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Certain new dynamic nonlinear inequalities in two independent variables and applications

A.A. El-Deeb[1*] and Zareen A. Khan[2]

*Correspondence:
ahmedeldeeb@azhar.edu.eg
[1]Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo, Egypt
Full list of author information is available at the end of the article

Abstract

Several inequalities were proved in 2018 by Boudeliou, in 2015 by Abdeldain and El-Deeb and in 1998 by Pachpatte. It is our aim in this paper to generalize these inequalities to time scales. Beside that, we also apply our inequalities to discrete and continuous calculus to obtain some new inequalities as special cases. Furthermore, we study the boundedness of some problems by applying our results.

Keywords: Gronwall-type inequality; Boundedness; Time scales

1 Introduction

In 2018, Boudeliou [9] discussed the following inequalities.

Theorem 1.1 *Suppose $a \in C(\hat{\Omega}, \mathbb{R}_+)$ is nondecreasing with respect to $(\breve{x}, \breve{y}) \in \hat{\Omega} = I_1 \times I_2$, let $\hat{\alpha}(\breve{x}) \in C^1(I_1, I_2)$ and $\hat{\beta}(\breve{y}) \in C^1(I_2, I_2)$ be nondecreasing functions with $\hat{\alpha}(\breve{x}) \le \breve{x}$ on I_1, $\hat{\beta}(\breve{y}) \le \breve{y}$, and g, u, p, $f \in C(\hat{\Omega}, \mathbb{R}_+)$. Furthermore, suppose $\bar{\psi}$, $\bar{\varphi} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $\{\bar{\psi}, \bar{\varphi}\}(u) > 0$ for $u > 0$, and $\lim_{u \to +\infty} \bar{\psi}(u) = +\infty$. If $u(\breve{x}, \breve{y})$ satisfies*

$$\begin{aligned} \bar{\psi}\big(u(\breve{x}, \breve{y})\big) \le a(\breve{x}, \breve{y}) &+ \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} \big[f(\breve{s}, \breve{t})\bar{\varphi}\big(u(\breve{s}, \breve{t})\big) + p(\breve{s}, \breve{t})\big]\, d\breve{t}\, d\breve{s} \\ &+ \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s}, \breve{t}) \left(\int_0^{\breve{s}} g(\breve{\tau}, \breve{t}) \bar{\varphi}\big(u(\breve{\tau}, \breve{t})\big)\, d\breve{\tau} \right) d\breve{t}\, d\breve{s}, \end{aligned}$$

for $(\breve{x}, \breve{y}) \in \hat{\Omega}$, then

$$u(\breve{x}, \breve{y}) \le \bar{\psi}^{-1} \left\{ \breve{G}^{-1} \left[\breve{G}\big(q(\breve{x}, \breve{y})\big) + \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s}, \breve{t}) \left(1 + \int_0^{\breve{s}} g(\breve{\tau}, \breve{t})\, d\breve{\tau} \right) d\breve{t}\, d\breve{s} \right] \right\},$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, where

$$q(\breve{x}, \breve{y}) = a(\breve{x}, \breve{y}) + \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} p(\breve{s}, \breve{t})\, d\breve{t}\, d\breve{s},$$

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$$\breve{G}(r) = \int_{r_0}^{r} \frac{d\breve{s}}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})}, \quad r \geq r_0 > 0, \qquad \breve{G}(+\infty) = \int_{r_0}^{+\infty} \frac{d\breve{s}}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})} = +\infty,$$

and $(\breve{x}_1, \breve{y}_1) \in \hat{\Omega}$ *is chosen so that*

$$\left(\breve{G}\big(q(\breve{x}, \breve{y})\big) + \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s}, \breve{t}) \left(1 + \int_0^{\breve{s}} g(\breve{\tau}, \breve{t})\, d\breve{\tau}\right) d\breve{t}\, d\breve{s}\right) \in \operatorname{Dom}\big(G^{-1}\big).$$

Theorem 1.2 *Assume that* $g, a, f, u, \hat{\beta}, \hat{\alpha}, , \bar{\psi}$ *and* $\bar{\varphi}$ *be as in Theorem* 1.1. *If* $u(\breve{x}, \breve{y})$ *satisfies*

$$\bar{\psi}\big(u(\breve{x}, \breve{y})\big) \leq a(\breve{x}, \breve{y}) + \left(\int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s}, \breve{t}) \bar{\varphi}\big(u(\breve{s}, \breve{t})\big)\, d\breve{t}\, d\breve{s}\right)^2 + \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s}, \breve{t}) \bar{\varphi}\big(u(\breve{s}, \breve{t})\big) \left(\int_0^{\breve{s}} g(\breve{\tau}, \breve{t}) \bar{\varphi}\big(u(\breve{\tau}, \breve{t})\big)\, d\breve{\tau}\right) d\breve{t}\, d\breve{s},$$

for $(\breve{x}, \breve{y}) \in \hat{\Omega}$, *then*

$$u(\breve{x}, \breve{y}) \leq \bar{\psi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\breve{x}, \breve{y})\big) + \breve{B}(\breve{x}, \breve{y}) + \left(\int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s}, \breve{t})\, d\breve{t}\, d\breve{s}\right)^2\right]\right\},$$

for $0 \leq \breve{x} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{y}_1$, *where*

$$\breve{B}(\breve{x}, \breve{y}) = \int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s}, \breve{t}) \left(\int_0^{\breve{s}} g(\breve{\tau}, \breve{t})\, d\breve{\tau}\right) d\breve{t}\, d\breve{s},$$

$$\breve{H}(r) = \int_{r_0}^{r} \frac{d\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})}, \quad r \geq r_0 > 0, \qquad \breve{H}(+\infty) = \int_{r_0}^{+\infty} \frac{d\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})} = +\infty,$$

and $(\breve{x}_1, \breve{y}_1) \in \hat{\Omega}$ *is chosen so that*

$$\left(\breve{H}\big(a(\breve{x}, \breve{y})\big) + B(\breve{x}, \breve{y}) + \left(\int_0^{\hat{\alpha}(\breve{x})} \int_0^{\hat{\beta}(\breve{y})} f(\breve{s}, \breve{t})\, d\breve{t}\, d\breve{s}\right)^2\right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

In 1988, Hilger [33] presented time scale theory to unify continuous and discrete analysis. For some Gronwall–Bellman-type integral, dynamic inequalities and other type inequalities on time scales, see Refs. [1–8, 13, 14, 16–32, 34–41]. For more details on time scales calculus see [15].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. We suppose throughout the article that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined for any $t \in \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined for any $t \in \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In the previous two definitions, we set $\inf \emptyset = \sup \mathbb{T}$ (i.e., if t is the maximum of $\mathbb{T}$, then $\sigma(t) = t$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., if t is the minimum of $\mathbb{T}$, then $\rho(t) = t$), where $\emptyset$ is the empty set.

A point $t \in \mathbb{T}$ with $\inf \mathbb{T} < t < \sup \mathbb{T}$ is said to be right-scattered if $\sigma(t) > t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and left-dense if $\rho(t) = t$. Points that are simultaneously right-dense and left-dense are called dense points. Points that are simultaneously right-scattered and left-scattered are called isolated points.

We define the forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ for any $t \in \mathbb{T}$ by $\mu(t) := \sigma(t) - t$.

Let $f : \mathbb{T} \to \mathbb{R}$ be a function. Then the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t))$, $\forall t \in \mathbb{T}$, that is, $f^\sigma = f \circ \sigma$. In a similar manner, the function $f^\rho : \mathbb{T} \to \mathbb{R}$ is defined by $f^\rho(t) = f(\rho(t))$, $\forall t \in \mathbb{T}$, that is, $f^\rho = f \circ \rho$.

