<p\le 2,$$

or

$$\varphi_q\big(\|x_1\|_{L^1}+\|x_2\|_{L^1}\big)<1, \quad \text{for } p>2.$$

The following lemmas are also needed to prove our main result.

Lemma 3.1 *The set* $\Omega_1=\{u\in\operatorname{dom}M: Mu=N_\lambda u \text{ for some } \lambda\in(0,1)\}$ *is bounded.*

Proof Let $u\in\Omega_1$ then $N_\lambda u\in\operatorname{Im}M=\ker Q$. Hence, $QN_\lambda u=0$ and $QNu=0$. It follows from H_2 that there exist $t_0,t_1\in[0,+\infty)$, such that $|u(t_0)|\le A_0$ and $|u'(t_1)|\le A_0$. From $u(t)=u(t_0)+\int_{t_0}^t u'(v)\,dv$, we have

$$|u(t)|=\left|u(t_0)-\int_{t_0}^t u'(s)\,ds\right|\le A_0+|t-t_0|\|u'\|_\infty.$$

Hence,

$$\|u\|_\infty=\sup_{t\to\infty}e^{-t}|u(t)|\le A_0+\|u'\|_\infty. \tag{3.5}$$

Also, from $Mu=N_\lambda u$, we get

$$\varphi_p\big(u'(t)\big)=-\int_{t_1}^t \lambda g\big(s,u(s),u'(s)\big)\,ds+\varphi_p\big(u(t_1)\big).$$

In view of (3.1), we have

$$\begin{aligned}\big|\big(u'(t)\big)\big| &\le \varphi_q\left(\varphi_p(A_0)+\int_0^{+\infty}\big(x_1(t)\big|\varphi_p\big(u(t)\big)\big|+x_2(t)\big|\varphi_p\big(u'\big)\big|+x_3(t)\big)\,dt\right)\\ &\le \varphi_q\big(\varphi_p(A_0)+\|x_1\|_{L^1}\varphi_p\big(\|u\|_\infty\big)+\|x_2\|_{L^1}\varphi_p\big(\|u'\|_\infty\big)+\|x_3\|_{L^1}\big)\\ &\le \varphi_q\big(\varphi_p(A_0)+\|x_1\|_{L^1}\varphi_p\big(A_0+\|u'\|_\infty\big)+\|x_2\|_{L^1}\varphi_p\big(\|u'\|_\infty\big)+\|x_3\|_{L^1}\big).\end{aligned} \tag{3.6}$$

If $1<p\le 2$, it follows from Lemma 2.1 that

$$\|u'\|_\infty\le\frac{2^{2q-4}[\varphi_q(\|x_3\|_{L^1})+A_0(1+2^{q-2}\|x_1\|_{L^1}}{1-2^{2q-4}(\|x_2\|_{L^1}+2^{q-2}\|x_1\|_{L^1})}. \tag{3.7}$$

If $p>2$ then, by Lemma 2.1, we get

$$\|u'\|_\infty\le\frac{A_0(1+\varphi_q(\|x_1\|_{L^1})+\varphi_q(\|x_3\|_{L^1})}{1-\varphi_q(\|x_1\|_{L^1}+\|x_2\|_{L^1})}. \tag{3.8}$$

Since $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\} \le A_0 + \|u'\|_\infty$, in view of (3.7) and (3.8), Ω_1 is bounded. □

Lemma 3.2 *If* $\Omega_2 = \{u \in \ker M : -\lambda u + (1-\lambda)JQNu = 0, \lambda \in [0,1]\}$, $J : \operatorname{Im} Q \to \ker M$ *is a homomorphism, then* Ω_2 *is bounded.*

Proof For $a, b \in R$, let $J : \operatorname{Im} Q \to \ker M$ be defined by

$$J(a+bt) = \frac{1}{C}\left[\delta_{11}|a| + \delta_{12}|b| + \left(\delta_{21}|a| + \delta_{22}|b|\right)t\right]e^{-t}. \tag{3.9}$$

