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The initial-boundary value problem for a class of third order pseudoparabolic equations

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Abstract

In this paper, a priori estimate for a linear third pseudoparabolic operator with bound is established, and applying the above result, the existence and uniqueness theorem of solutions for a class of nonlinear pseudoparabolic equations is obtained with the help of the homeomorphism method and the initial value method. Furthermore, an existence and uniqueness theorem of the semilinear equation is obtained as a corollary.

Keywords: Third order pseudoparabolic equations; Homeomorphism method; Initial-boundary value problem; Existence and uniqueness theorem

1 Introduction

Consider the initial-boundary value problem

$$(a_{i,j}(u + u_t)_{x_j})_{x_i} + b_i u_{x_i} - u_t + F(x, t, u, u_{x_i}) = f(x, t, u), \quad (x, t) \in D, \quad (1)$$

$$u|_{t=0} = 0, \quad x \in \Omega, \quad (2)$$

$$u|_{\partial\Omega} = 0, \quad (x, t) \in \partial\Omega \times [0, T], \quad (3)$$

where Ω is a connected bounded subset of n -dimensional space, the boundary of Ω is piecewise smooth and has nonnegative mean curvature everywhere, $D = \Omega \times [0, T]$ belongs to the Hilbert space $W_2^{2,1}(D)$, and $a_{i,j}$, b_i are bounded measurable functions.

Using a continuous method, Sigillito [1] explored the solution for the heat equation. Elcart and Sigillito [2, 3] had derived an explicit coercivity inequality and discussed the convergence of the algorithm for a semilinear third order pseudoparabolic equation of the following type:

$$\Delta(u + u_t) - u_t = f(x, t, u), \quad (x, t) \in D,$$

since then, there have been some further studies of other forms of parabolic equations, most of the results have focused on the discussion of algorithms [4].

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In 2004, Bouziani [5, 6] had derived an explicit coercivity inequality and given a sufficient condition for the existence and uniqueness of a solution to the first order parabolic equation.

Motivated by the spirit of this work and results by Brown and Lin [7], the explicit coercivity inequalities of a linear third pseudoparabolic operator with bound are obtained in Sect. 3. By using these estimates, we shall utilize the homeomorphism method and the initial value method to give a new set of sufficient conditions for the existence and uniqueness of the third order pseudoparabolic equation in this paper, which can be found in Sect. 4.

2 Preliminaries and lemmas

In this section, we will state some lemmas which are useful to our results.

Firstly, we will give sufficient conditions for f to be a global homeomorphism of D onto Y .

Definition 2.1 ([8]) Let X, Y be Banach spaces, $D \subseteq X$ be open and connected, the continuous mapping $f : D \subset X \rightarrow Y$ satisfies condition (C) if and only if for any continuous function $r : [0, a) \rightarrow D \subseteq X$ such that

$$f(r(t)) = q(t), \quad t \in [0, a),$$

where $q : [0, 1] \rightarrow Y$ is any line in Y , there is a sequence $\{t_n\}$ such that $t_n \rightarrow a, n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} r(t_n)$$

exists and is in D .

In the following for convenience, with no loss of generality, for the function q one may assume that $q(t) = (1 - t)f(x_0) + ty, t \in [0, 1]$, for arbitrary $x_0 \in D$ and $y \in Y$.

Theorem 2.1 (Plascock [8]) Let $f : D \subset X \rightarrow Y$ be a local homeomorphism.

Then f is a global homeomorphism of D onto Y if and only if f satisfies condition (C).

Secondly, the comparison theorem plays an important role to prove the sufficient condition for the existence of a unique solution of the problem (1).

Let E be an open (t, x) -set in R^2 and $g \in C[E, R]$. Consider the scalar differential equation with an initial condition

$$\begin{cases} u' = g(t, u), \\ u(t_0) = u_0. \end{cases} \tag{4}$$

Definition 2.2 ([9]) Let $y(t)$ be a solution of the scalar differential equation (4) on $[t_0, t_0 + a)$, then $y(t)$ is said to be a maximal solution of (4) if, for every solution $u(t)$ of (4) existing on $[t_0, t_0 + a)$, we have the inequality

$$u(t) \leq y(t), \quad t \in [t_0, t_0 + a)$$

holds.