We introduce the set $\mathbb{T}^\kappa$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

The interval $[a, b]$ in $\mathbb{T}$ is defined by

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}.$$

Open intervals and half-closed interval are defined similarly.

Suppose $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$. Then we say that $f^\Delta(t) \in \mathbb{R}$ is the delta derivative of f at t if for any $\varepsilon > 0$ there exists a neighborhood U of t such that, for all $s \in U$, we have

$$\left| \left[f\left(\sigma(t)\right) - f(s) \right] - f^\Delta(t)\left[\sigma(t) - s\right] \right| \le \varepsilon \left| \sigma(t) - s \right|.$$

Furthermore, f is said to be delta differentiable on $\mathbb{T}^\kappa$ if it is delta differentiable at each $t \in \mathbb{T}^\kappa$.

If $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable functions at $t \in \mathbb{T}^\kappa$, then

1. $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$;
2. $(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t))$;
3. $(\frac{f}{g})^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}$ provided $g(t)g(\sigma(t)) \neq 0$.

A function $g : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous (rd-continuous) if g is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is said to be a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. In this case, the definite delta integral of f is defined by

$$\int_a^b f(\eta)\Delta\eta = F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

If $g \in C_{\mathrm{rd}}(\mathbb{T})$ and $t, t_0 \in \mathbb{T}$, then the definite integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, and $G^\Delta(t) = g(t)$ holds.

Assume that $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and f, g be continuous functions on $[a, b]_{\mathbb{T}}$. Then

1. $\int_a^b [f(t) + g(t)]\Delta\eta = \int_a^b f(\eta)\Delta\eta + \int_a^b g(\eta)\Delta\eta$;
2. $\int_a^b \alpha f(\eta)\Delta\eta = \alpha \int_a^b f(\eta)\Delta\eta$;
3. $\int_a^b f(\eta)\Delta\eta = \int_a^c f(\eta)\Delta\eta + \int_c^b f(\eta)\Delta\eta$;
4. $\int_a^b f(\eta)\Delta\eta = -\int_b^a f(\eta)\Delta\eta$;
5. $\int_a^a f(\eta)\Delta\eta = 0$;
6. if $f(t) \ge g(t)$ on $[a, b]_{\mathbb{T}}$, then $\int_a^b f(\eta)\Delta\eta \ge \int_a^b g(\eta)\Delta\eta$.

We will need the following important relations between calculus on time scales $\mathbb{T}$ and either continuous calculus on $\mathbb{R}$ or discrete calculus on $\mathbb{Z}$. Note that:

1. If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = t, \qquad \mu(t) = 0, \qquad f^{\Delta}(t) = f'(t), \qquad \int_a^b f(\eta)\Delta\eta = \int_a^b f(t)\,dt.$$

2. If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = t + 1, \qquad \mu(t) = 1, \qquad f^{\Delta}(t) = f(t+1) - f(t), \qquad \int_a^b f(\eta)\Delta\eta = \sum_{t=a}^{b-1} f(t).$$

In the following, we present the basic theorems that will be needed in the proofs of our main results.

Theorem 1.3 *If $\hat{f}$ is $\hat{\Delta}$-integrable on $[a,b]$, then so is $|\hat{f}|$, and*

$$\left|\int_a^b \hat{f}(\check{t})\hat{\Delta}\check{t}\right| \le \int_a^b \left|\hat{f}(\check{t})\right|\hat{\Delta}\check{t}.$$

Theorem 1.4 (Chain rule on time scales [15]) *Assume $\hat{g} : \mathbb{R} \to \mathbb{R}$ is continuous, $\hat{g} : \breve{\mathbb{T}} \to \mathbb{R}$ is $\hat{\Delta}$-differentiable on $\mathbb{T}^{\kappa}$, and $\hat{f} : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists $c \in [\check{t}, \sigma(\check{t})]$ with*

$$(\hat{f} \circ \hat{g})^{\hat{\Delta}}(\check{t}) = \hat{f}'\big(\hat{g}(c)\big)\hat{g}^{\hat{\Delta}}(\check{t}). \tag{1.1}$$

Theorem 1.5 (Chain rule on time scales [15]) *Let $\hat{f} : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $\hat{g} : \breve{\mathbb{T}} \to \mathbb{R}$ is $\hat{\Delta}$-differentiable. Then $f \circ \hat{g} : \breve{\mathbb{T}} \to \mathbb{R}$ is $\hat{\Delta}$-differentiable and the formula*

$$(\hat{f} \circ \hat{g})^{\hat{\Delta}}(\check{t}) = \left\{\int_0^1 \left[\hat{f}'\big(h\hat{g}^{\sigma}(\check{t}) + (1-h)\hat{g}(\check{t})\big)\right] dh\right\}\hat{g}^{\hat{\hat{\Delta}}}(\check{t}) \tag{1.2}$$

holds.

This paper gives us the time scale versions of the results provided in [9]. These inequalities, proved here, extend some known results in the literature, and they are also unify the continuous and the discrete case.

2 Main results

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, +\infty)$, $\breve{\mathbb{T}}_1$, $\breve{\mathbb{T}}_2$ are two time scales and we put $\Omega = \breve{\mathbb{T}}_1 \times \breve{\mathbb{T}}_2 = \{(\check{t}, \check{s}) : \check{t} \in \breve{\mathbb{T}}_1, \check{s} \in \breve{\mathbb{T}}_2\}$ which is a complete metric space with the metric $\breve{\rho}$ defined by

$$\breve{\rho}\big((\check{t},\check{s}),(\acute{t},\acute{s})\big) = \sqrt{(\check{t}-\acute{t})^2 + (\check{s}-\acute{s})^2}, \quad \forall (\check{t},\check{s}),(\acute{t},\acute{s}) \in \breve{\mathbb{T}}_1 \times \breve{\mathbb{T}}_2.$$

$C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$ denotes the set of all right-dense continuous functions from Ω into $\mathbb{R}_+$ and $C^1_{\mathrm{rd}}(\breve{\mathbb{T}}_i, \breve{\mathbb{T}}_i)$ denotes the set of all right-dense continuously delta-differentiable functions from $\breve{\mathbb{T}}_i$ into $\breve{\mathbb{T}}_i$, $i = 1, 2$. The two-variables time scales calculus and multiple integration on time scales were introduced in [10, 11] (see also [12]).