If (3.3) holds, for any $u(t) = a + bt \in \Omega_3$, from $-\lambda u + (1-\lambda)JQNu = 0$, we obtain

$$\begin{cases} \delta_{11}(-\lambda|a| + (1-\lambda)Q_1N(a+bt)) + \delta_{12}(-\lambda|b| + (1-\lambda)Q_2N(a+bt)) = 0, \\ \delta_{21}(-\lambda|a| + (1-\lambda)Q_1N(a+bt)) + \delta_{22}(-\lambda|b| + (1-\lambda)Q_2N(a+bt)) = 0. \end{cases}$$

Since $C \neq 0$,

$$\begin{aligned} \lambda|a| &= (1-\lambda)Q_1N(a+bt), \\ \lambda|b| &= (1-\lambda)Q_2N(a+bt). \end{aligned} \tag{3.10}$$

From (3.10), when $\lambda = 1$, $a = b = 0$. When $\lambda = 0$,

$$Q_1N(a+bt) + Q_2N(a+bt) = 0,$$

which contradicts (3.3) and (3.4), hence from (H_3), $|a| \le l$ and $|b| \le l$. For $\lambda \in (0,1)$, in view of (3.3) and (3.10), we have

$$0 \le \lambda\left(|a| + |b|\right) = (1-\lambda)\left[Q_1N(a+bt) + Q_2N(a+bt)\right] < 0,$$

which contradicts $\lambda(|a|+|b|) \ge 0$. Hence, (H_3), $|a| \le l$ and $|b| \le l$, thus $\|u\| \le 2l$. Therefore Ω_2 is bounded. □

Proof of Theorem 3.1 Since M is quasi-linear, condition (A_1) of Theorem 2.1 holds, Lemma 2.2 proved (A_2), while Lemma 3.1 shows that (A_3) holds.

Let $\Omega \supset \Omega_1 \cup \Omega_2$ be a nonempty, open and bounded set, $u \in \operatorname{dom} M \cap \partial\Omega$, $H(u,\lambda) = -\lambda u + (1-\lambda)JQNu$, and J be as defined in Lemma 3.2 then $H(u,\lambda) \neq 0$. Therefore by the homotopy property of the Brouwer degree

$$\begin{aligned} \deg\{JQN|_{\overline{\Omega}\cap\ker M}, \Omega \cap \ker M, 0\} &= \deg\left\{H(\cdot,0), \Omega \cap \ker M, 0\right\} \\ &= \deg\left\{H(\cdot,1), \Omega \cap \ker M, 0\right\} \\ &= \deg\{-I, \Omega \cap \ker M, 0\} \neq 0. \end{aligned}$$

Hence, condition (A_4) of Theorem 2.1 also holds. □

Since all the conditions of Theorem 2.1 are satisfied, the abstract equation $Mu = Nu$ has at least one solution in $\overline{\Omega} \cap \operatorname{dom} M$. Hence, (1.1) has at least one solution.

4 Example

Consider the following boundary value problem:

$$\begin{cases} (\varphi_4(u'(t)))' + e^{-t-2}\sin t\cdot u^3 + e^{-t-3}\cos t\cdot u'^3 + \frac{1}{6}e^{-6t} = 0, \quad t\in(0,+\infty), \\ \varphi_4(u'(0)) = \int_0^{+\infty} 2e^{-2t}\varphi_4(u'(t))\,dt, \qquad \varphi_4(u'(+\infty)) = 9\int_0^{1/9}\varphi_4(u'(t))\,dt. \end{cases} \tag{4.1}$$

Here $v(t) = 2e^{-2t}$, $p = 4$, $q = \frac{4}{3}$, $\beta_1 = 9$, $\eta_1 = \frac{1}{9}$, $x_1 = e^{-t-2}\sin t$ and $x_2 = e^{-t-3}\cos t$. Therefore, $\sum_{j=1}^{1}\beta_j\eta_j = 1$, $\int_0^{+\infty} v(t)\,dt = 1$, $C \neq 0$ and $\varphi_q(\|x_1\|_{L^1} + \|x_2\|_{L^2}) < 1$. It can easily be seen that conditions (H_1)–(H_3) hold. Hence, (4.1) has at least one solution.

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Authors' contributions
OF conceived the idea. SA supervised the work. All authors discussed and contributed to the final manuscript.

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