Theorem 2.2 (Plascock [9]) *Let $g \in C[R_0, R]$, where R_0 is the rectangle $t_0 \leq t \leq t_0 + a$, $|u - u_0| \leq b$, and $|g(t, u)| \leq M$ on R_0 . Then there exist a maximal solution and a minimal solution of (4) on $[t_0, t_0 + a]$, where $\alpha = \min(a, b \setminus 2(M + b))$.*

Theorem 2.3 (Comparison theorem [9]) *In the setting of the above, suppose that $[t_0, t_0 + b)$ is the largest interval in which the maximal solution $y(t)$ of (4) exists. Let*

$$m \in C[[t_0, t_0 + b), R], \quad (t, m(t)) \in E \quad \text{for } t \in [t_0, t_0 + b), m(t_0) \leq u_0,$$

and for a fixed Dini derivative

$$Dm(t) \leq g(t, m(t)), \quad t \in [t_0, t_0 + b) \setminus T,$$

then

$$m(t) \leq y(t), \quad t \in [t_0, t_0 + b),$$

where T denotes an almost countable subset of $t \in [t_0, t_0 + b)$.

3 The coercivity inequality

Let $W_0(D)$ denote the Hilbert space with the norm

$$(\|u\|_{2,1})^2 = \int_D (u^2 + |\nabla u|^2 + |D^2 u|^2 + u_t^2) \, dx \, dt,$$

here $|D^2 u|^2$ represents the sum of the squares of all the second derivatives with respect to space variables. In this section we derive a coercivity inequality,

$$\| \|u\| \leq \text{const} \|Lu\|,$$

for the pseudoparabolic operator defined by

$$Lu = (a_{ij}(u + u_t)_{x_i})_{x_j} + b_i u_{x_i} - au - u_t. \tag{5}$$

The norm $\| \cdot \|$ on W_0 is defined by

$$\left[(\|u\|_{2,1})^2 + \int_0^T \|u_t\|_2^2 \, dt \right]^{\frac{1}{2}},$$

where $\| \cdot \|_2$ is the norm on $W_2^2(\Omega)$, $\| \cdot \|$ is the norm on $L_2(D)$, $a : W_0(D) \rightarrow L_2(D)$ is continuous and a bounded function on t, x_1, \dots, x_n, u .

We assume that a_{ij} is a symmetric matrix of measurable functions satisfying the inequality

$$\tau^2 |\xi| \leq a_{i,j} \xi_i \xi_j$$

for some positive constant τ , all n -dimensional vectors ξ and all x in D . We also assume that the functions a_{ij} are sufficiently regular to ensure the validity of the identity

$$\int_{\Omega} u(a_{ij}u_{x_j})_{x_i} dx = - \int_{\Omega} a_{ij}u_{x_j}u_{x_i} dx$$

for u in $W_2^2(\Omega)$. From (5), we have

$$\begin{aligned} -uLu &= -u(a_{i,j}(u + u_t)_{x_i})_{x_j} - b_i u_{x_i} u + au^2 + u_t u \\ &= (a_{i,j})_{x_j} (u + u_t)_{x_i} u + a_{i,j} (u + u_t)_{x_i} (u + u_t)_{x_j} - b_i u_{x_i} u + au^2 + u_t u. \end{aligned}$$

Using the inequality

$$2bc \leq \varepsilon b^2 + \varepsilon^{-1} c^2, \tag{6}$$

and the inequality

$$\tau^2 |\xi|^2 \leq a_{i,j} \xi_i \xi_j$$

for $\varepsilon > 0, \alpha > 0$, the inequality

$$\begin{aligned} \int_D (Lu)^2 dx dt &\geq 2\varepsilon\tau^2(1 - \alpha) \int_D |\nabla u|^2 dx dt du \\ &\quad + \left(a_0 - \frac{\varepsilon}{2} - \frac{S^2}{4\tau^2\alpha} \right) 2\varepsilon \int_D u^2 dx dt \end{aligned} \tag{7}$$

is valid for u in W_0 .

The next two lemmas are obtained from (5) by evident choices of ε and α . In order to facilitate statements to be made below, we define $S = \sup |b_i - (a_{i,j})_{x_j}|, a_0 = \inf_D a(x, t)$.

Lemma 3.1 *The inequality*

$$\int_D (Lu)^2 dx dt \geq \left(a_0 - \frac{S^2}{4\tau^2} \right)^2 \int_D u^2 dx dt \tag{8}$$

is valid for u in W_0 .

Lemma 3.2 *The inequality*

$$\int_D (Lu)^2 dx dt \geq (2\tau\sqrt{a_0(1-S)})^2 \int_D |\nabla u|^2 dx dt \tag{9}$$

is valid for u in W_0 .