Theorem 2.1 *Suppose that $a \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$ is nondecreasing with respect to $(\breve{x}, \breve{y}) \in \Omega$, and g, u, p, $f \in C_{\mathrm{rd}}(\Omega, \mathbb{R}_+)$. Furthermore, suppose that $\bar{\psi}$, $\bar{\varphi} \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $\{\bar{\psi}, \bar{\varphi}\}(u) > 0$ for $u > 0$, and $\lim_{u \to +\infty} \bar{\psi}(u) = +\infty$. If $u(\breve{x}, \breve{y})$ satisfies*

$$\begin{aligned}\bar{\psi}\big(u(\breve{x},\breve{y})\big) \le a(\breve{x},\breve{y}) &+ \int_0^{\breve{x}}\int_0^{\breve{y}} \big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\ &+ \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bigg(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\bigg)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},\end{aligned}$$

for $(\breve{x}, \breve{y}) \in \Omega$, then

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\bigg\{\breve{G}^{-1}\bigg[\breve{G}\big(q(\breve{x},\breve{y})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bigg(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\bigg)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\bigg]\bigg\} \tag{2.1}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, where

$$q(\breve{x},\breve{y}) = a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} p(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \tag{2.2}$$

$$\breve{G}(r) = \int_{r_0}^{r} \frac{\hat{\Delta}\breve{s}}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})}, \quad r \ge r_0 > 0, \qquad \breve{G}(+\infty) = \int_{r_0}^{+\infty} \frac{\hat{\Delta}\breve{s}}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})} = +\infty, \tag{2.3}$$

and $(\breve{x}_1, \breve{y}_1) \in \Omega$ is chosen so that

$$\bigg(\breve{G}\big(q(\breve{x},\breve{y})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bigg(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\bigg)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\bigg) \in \mathrm{Dom}\big(G^{-1}\big).$$

Proof Assume that $a(\breve{x}, \breve{y}) > 0$. Since $q \ge 0$ and it is nondecreasing, fixing an arbitrary point $(\breve{\xi}, \breve{\zeta}) \in \Omega$ and defining $z(\breve{x}, \breve{y})$ by

$$\begin{aligned}z(\breve{x},\breve{y}) = q(\breve{\xi},\breve{\zeta}) &+ \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\ &+ \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bigg(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\bigg)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},\end{aligned}$$

which is a positive and nondecreasing function for $0 \le \breve{x} \le \breve{\xi} \le \breve{x}_1$, $0 \le \breve{x} \le \breve{\zeta} \le \breve{y}_1$, we have $z(0, \breve{y}) = z(\breve{x}, 0) = q(\breve{\xi}, \breve{\zeta})$ and

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big). \tag{2.4}$$

Differentiating $z(\breve{x}, \breve{y})$, with respect to $\breve{x}$ and using (2.4), we get

$$\begin{aligned}z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y}) &= \int_0^{\breve{y}} f(\breve{x},\breve{t})\bigg[\bar{\varphi}\big(u(\breve{x},\breve{t})\big) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\bigg]\hat{\Delta}\breve{t} \\ &\le \int_0^{\breve{y}} f(\breve{x},\breve{t})\bigg[\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\bigg]\hat{\Delta}\breve{t},\end{aligned}$$

since $\bar{\varphi} \circ \bar{\psi}^{-1}$ is nondecreasing with respect to $(\breve{x}, \breve{y}) \in \mathbb{R}_+ \times \mathbb{R}_+$, we have

$$\begin{aligned} z^{\hat{\Delta}\breve{x}}(\breve{x}, \breve{y}) &\leq \int_0^{\breve{y}} f(\breve{x}, \breve{t}) \Big[\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x}, \breve{t})\big) + \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x}, \breve{t})\big) \int_0^{\breve{x}} g(\breve{\tau}, \breve{t}) \hat{\Delta}\breve{\tau} \Big] \hat{\Delta}\breve{t} \\ &\leq \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x}, \breve{y})\big) \int_0^{\breve{y}} f(\breve{x}, \breve{t}) \Big[1 + \int_0^{\breve{x}} g(\breve{\tau}, \breve{t}) \hat{\Delta}\breve{\tau} \Big] \hat{\Delta}\breve{t}, \end{aligned} \tag{2.5}$$

and from (2.5) we get

$$\frac{z^{\hat{\Delta}\breve{x}}(\breve{x}, \breve{y})}{\bar{\varphi} \circ \bar{\psi}^{-1}(z(\breve{x}, \breve{y}))} \leq \int_0^{\breve{y}} f(\breve{x}, \breve{t}) \Big(1 + \int_0^{\breve{x}} g(\breve{\tau}, \breve{t}) \hat{\Delta}\breve{\tau} \Big) \hat{\Delta}\breve{t}. \tag{2.6}$$

From (2.6) we get

$$\breve{G}\big(z(\breve{x}, \breve{y})\big) \leq \breve{G}\big(q(\breve{\xi}, \breve{\zeta})\big) + \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s}, \breve{t}) \Big(1 + \int_0^{\breve{s}} g(\breve{\tau}, \breve{t}) \hat{\Delta}\breve{\tau} \Big) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s}.$$

Since $(\breve{\xi}, \breve{\zeta}) \in \Omega$ is chosen arbitrarily,

$$z(\breve{x}, \breve{y}) \leq \breve{G}^{-1} \Big[\breve{G}\big(q(\breve{x}, \breve{y})\big) + \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s}, \breve{t}) \Big(1 + \int_0^{\breve{s}} g(\breve{\tau}, \breve{t}) \hat{\Delta}\breve{\tau} \Big) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \Big]. \tag{2.7}$$

So from (2.7) and (2.4) we get the desired inequality in (2.1). For $a(\breve{x}, \breve{y}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\breve{x}, \breve{y})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Corollary 2.2 *If we take $\breve{\mathbb{T}} = \mathbb{R}$ in Theorem 2.1, then the inequality*

$$\begin{aligned} \bar{\psi}\big(u(\breve{x}, \breve{y})\big) &\leq a(\breve{x}, \breve{y}) + \int_0^{\breve{x}} \int_0^{\breve{y}} \big[f(\breve{s}, \breve{t}) \bar{\varphi}\big(u(\breve{s}, \breve{t})\big) + p(\breve{s}, \breve{t}) \big] \, d\breve{t} \, d\breve{s} \\ &\quad + \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s}, \breve{t}) \Big(\int_0^{\breve{s}} g(\breve{\tau}, \breve{t}) \bar{\varphi}\big(u(\breve{\tau}, \breve{t})\big) \, d\breve{\tau} \Big) \, d\breve{t} \, d\breve{s}, \end{aligned}$$

for $(\breve{x}, \breve{y}) \in \Omega$, implies

$$u(\breve{x}, \breve{y}) \leq \bar{\psi}^{-1} \Big\{ \breve{G}^{-1} \Big[\breve{G}\big(q(\breve{x}, \breve{y})\big) + \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s}, \breve{t}) \Big(1 + \int_0^{\breve{s}} g(\breve{\tau}, \breve{t}) \, d\breve{\tau} \Big) \, d\breve{t} \, d\breve{s} \Big] \Big\}$$

for $0 \leq \breve{x} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{y}_1$, where

$$\begin{aligned} &q(\breve{x}, \breve{y}) = a(\breve{x}, \breve{y}) + \int_0^{\breve{x}} \int_0^{\breve{y}} p(\breve{s}, \breve{t}) \, d\breve{t} \, d\breve{s}, \\ &\breve{G}(r) = \int_{r_0}^{r} \frac{d\breve{s}}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})}, \quad r \geq r_0 > 0, \qquad \breve{G}(+\infty) = \int_{r_0}^{+\infty} \frac{d\breve{s}}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})} = +\infty, \end{aligned} \tag{2.8}$$

and $(\breve{x}_1, \breve{y}_1) \in \Omega$ is chosen so that

$$\Big(\breve{G}\big(q(\breve{x}, \breve{y})\big) + \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s}, \breve{t}) \Big(1 + \int_0^{\breve{s}} g(\breve{\tau}, \breve{t}) \, d\breve{\tau} \Big) \, d\breve{t} \, d\breve{s} \Big) \in \operatorname{Dom}\big(G^{-1}\big).$$

Corollary 2.3 *The discrete form can be obtained by letting* $\breve{\mathbb{T}} = \mathbb{Z}$ *in Theorem* 2.1:

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \le a(\breve{x},\breve{y}) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1}\big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big] + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\left(\sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\right)$$

for $(\breve{x},\breve{y}) \in \Omega$, *which implies*

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\left\{\breve{G}^{-1}\left[\breve{G}\big(q(\breve{x},\breve{y})\big) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\left(1 + \sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\right)\right]\right\}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, *where*

$$q(\breve{x},\breve{y}) = a(\breve{x},\breve{y}) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} p(\breve{s},\breve{t}),$$

$$\breve{G}(r) = \sum_{\breve{s}=r_0}^{r-1} \frac{1}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})}, \quad r \ge r_0 > 0, \qquad \breve{G}(+\infty) = \sum_{\breve{s}=r_0}^{+\infty} \frac{1}{\bar{\varphi} \circ \bar{\psi}^{-1}(\breve{s})} = +\infty, \tag{2.9}$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{G}\big(q(\breve{x},\breve{y})\big) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\left(1 + \sum_{\breve{\tau}_0}^{\breve{s}} g(\breve{\tau},\breve{t})\right)\right) \in \mathrm{Dom}\big(G^{-1}\big).$$

Theorem 2.4 *Assume that* $h, b \in C_{\mathrm{rd}}(\Omega,\mathbb{R}_+)$. *Let* $g, f, p, a, u, \bar{\psi}$ *and* $\bar{\varphi}$ *be as in Theorem* 2.1, *if* $u(\breve{x},\breve{y})$ *satisfies*

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \le a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}}\big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} + \int_0^{\breve{x}}\int_0^{\breve{y}} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}$$

for $(\breve{x},\breve{y}) \in \Omega$, *then*

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\left\{\breve{G}^{-1}\left[\breve{G}\big(q(\breve{x},\breve{y})\big) + A(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right]\right\} \tag{2.10}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, *where* q, $\breve{G}$ *are defined by* (2.2) *and* (2.3), *respectively, and*

$$\breve{A}(\breve{x},\breve{y}) = \int_0^{\breve{x}}\int_0^{\breve{y}} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t}) + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \tag{2.11}$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{G}\big(q(\breve{x},\breve{y})\big) + \breve{A}(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right) \in \mathrm{Dom}\big(\breve{G}^{-1}\big).$$

Proof Assume that $a(\breve{x},\breve{y}) > 0$. Fixing an arbitrary $(\breve{\xi},\breve{\zeta}) \in \Omega$, we define positive and nondecreasing function $z(\breve{x},\breve{y})$ by

$$\begin{aligned} z(\breve{x},\breve{y}) &= q(\breve{\xi},\breve{\zeta}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\ &\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \end{aligned}$$

for $0 \le \breve{x} \le \breve{\xi} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{\zeta} \le y_1$, then $z(0,\breve{y}) = z(\breve{x},0) = q(\breve{\xi},\breve{\zeta})$ and

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big);$$

then we have

$$\begin{aligned} z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y}) &= \int_0^{\breve{y}} f(\breve{x},\breve{t})\bar{\varphi}\big(u(\breve{x},\breve{t})\big)\hat{\Delta}\breve{t} \\ &\quad + \int_0^{\breve{y}} b(\breve{x},\breve{t})\left(h(\breve{x},\breve{t})\bar{\varphi}\big(u(\breve{x},\breve{t})\big) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t} \\ &\le \int_0^{\breve{y}} f(\breve{x},\breve{t})\bar{\varphi}\circ\bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big)\hat{\Delta}\breve{t} + \int_0^{\breve{y}} b(\breve{x},\breve{t}) \\ &\quad \times \left(h(\breve{x},\breve{t})\bar{\varphi}\circ\bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi}\circ\bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t} \\ &\le \bar{\varphi}\circ\bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big)\left[\int_0^{\breve{y}} f(\breve{x},\breve{t})\hat{\Delta}\breve{t}\right. \\ &\quad \left. + \int_0^{\breve{y}} b(\breve{x},\breve{t})\left(h(\breve{x},\breve{t}) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right)\right]\hat{\Delta}\breve{t}, \end{aligned}$$

then

$$\frac{z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y})}{\bar{\varphi}\circ\bar{\psi}^{-1}(z(\breve{x},\breve{y}))} \le \left[\int_0^{\breve{y}} f(\breve{x},\breve{t})\hat{\Delta}\breve{t} + \int_0^{\breve{y}} b(\breve{x},\breve{t})\left(h(\breve{x},\breve{t}) + \int_0^{\breve{x}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right)\right]\hat{\Delta}\breve{t}. \tag{2.12}$$

Integrating (2.12) and using (2.3) and (2.11), we get

$$\breve{G}\big(z(\breve{x},\breve{y})\big) \le \breve{G}\big(q(\breve{\xi},\breve{\zeta})\big) + \breve{A}(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}.$$

Since $(\breve{\xi},\breve{\zeta}) \in \Omega$ is chosen arbitrarily,

$$z(\breve{x},\breve{y}) \le \breve{G}^{-1}\left[\breve{G}\big(q(\breve{x},\breve{y})\big) + \breve{A}(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right]. \tag{2.13}$$

From (2.13) and $u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}(z(\breve{x},\breve{y}))$, we get the required inequality in (2.10). For $a(\breve{x},\breve{y}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\breve{x},\breve{y})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Corollary 2.5 *If we take $\breve{\mathbb{T}} = \mathbb{R}$ in Theorem 2.4, then the inequality*

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \leq a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big]\,d\breve{t}\,d\breve{s}$$
$$+ \int_0^{\breve{x}}\int_0^{\breve{y}} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\,d\breve{\tau}\right]d\breve{t}\,d\breve{s},$$

for $(\breve{x},\breve{y}) \in \Omega$, implies

$$u(\breve{x},\breve{y}) \leq \bar{\psi}^{-1}\left\{\breve{G}^{-1}\left[\breve{G}\big(q(\breve{x},\breve{y})\big) + A(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\,d\breve{t}\,d\breve{s}\right]\right\}$$

for $0 \leq \breve{x} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{y}_1$, where $\breve{G}$ is defined by (2.8) and

$$\breve{A}(\breve{x},\breve{y}) = \int_0^{\breve{x}}\int_0^{\breve{y}} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t}) + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\,d\breve{\tau}\right]d\breve{t}\,d\breve{s}$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ is chosen so that

$$\left(\breve{G}\big(q(\breve{x},\breve{y})\big) + \breve{A}(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\,d\breve{t}\,d\breve{s}\right) \in \mathrm{Dom}\big(\breve{G}^{-1}\big).$$

Corollary 2.6 *The discrete form can be obtained by letting $\breve{\mathbb{T}} = \mathbb{Z}$ in Theorem 2.4:*

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \leq a(\breve{x},\breve{y}) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} \big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big]$$
$$+ \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + \sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\right],$$

for $(\breve{x},\breve{y}) \in \Omega$, implies

$$u(\breve{x},\breve{y}) \leq \bar{\psi}^{-1}\left\{G^{-1}\left[G\big(q(\breve{x},\breve{y})\big) + A(\breve{x},\breve{y}) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\right]\right\}$$

for $0 \leq \breve{x} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{y}_1$, where $\breve{G}$ is defined by (2.9) and

$$\breve{A}(\breve{x},\breve{y}) = \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} b(\breve{s},\breve{t})\left[h(\breve{s},\breve{t}) + \sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\right]$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ is chosen so that

$$\left(\breve{G}\big(q(\breve{x},\breve{y})\big) + \breve{A}(\breve{x},\breve{y}) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\right) \in \mathrm{Dom}\big(\breve{G}^{-1}\big).$$