We define

$$Pu = a_{i,j}u_{x_j}u_{x_i}, \quad Qu = (a_{i,j}u_{x_i})_{x_j}, \quad a_1 = \sup |b_i|, \quad \bar{M} = \max_i \sum_j a_{i,j}^2,$$

and from [2] we have the inequality

$$\frac{\tau^2}{2} \int_D \sum |D^2 u|^2 \, dx \, dt \leq \int_D P^2 u \, dx \, dt + \frac{n^4 B^2}{2\tau^2} \int_D |\nabla u|^2 \, dx \, dt.$$

From $Pu = Qu - (a_{ij})_{x_j} u_{x_i}$, $u \in W_0$, we have

$$\begin{aligned} \int_D P^2 u \, dx \, dt &\leq 2 \int_D Q^2 u \, dx \, dt + 2 \int_D ((a_{ij})_{x_j} u_{x_i})^2 \, dx \, dt \\ &\leq 2 \int_D Q^2 u \, dx \, dt + 2\bar{M} \int_D |\nabla u|^2 \, dx \, dt, \end{aligned}$$

so

$$\begin{aligned} \frac{\tau^2}{2} \int_D |D^2 u|^2 \, dx \, dt &\leq 2 \int_D Q^2 u \, dx \, dt + \left(\frac{n^4 B^2}{2\tau^2} + 2\bar{M} \right) \int_D |\nabla u|^2 \, dx \, dt \\ &\leq 2 \int_D Q^2(u + u_t) \, dx \, dt \\ &\quad + \left(\frac{n^4 B^2}{2\tau^2} + 2\bar{M} \right) \int_D |\nabla u|^2 \, dx \, dt. \end{aligned}$$

Remark 3.1 Results analogous in the present situation are in Lemma 3.3.

Lemma 3.3 *The inequality*

$$\begin{aligned} \frac{\tau^2}{2} \int_D |D^2 u|^2 \, dx \, dt &\leq 2 \int_D Q^2(u + u_t) \, dx \, dt \\ &\quad + \left(\frac{n^4 B^2}{2\tau^2} + 2\bar{M} \right) \int_D |\nabla u|^2 \, dx \, dt \end{aligned} \tag{10}$$

is valid for u in W_0 .

From $Qu = Pu + (a_{ij})_{x_j} u_{x_i}$, we also have

$$\begin{aligned} \int_D Q^2(u + u_t) \, dx \, dt &\leq \int_D (Lu)^2 \, dx \, dt + \left(\frac{1}{4} + \frac{S}{2} + a_1 \right) \int_D u^2 \, dx \, dt \\ &\quad + \frac{S}{2} \int_D |\nabla u|^2 \, dx \, dt. \end{aligned} \tag{11}$$

The inequalities (10) and (11) imply that

$$\begin{aligned} \frac{\tau^2}{2} \int_D |D^2 u|^2 \, dx \, dt &\leq \int_D (Lu)^2 \, dx \, dt + \left(\frac{1}{4} + \frac{S}{2} + a_1 \right) \\ &\quad \int_D u^2 \, dx \, dt + \left(\frac{S}{2} + \frac{n^4 B^2}{2\tau^2} + 2\bar{M} \right) \int_D |\nabla u|^2 \, dx \, dt. \end{aligned} \tag{12}$$

Denote

$$L_0 u = (a_{ij}(u + u_t)_{x_i})_{x_j} + b_i u_{x_i} - au,$$

and by further application of the arithmetic–geometric mean inequality to

$$-\int_D uu_t \, dx \, dt = -\int_D uLu \, dx \, dt + \int_D uL_0u \, dx \, dt,$$

we obtain

$$\int_D u_t^2 \, dx \, dt \leq \int_D (Lu)^2 \, dx \, dt - 4\left(a_0 + \frac{1}{2} + \frac{S}{4\tau^2}\right) \int_D u^2 \, dx \, dt. \tag{13}$$

Combining (8), (9), (12) and (13), we have Lemma 3.4.

Lemma 3.4 *If $a_0 - \frac{S^2}{4\tau^2} > 0$, the inequality*

$$\|u\| \leq C \|Lu\|$$

is valid for u in W_0 , where

$$C^2 = \frac{2 + \tau^2}{\tau^2} + \frac{\tau^2 + S + 4\bar{M}}{4\tau^4 a_0(1 - S)} + \frac{B^2 n^4}{4\tau^6 a_0(1 - S)} + \frac{8\tau^2(1 + 4a_1) - 16\tau^4(4a_0 + 1)}{(4a_0\tau^2 - S^2)^2}.$$

4 The coercivity inequality

Denote

$$Mu = (a_{ij}(u + u_t)_{x_i})_{x_j} + b_i u_{x_i} - u_t,$$

then M is a linear operator from $W_0(D)$ to $L_2(D)$. Now let us turn our attention to the following operator equation:

$$Au = Mu - fu.$$

For all $u, \phi \in W_0(D)$, we have

$$A'\phi = M\phi - f_u(x, u(x))\phi.$$

If $\inf_{\Omega} f_u > \frac{S^2}{4\tau^2}$, then zero is not an eigenvalue of $M\phi - f_u(x, u(x))\phi$, so for every $u \in W_0(D)$, the operator $A'(u) = M - f_u I$ is invertible and A is a local homeomorphism from $W_0(D)$ onto $L_2(D)$, where I denotes the identical operator. Furthermore, an upper bound for $\| [A'(x)]^{-1} \|$ is provided by Lemma 3.4 if the coefficient $a(x)$ is identified with $f_u(x, u)$, it implies that

$$\| [A'(x)]^{-1} h \| \leq \left(\alpha \sup_{\|x\| \leq s} |f_u(x, u(x))| + \beta \right) \|h\|$$

for positive constant α, β .

Denote

$$\lambda(s) = \alpha \sup_{\|x\| \leq s} |f_u(x, u(x))| + \beta,$$

then

$$\| [A'(x)]^{-1}h \| = \| (M - f_u)^{-1}h \| \leq \lambda(\|u\|)\|h\|, \quad u \in W_0(D). \tag{14}$$

We may express the first line of (1) in the form

$$Pu = Mu - fu + Fu = 0$$

for $u, \phi \in W_0(D)$, we have

$$P'(u)(\phi) = M\phi - f_u\phi + F_u\phi.$$

We can state and prove our main theorem.

Theorem 4.1 *In the setting above, for Eqs. (1), (2) and (3) there exists a unique solution if the following conditions hold:*

- (1) $\inf_{\Omega} f_u > \frac{S^2}{4r^2}$;
- (2) for each $\mu \in R$, the maximum solution y of the initial value problem

$$\begin{cases} y'(t) = \mu\delta(y(t)), & t \in [0, a), \\ y(0) = 0, \end{cases} \tag{15}$$

is defined on $[0, a]$ and there exists a sequence $t_n \rightarrow a$ as $n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} y(t_n) = y^*$ is finite;

- (3) F is continuously differentiable and

$$\left| \frac{\partial F_i}{\partial x_j}(w) \right|, \left| \frac{\partial F_i}{\partial y_j}(w) \right| \leq \frac{c}{n\delta(\|w\|)}, \quad c < 1, w \in R^n. \tag{16}$$

Proof Firstly, we prove $[M - f_u(u) + F_u(u)]^{-1} \leq \delta(\|u\|)$. For $u, v \in D$, it is obvious that F is continuously Frechet differentiable with

$$[F'(u)v](t) = - \left(\frac{\partial F_i}{\partial x_j} u(t) \right) v(t) - \left(\frac{\partial F_i}{\partial y_j} u(t) \right) v'(t).$$

It follows from the above assumption and (16) that

$$\|F'(u)v\| \leq n \frac{c}{n\lambda(\|u\|)} (\|v\| + \|v'\|) \leq \frac{c}{\lambda(\|u\|)} (\|v\|). \tag{17}$$

Now

$$Mu - f_u + F_u = (I + F_u[M - f_u]^{-1})(M - f_u).$$

Let $Q : W_0(D) \rightarrow W_0(D)$ be defined by

$$Q = F'(u)[M - f_u]^{-1},$$

by (14) and (16), we have

$$\|Qh\| \leq \frac{c}{\lambda(\|w\|)} \lambda(\|w\|) \|h\| = c\|h\|, \quad h \in W_0(D).$$

So $I + Q$ is invertible with

$$\|(I + Q)^{-1}\| \leq \frac{1}{1 - c}.$$

Hence, $M - f_u(u) + F_u(u) : D \rightarrow W_0(D)$ is invertible with

$$[M - f_u(u) + F_u(u)]^{-1} = [M - f_u(u)]^{-1}(I + Q)^{-1},$$

and so

$$[M - f_u(u) + F_u(u)]^{-1} \leq \frac{\lambda(\|u\|)}{1 - c}.$$

Denote $\delta(\|u\|) = \frac{\lambda(\|u\|)}{1 - c}$, then

$$[M - f_u(u) + F_u(u)]^{-1} \leq \delta(\|u\|).$$

It implies that P is invertible at every $u \in W_0(D)$, hence, P is a local homeomorphism of $W_0(D)$.