Theorem 2.7 *Assume that* $g, a, u, f, p, \bar{\psi}$ *and* $\bar{\varphi}$ *are as in Theorem* 2.1. *If* $u(\breve{x},\breve{y})$ *satisfies*

$$\begin{aligned}\bar{\psi}\bigl(u(\breve{x},\breve{y})\bigr) &\leq a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \bar{\varphi}\bigl(u(\breve{s},\breve{t})\bigr)\bigl[f(\breve{s},\breve{t})\bar{\varphi}\bigl(u(\breve{s},\breve{t})\bigr) + p(\breve{s},\breve{t})\bigr]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\ &\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi}\bigl(u(\breve{s},\breve{t})\bigr)\biggl(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\bigl(u(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\biggr)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},\end{aligned}$$

for $(\breve{x},\breve{y}) \in \Omega$, *then*

$$\begin{aligned}u(\breve{x},\breve{y}) &\leq \bar{\psi}^{-1}\biggl\{\breve{G}^{-1}\biggl(\breve{F}^{-1}\biggl[\breve{F}\bigl(q_1(\breve{x},\breve{y})\bigr) \\ &\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\biggl(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\biggr)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\biggr]\biggr)\biggr\},\end{aligned} \tag{2.14}$$

for $0 \leq \breve{x} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{y}_1$, *where* $\breve{G}$ *is defined in* (2.3) *and*

$$q_1(\breve{x},\breve{y}) = \breve{G}\bigl(a(\breve{x},\breve{y})\bigr) + \int_0^{\breve{x}}\int_0^{\breve{y}} p(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \tag{2.15}$$

$$\begin{aligned}\breve{F}(r) &= \int_{r_0}^{r} \frac{\hat{\Delta}\breve{s}}{((\bar{\varphi}\circ\bar{\psi}^{-1})\circ\breve{G}^{-1})(\breve{s})}, \quad r \geq r_0 > 0, \\ \breve{F}(+\infty) &= \int_{r_0}^{+\infty} \frac{\hat{\Delta}\breve{s}}{(\bar{\varphi}\circ\bar{\psi}^{-1})\circ\breve{G}^{-1}(\breve{s})} = +\infty,\end{aligned} \tag{2.16}$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\biggl(\breve{F}\bigl(q_1(\breve{x},\breve{y})\bigr) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\biggl(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\biggr)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\biggr) \in \operatorname{Dom}\bigl(\breve{F}^{-1}\bigr).$$

Proof Suppose that $a(\breve{\xi},\breve{\zeta}) > 0$. Fixing an arbitrary $(\breve{\xi},\breve{\zeta}) \in \Omega$, we define a positive and nondecreasing function $z(\breve{x},\breve{y})$ by

$$\begin{aligned}z(\breve{x},\breve{y}) &= a(\breve{\xi},\breve{\zeta}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \bar{\varphi}\bigl(u(\breve{s},\breve{t})\bigr)\bigl[f(\breve{s},\breve{t})\bar{\varphi}\bigl(u(\breve{s},\breve{t})\bigr) + p(\breve{s},\breve{t})\bigr]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\ &\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi}\bigl(u(\breve{s},\breve{t})\bigr)\biggl(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\bigl(u(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\biggr)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},\end{aligned}$$

for $0 \leq \breve{x} \leq \breve{\xi} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{\zeta} \leq \breve{y}_1$, then $z(0,\breve{y}) = z(\breve{x},0) = a(\breve{\xi},\breve{\zeta})$ and

$$u(\breve{x},\breve{y}) \leq \bar{\psi}^{-1}\bigl(z(\breve{x},\breve{y})\bigr),$$

then we have

$$\begin{aligned}z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y}) &= \int_0^{\breve{y}} \bar{\varphi}\bigl(u(\breve{x},\breve{t})\bigr)\bigl[f(\breve{x},\breve{t})\bar{\varphi}\bigl(u(\breve{x},\breve{t})\bigr) + p(\breve{x},\breve{t})\bigr]\hat{\Delta}\breve{t} \\ &\quad + \int_0^{\breve{y}} f(\breve{x},\breve{t})\bar{\varphi}\bigl(u(\breve{x},\breve{t})\bigr)\biggl(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi}\bigl(u(\breve{\tau},\breve{t})\bigr)\hat{\Delta}\breve{\tau}\biggr)\hat{\Delta}\breve{t}\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^{\breve{y}} \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big)\big[f(\breve{x},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big) + p(\breve{x},\breve{t})\big]\hat{\Delta}\breve{t} \\
&\quad + \int_0^{\breve{y}} f(\breve{x},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big)\left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t} \\
&\leq \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big) \int_0^{\breve{y}} \big[f(\breve{x},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big) + p(\breve{x},\breve{t})\big]\hat{\Delta}\breve{t} \\
&\quad + \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big) \int_0^{\breve{y}} f(\breve{x},\breve{t})\left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t},
\end{aligned}$$

or

$$\begin{aligned}
\frac{z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y})}{\bar{\varphi} \circ \bar{\psi}^{-1}(z(\breve{x},\breve{y}))} &\leq \int_0^{\breve{y}} \big[f(\breve{x},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big) + p(\breve{x},\breve{t})\big]\hat{\Delta}\breve{t} \\
&\quad + \int_0^{\breve{y}} f(\breve{x},\breve{t})\left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}. \qquad (2.17)
\end{aligned}$$

Integrating (2.17) and using (2.3), we get

$$\begin{aligned}
\breve{G}\big(z(\breve{x},\breve{y})\big) &\leq \breve{G}\big(a(\breve{\xi},\breve{\zeta})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} \big[f(\breve{s},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\
&\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}.
\end{aligned}$$

$(\breve{\xi},\breve{\zeta}) \in \Omega$ is chosen arbitrarily, then from (2.15) we have

$$\begin{aligned}
\breve{G}\big(z(\breve{x},\breve{y})\big) &\leq q_1(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{s},\breve{t})\big)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\
&\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}.
\end{aligned}$$

Since $q_1(\breve{x},\breve{y}) > 0$ is a nondecreasing function, fixing an arbitrary point $(\breve{\xi},\breve{\zeta}) \in \Omega$ and defining $v(\breve{x},\breve{y}) > 0$ to be a nondecreasing function by

$$\begin{aligned}
v(\breve{x},\breve{y}) &= q_1(\breve{\xi},\breve{\zeta}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{s},\breve{t})\big)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\
&\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},
\end{aligned}$$

for $0 \leq \breve{x} \leq \breve{\xi} \leq \breve{x}_1$, $0 \leq \breve{y} \leq \breve{\zeta} \leq y_1$, we have $v(0,\breve{y}) = v(\breve{x},0) = q_1(\breve{\xi},\breve{\zeta})$ and

$$z(\breve{x},\breve{y}) \leq \breve{G}^{-1}\big(v(\breve{x},\breve{y})\big); \qquad (2.18)$$

then we have

$$\begin{aligned}
v^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y}) &= \int_0^{\breve{y}} f(\breve{x},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big)\hat{\Delta}\breve{t} \\
&\quad + \int_0^{\breve{y}} f(\breve{x},\breve{t})\left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t})\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^{\breve y} f(\breve x,\breve t)\bar\varphi\circ\bar\psi^{-1}\big(G^{-1}\big(\nu(\breve x,\breve t)\big)\big)\hat\Delta\breve t\\
&\quad+\int_0^{\breve y} f(\breve x,\breve t)\bigg(\int_0^{\breve x} g(\breve\tau,\breve t)\bar\varphi\circ\bar\psi^{-1}\big(G^{-1}\big(\nu(\breve\tau,\breve t)\big)\big)\hat\Delta\breve\tau\bigg)\hat\Delta\breve t\\
&\le \big(\bar\varphi\circ\bar\psi^{-1}\big)\circ\breve G^{-1}\big(\nu(\breve x,\breve y)\big)\bigg[\int_0^{\breve y} f(\breve x,\breve t)\hat\Delta\breve t+\int_0^{\breve y} f(\breve x,\breve t)\bigg(\int_0^{\breve x} g(\breve\tau,\breve t)\hat\Delta\breve\tau\bigg)\hat\Delta\breve t\bigg],
\end{aligned}
$$