Secondly, in view of Theorem 2.1, we need only show that P has the property (C) for any continuous function $q : [0, 1] \rightarrow L_2(D)$. For a given $y \in L_2(D)$ and an arbitrary $x_0 \in W_0(D)$, let

$$q(t) = (1 - t)y + tP(x_0),$$

suppose that there exists a continuous function $r : [0, a) \rightarrow D \subseteq W_0(D)$ such that

$$P(r(t)) = q(t), \quad t \in [0, a), \text{ for } 0 < a \leq 1. \tag{18}$$

Now we need to prove that there exists a real sequence $\{t_n\}$ such that $t_n \rightarrow a, n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} r(t_n) = r^*$$

exists and is in $W_0(D)$.

It is clear that r is differentiable in this case. We have from (18)

$$\begin{cases} r'(t) = -P'(r(t))^{-1}(P(x_0) - P(r(t))), & t \in [0, a), \\ r(0) = x_0. \end{cases} \tag{19}$$

Denote by $D\|r(t)\|$ the Dini derivative of $\|r(t)\|$ and set $\mu = P(r(t)) - P(x_0)$, and we have

$$D\|r(t)\| \leq \|r'(t)\| = \|P'(r(t))^{-1}\| \|P(x) - P(x_0)\| \leq \mu \delta(\|r(t)\|).$$

By the assumption (2), we know the maximum solution $y(t)$ of (15) is defined on $[0, a]$ and there exists a sequence $t_n \rightarrow a$ as $n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} y(t_n) = y^*$$

is finite. It follows that $y(t)$ is continuous on $[0, a]$ and there is a constant K such that $|y(t)| \leq K, t \in [0, a]$. By the comparison theorem, Theorem 2.3, we have

$$\|r(t)\| \leq |y(t)| \leq K, \quad t \in [0, a].$$

For $t_1, t_2 \in [0, a]$, we have

$$\begin{aligned} \|r(t_1) - r(t_2)\| &\leq \int_{t_1}^{t_2} \|r'(s)\| ds \leq \mu \int_{t_1}^{t_2} \delta(\|r(s)\|) ds \\ &\leq \mu \int_{t_1}^{t_2} \delta(\|y(s)\|) ds \leq \mu \delta(K) |t_1 - t_2|. \end{aligned}$$

So $\{r(t_n)\}$ is a Cauchy sequence and consequently for the real sequence $t_n \rightarrow a$ as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} r(t_n) = r^*$$

exists. This proves that $r^* \in W_0(D)$ and P satisfies the condition (C). The theorem is proved. □

5 Related results

Using a similar technique to the one of the theorem, we can prove the following conclusion as regards the initial-boundary problem for pseudoparabolic equations.

Corollary 5.1 *Let the condition (1) and the condition (2) hold and (3) F be continuously differentiable and*

$$\left| \frac{\partial F_i}{\partial x_j}(w) \right|, \left| \frac{\partial F_i}{\partial y_j}(w) \right| \leq \frac{c}{nW}, \quad c < 1, w \in R^n,$$

where W is constant. Then Eq. (1) has a unique solution.

Corollary 5.2 *Let the condition (1) and the condition (2) hold. Assume that f is continuous and has continuous partial derivatives with respect to u through the third order. Then the semilinear equation*

$$\begin{aligned} (a_{i,j}(u + u_t)_{x_i})_{x_j} + b_i u_{x_i} - u_t &= f(x, t, u), \quad (x, t) \in D, \\ u|_{t=0} &= 0, \quad x \in \Omega, \\ u|_{\partial\Omega} &= 0, \quad (x, t) \in \partial\Omega \times [0, T] \end{aligned}$$

has a unique solution.

Especially when $a_{i,j} = 1$, $b_i = 0$, we get the equation in [3]:

$$\Delta(u + u_t) - u_t = f(x, t, u), \quad (x, t) \in D,$$

and the conclusion of [3] but with a different method.

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The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscripts.

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