or

$$
\frac{\nu^{\hat\Delta\breve x}(\breve x,\breve y)}{(\bar\varphi\circ\bar\psi^{-1})\circ\breve G^{-1}(\nu(\breve x,\breve y))}\le\bigg[\int_0^{\breve y} f(\breve x,\breve t)\hat\Delta\breve t+\int_0^{\breve y} f(\breve x,\breve t)\bigg(\int_0^{\breve x} g(\breve\tau,\breve t)\hat\Delta\breve\tau\bigg)\hat\Delta\breve t\bigg]. \tag{2.19}
$$

Integrating (2.19) and using (2.16), we get

$$
\breve F\big(\nu(\breve x,\breve y)\big)\le\breve F\big(q_1(\breve\xi,\breve\zeta)\big)+\int_0^{\breve x}\int_0^{\breve y} f(\breve s,\breve t)\bigg[1+\int_0^{\breve s} g(\breve\tau,\breve t)\hat\Delta\breve\tau\bigg]\hat\Delta\breve t\hat\Delta\breve s.
$$

Since we can choose $(\breve\xi,\breve\zeta)\in\Omega$ arbitrarily, we have

$$
\nu(\breve x,\breve y)\le\breve F^{-1}\bigg[\breve F\big(q_1(\breve x,\breve y)\big)+\int_0^{\breve x}\int_0^{\breve y} f(\breve s,\breve t)\bigg[1+\int_0^{\breve s} g(\breve\tau,\breve t)\hat\Delta\breve\tau\bigg]\hat\Delta\breve t\hat\Delta\breve s\bigg]. \tag{2.20}
$$

From (2.20), (2.18) and $u(\breve x,\breve y)\le\bar\psi^{-1}(z(\breve x,\breve y))$ we get the desired inequality in (2.14). For $a(\breve x,\breve y)=0$, we carry out the above procedure with $\epsilon>0$ instead of $a(\breve x,\breve y)$ and subsequently let $\epsilon\to 0$. This completes the proof. □

Corollary 2.8 *If we take $\breve{\mathbb T}=\mathbb R$ in Theorem 2.7, then the inequality*

$$
\begin{aligned}
\bar\psi\big(u(\breve x,\breve y)\big)&\le a(\breve x,\breve y)+\int_0^{\breve x}\int_0^{\breve y}\bar\varphi\big(u(\breve s,\breve t)\big)\big[f(\breve s,\breve t)\bar\varphi\big(u(\breve s,\breve t)\big)+p(\breve s,\breve t)\big]\,d\breve t\,d\breve s\\
&\quad+\int_0^{\breve x}\int_0^{\breve y} f(\breve s,\breve t)\bar\varphi\big(u(\breve s,\breve t)\big)\bigg(\int_0^{\breve s} g(\breve\tau,\breve t)\bar\varphi\big(u(\breve\tau,\breve t)\big)\hat\Delta\breve\tau\bigg)\,d\breve t\,d\breve s,
\end{aligned}
$$

for $(\breve x,\breve y)\in\Omega$, implies

$$
u(\breve x,\breve y)\le\bar\psi^{-1}\bigg\{\breve G^{-1}\bigg(\breve F^{-1}\bigg[\breve F\big(q_2(\breve x,\breve y)\big)+\int_0^{\breve x}\int_0^{\breve y} f(\breve s,\breve t)\bigg(1+\int_0^{\breve s} g(\breve\tau,\breve t)\,d\breve\tau\bigg)\,d\breve t\,d\breve s\bigg]\bigg)\bigg\},
$$

for $0\le\breve x\le\breve x_1$, $0\le\breve y\le\breve y_1$, where $\breve G$ is as defined in (2.8) and

$$
\begin{aligned}
&q_2(\breve x,\breve y)=\breve G\big(a(\breve x,\breve y)\big)+\int_0^{\breve x}\int_0^{\breve y} p(\breve s,\breve t)\,d\breve t\,d\breve s,\\
&\breve F(r)=\int_{r_0}^{r}\frac{d\breve s}{((\bar\varphi\circ\bar\psi^{-1})\circ\breve G^{-1})(\breve s)},\quad r\ge r_0>0,\\
&\breve F(+\infty)=\int_{r_0}^{+\infty}\frac{d\breve s}{(\bar\varphi\circ\bar\psi^{-1})\circ\breve G^{-1}(\breve s)}=+\infty,
\end{aligned}
$$

and $(\breve{x}_1, \breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{F}\big(q_2(\breve{x},\breve{y})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\,d\breve{\tau}\right)d\breve{t}\,d\breve{s}\right) \in \mathrm{Dom}\big(\breve{F}^{-1}\big).$$

Corollary 2.9 *The discrete form of Theorem 2.7 can be obtained by letting* $\breve{\mathbb{T}} = \mathbb{Z}$:

$$\begin{aligned}\bar{\psi}\big(u(\breve{x},\breve{y})\big) \le{}& a(\breve{x},\breve{y}) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} \bar{\varphi}\big(u(\breve{s},\breve{t})\big)\big[f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big) + p(\breve{s},\breve{t})\big] \\ &+ \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\left(\sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\right),\end{aligned}$$

for $(\breve{x},\breve{y}) \in \Omega$, *implies*

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\left\{\bar{G}^{-1}\left(\bar{F}^{-1}\left[\bar{F}\big(\bar{q}_2(\breve{x},\breve{y})\big) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\left(1 + \sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\right)\right]\right)\right\},$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, *where* $\breve{G}$ *is as defined in* (2.9) *and*

$$\begin{aligned}&\bar{q}_2(\breve{x},\breve{y}) = \breve{G}\big(a(\breve{x},\breve{y})\big) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} p(\breve{s},\breve{t}), \\ &\bar{F}(r) = \sum_{\breve{s}=r_0}^{r-1} \frac{1}{((\bar{\varphi}\circ\bar{\psi}^{-1})\circ\bar{G}^{-1})(\breve{s})}, \qquad r \ge r_0 > 0, \\ &\bar{F}(+\infty) = \sum_{\breve{s}=r_0}^{+\infty} \frac{1}{(\bar{\varphi}\circ\bar{\psi}^{-1})\circ\bar{G}^{-1}(\breve{s})} = +\infty,\end{aligned}$$

and $(\breve{x}_1, \breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\bar{F}\big(\bar{q}_2(\breve{x},\breve{y})\big) + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\left(1 + \sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\right)\right) \in \mathrm{Dom}\big(\bar{F}^{-1}\big).$$

Theorem 2.10 *Assume that* $g, a, f, u, , \bar{\psi}$ *and* $\bar{\varphi}$ *be as in Theorem* 2.1. *If* $u(\breve{x},\breve{y})$ *satisfies*

$$\begin{aligned}\bar{\psi}\big(u(\breve{x},\breve{y})\big) \le{}& a(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right)^2 \\ &+ \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},\end{aligned}$$

for $(\breve{x},\breve{y}) \in \Omega$, *then*

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\breve{x},\breve{y})\big) + \breve{B}(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right)^2\right]\right\}, \tag{2.21}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, *where*

$$\breve{B}(\breve{x},\breve{y}) = \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t}) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s}, \tag{2.22}$$

$$\begin{aligned} \breve{H}(r) &= \int_{r_0}^{r} \frac{\hat{\Delta}\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})}, \quad r \ge r_0 > 0, \\ \breve{H}(+\infty) &= \int_{r_0}^{+\infty} \frac{\hat{\Delta}\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})} = +\infty, \end{aligned} \tag{2.23}$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{H}\big(a(\breve{x},\breve{y})\big) + B(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right)^2 \right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

Proof Assume that $a(\breve{x},\breve{y}) > 0$. Taking $(\breve{\xi},\breve{\zeta}) \in \Omega$ as a fixed arbitrary point, we define $z(\breve{x},\breve{y}) > 0$ to be a nondecreasing function by

$$\begin{aligned} z(\breve{x},\breve{y}) = {}& a(\breve{\xi},\breve{\zeta}) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right)^2 \\ & + \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t}) \bar{\varphi}\big(u(\breve{\tau},\breve{t})\big) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s}, \end{aligned} \tag{2.24}$$

for $0 \le \breve{x} \le \breve{\xi} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{\zeta} \le \breve{y}_1$, hence $z(0,\breve{y}) = z(\breve{x},0) = a(\breve{\xi},\breve{\zeta})$ and

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big).$$

From (2.24), and applying the chain rule on time scales, Theorem 1.4, we get

$$\begin{aligned} z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y}) = {}& 2\left(\int_0^{c} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right) \int_0^{\breve{y}} f(\breve{x},\breve{t}) \bar{\varphi}\big(u(\breve{x},\breve{t})\big) \hat{\Delta}\breve{t} \\ & + \int_0^{\breve{y}} f(\breve{x},\breve{t}) \bar{\varphi}\big(u(\breve{x},\breve{t})\big) \left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t}) \bar{\varphi}\big(u(\breve{\tau},\breve{t})\big) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t} \\ \le {}& 2\left(\int_0^{c} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{s},\breve{t})\big) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right) \int_0^{\breve{y}} f(\breve{x},\breve{t}) \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big) \hat{\Delta}\breve{t} \\ & + \int_0^{\breve{y}} f(\breve{x},\breve{t}) \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{t})\big) \left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t}) \bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{\tau},\breve{t})\big) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t} \\ \le {}& 2\big(\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big)\big)^2 \left(\int_0^{c} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right) \int_0^{\breve{y}} f(\breve{x},\breve{t}) \hat{\Delta}\breve{t} \\ & + \big(\bar{\varphi} \circ \bar{\psi}^{-1}\big(z(\breve{x},\breve{y})\big)\big)^2 \int_0^{\breve{y}} f(\breve{x},\breve{t}) \left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t}) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t}, \end{aligned}$$

thus, we have

$$\begin{aligned} \frac{z^{\hat{\Delta}\breve{x}}(\breve{x},\breve{y})}{(\bar{\varphi} \circ \bar{\psi}^{-1}(z(\breve{x},\breve{y})))^2} \le {}& 2\left(\int_0^{c} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t} \hat{\Delta}\breve{s} \right) \int_0^{\breve{y}} f(\breve{x},\breve{t}) \hat{\Delta}\breve{t} \\ & + \int_0^{\breve{y}} f(\breve{x},\breve{t}) \left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t}) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t} \end{aligned}$$

$$= \left[\left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \right)^2 \right]^{\hat{\Delta}_{\breve{x}}} + \int_0^{\breve{y}} f(\breve{x},\breve{t}) \left(\int_0^{\breve{x}} g(\breve{\tau},\breve{t}) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t}. \tag{2.25}$$

Integrating (2.25) and using (2.23), we get

$$\breve{H}\big(z(\breve{x},\breve{y})\big) \le \breve{H}\big(a(\breve{\xi},\breve{\zeta})\big) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \right)^2 + \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t}) \hat{\Delta}\breve{\tau} \right) \hat{\Delta}\breve{t}\hat{\Delta}\breve{s}.$$

Since $(\breve{\xi},\breve{\zeta}) \in \Omega$ is chosen arbitrarily,

$$z(\breve{x},\breve{y}) \le \breve{H}^{-1} \left[\breve{H}\big(a(\breve{x},\breve{y})\big) + \breve{B}(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \right)^2 \right]. \tag{2.26}$$

From (2.26) and $u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}(z(\breve{x},\breve{y}))$, we get the desired inequality (2.21). For $a(\breve{x},\breve{y}) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\breve{x},\breve{y})$ and subsequently let $\epsilon \to 0$. This completes the proof. □

Corollary 2.11 *If we take* $\breve{\mathbb{T}} = \mathbb{R}$ *in Theorem* 2.10, *then the inequality*

$$\bar{\psi}\big(u(\breve{x},\breve{y})\big) \le a(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big)\, d\breve{t}\, d\breve{s} \right)^2 + \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \bar{\varphi}\big(u(\breve{s},\breve{t})\big) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t}) \bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\, d\breve{\tau} \right) d\breve{t}\, d\breve{s},$$

for $(\breve{x},\breve{y}) \in \Omega$, *implies*

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1} \left\{ \breve{H}^{-1} \left[\breve{H}\big(a(\breve{x},\breve{y})\big) + \breve{B}(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t})\, d\breve{t}\, d\breve{s} \right)^2 \right] \right\},$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, *where*

$$\breve{B}(\breve{x},\breve{y}) = \int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t}) \left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\, d\breve{\tau} \right) d\breve{t}\, d\breve{s},$$

$$\breve{H}(r) = \int_{r_0}^{r} \frac{d\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})}, \quad r \ge r_0 > 0, \qquad \breve{H}(+\infty) = \int_{r_0}^{+\infty} \frac{d\breve{s}}{(\bar{\varphi} \circ \bar{\psi}^{-1})^2(\breve{s})} = +\infty,$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{H}\big(a(\breve{x},\breve{y})\big) + B(\breve{x},\breve{y}) + \left(\int_0^{\breve{x}} \int_0^{\breve{y}} f(\breve{s},\breve{t})\, d\breve{t}\, d\breve{s} \right)^2 \right) \in \mathrm{Dom}\big(\breve{H}^{-1}\big).$$

Corollary 2.12 *The discrete form can be obtained by letting* $\breve{\mathbb{T}} = \mathbb{Z}$ *in Theorem* 2.10:

$$\begin{aligned}\bar{\psi}\big(u(\breve{x},\breve{y})\big) &\le a(\breve{x},\breve{y}) + \left(\sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\right)^2 \\ &\quad + \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\bar{\varphi}\big(u(\breve{s},\breve{t})\big)\left(\sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\bar{\varphi}\big(u(\breve{\tau},\breve{t})\big)\right),\end{aligned}$$

for $(\breve{x},\breve{y}) \in \Omega$, *implies*

$$u(\breve{x},\breve{y}) \le \bar{\psi}^{-1}\left\{\breve{H}^{-1}\left[\breve{H}\big(a(\breve{x},\breve{y})\big) + \breve{B}(\breve{x},\breve{y}) + \left(\sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\right)^2\right]\right\},$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, *where*

$$\breve{B}(\breve{x},\breve{y}) = \sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}} f(\breve{s},\breve{t})\left(\sum_{\breve{\tau}=0}^{\breve{s}-1} g(\breve{\tau},\breve{t})\right),$$

$$\breve{H}(r) = \sum_{\breve{s}=r_0}^{r-1}\frac{1}{(\bar{\varphi}\circ\bar{\psi}^{-1})^2(\breve{s})}, \quad r \ge r_0 > 0, \qquad \breve{H}(+\infty) = \sum_{\breve{s}=r_0}^{+\infty}\frac{1}{(\bar{\varphi}\circ\bar{\psi}^{-1})^2(\breve{s})} = +\infty,$$

and $(\breve{x}_1,\breve{y}_1) \in \Omega$ *is chosen so that*

$$\left(\breve{H}\big(a(\breve{x},\breve{y})\big) + B(\breve{x},\breve{y}) + \left(\sum_{\breve{s}=0}^{\breve{x}-1}\sum_{\breve{t}=0}^{\breve{y}-1} f(\breve{s},\breve{t})\right)^2\right) \in \operatorname{Dom}\big(\breve{H}^{-1}\big).$$

3 Applications

The present section illustrates how Theorems 2.7 and 2.1 can be used to study the boundedness of the solutions of some initial boundary value problem for partial dynamic equations in two independent variables.

Let us consider the problem

$$u^{\hat{\Delta}\breve{x}\hat{\Delta}\breve{y}}(\breve{x},\breve{y}) = \breve{F}\left(\breve{x},\breve{y},u(\breve{x},\breve{y}),\int_0^{\breve{x}}\breve{k}\big(\breve{s},\breve{y},u(s,\breve{y})\big)\hat{\Delta}\breve{s}\right), \tag{3.1}$$

$$u(\breve{x},0) = a_1(\breve{x}), \qquad u(0,\breve{y}) = a_2(\breve{y}), \qquad a_1(0) = a_2(0) = 0, \tag{3.2}$$

for any $(\breve{x},\breve{y}) \in \Omega$, where $\breve{k} \in C_{\mathrm{rd}}(\Omega \times \mathbb{R},\mathbb{R})$, $\breve{F} \in C_{\mathrm{rd}}(\Omega \times \mathbb{R} \times \mathbb{R},\mathbb{R})$, $a_1 \in C_{\mathrm{rd}}(\breve{\mathbb{T}}_1,\mathbb{R})$ and $a_2 \in C_{\mathrm{rd}}(\breve{\mathbb{T}}_2,\mathbb{R})$.

Theorem 3.1 *Suppose that the functions* $\breve{k}, \breve{F}, a_2, a_1$ *in* (3.1) *and* (3.2) *satisfy the conditions*

$$\begin{aligned}\big|\breve{F}(\breve{x},\breve{y},u(\breve{x},\breve{y},\nu)\big| &\le \bar{\varphi}\big(\big|u(\breve{x},\breve{y})\big|\big)\big[f(\breve{x},\breve{y})\bar{\varphi}\big(\big|u(\breve{x},\breve{y})\big|\big) + p(\breve{x},\breve{y})\big] \\ &\quad + f(\breve{x},\breve{y})\bar{\varphi}\big(\big|u(\breve{x},\breve{y})\big|\big)\nu,\end{aligned} \tag{3.3}$$

$$\big|\breve{k}\big(\breve{x},\breve{y},u(\breve{x},\breve{y})\big)\big| \le g(\breve{x},\breve{y})\bar{\varphi}\big(\big|u(\breve{x},\breve{y})\big|\big), \tag{3.4}$$

$$\big|a_1(\breve{x}) + a_2(\breve{y})\big| \le a(\breve{x},\breve{y}), \tag{3.5}$$

where the functions p, g, a, f, *and* $\bar{\varphi}$ *are defined as in Theorem 2.7 with* $a(\breve{x},\breve{y}) > 0$, *for all* $(\breve{x},\breve{y}) \in \Omega$, *then*

$$\left|u(\breve{x},\breve{y})\right| \le \breve{G}^{-1}\left(\breve{F}^{-1}\left[\breve{F}\big(q_2(\breve{x},\breve{y})\big) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left[1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right]\right), \tag{3.6}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$, *where* F, q_2 *and* G *are defined as in Theorem 2.7.*

Proof If the problem (3.1) and (3.2) has a solution $u(\breve{x},\breve{y})$, it can be written as

$$u(\breve{x},\breve{y}) = a_1(\breve{x}) + a_2(\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \breve{F}\left(\breve{s},\breve{t},u(\breve{s},\breve{t}),\int_0^{\breve{s}} \breve{k}\big(\breve{\tau},\breve{t},u(\breve{\tau},t)\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \tag{3.7}$$

for any $(\breve{x},\breve{y}) \in \Omega$. Using the conditions (3.3), (3.4) and (3.5) in (3.7), we get

$$\begin{aligned}\left|u(\breve{x},\breve{y})\right| \le{}& a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} \bar{\varphi}\big(\left|u(\breve{s},\breve{t})\right|\big)\big[f(\breve{s},\breve{t})\bar{\varphi}\big(\left|u(\breve{s},\breve{t})\right|\big) + p(\breve{s},\breve{t})\big]\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\ &+ \int_0^{\breve{x}}\int_0^{\breve{y}} f(s,t)\bar{\varphi}\big(\left|u(s,t)\right|\big)\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\bar{\varphi}\big(\left|u(\breve{\tau},\breve{t})\right|\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s},\end{aligned} \tag{3.8}$$

for any $(\breve{x},\breve{y}) \in \Omega$. Now, an application of Theorem 2.7 to (3.8) yields the required inequality in (3.6) where $\bar{\psi}(u) = u$. □

Let us consider the initial boundary value problem of the form

$$\big(z^q\big)^{\hat{\Delta}\breve{x}\hat{\Delta}\breve{y}}(\breve{x},\breve{y}) = \breve{A}\left(\breve{x},\breve{y},z(\breve{x},\breve{y}),\int_0^{\breve{x}} h\big(\breve{s},\breve{y},z(\breve{s},\breve{y})\big)\hat{\Delta}\breve{s}\right) \tag{3.9}$$

$$z(\breve{x},0) = a_1(\breve{x}), \qquad z(0,\breve{y}) = a_2(\breve{y}), \qquad a_1(0) = a_2(0) = 0, \tag{3.10}$$

for any $(\breve{x},\breve{y}) \in \Omega$.

Theorem 3.2 *Assume that the functions* h, A, a_2, a_1 *in* (3.9) *and* (3.10) *satisfy the conditions*

$$\left|A\big(\breve{x},\breve{y},z(\breve{x},\breve{y},v)\right| \le f(\breve{x},\breve{y})\left|z^r(\breve{x},\breve{y})\right| + f(\breve{x},\breve{y})v, \tag{3.11}$$

$$\left|h\big(\breve{x},\breve{y},z(\breve{x},\breve{y})\big)\right| \le g(\breve{x},\breve{y})\left|z^r(\breve{x},\breve{y})\right|, \tag{3.12}$$

$$\left|a_1(\breve{x}) + a_2(\breve{y})\right| \le a(\breve{x},\breve{y}), \tag{3.13}$$

where $r \ge q > 0$, *then*

$$\left|z(\breve{x},\breve{y})\right| \le \left[\big(a(\breve{x},\breve{y})\big)^{\frac{q-r}{q}} + \frac{q-r}{q}\int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(1 + \int_0^{\breve{s}} g(\breve{\tau},\breve{t})\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}\right]^{\frac{1}{q-r}}, \tag{3.14}$$

for $0 \le \breve{x} \le \breve{x}_1$, $0 \le \breve{y} \le \breve{y}_1$.

Proof If the problem (3.9) and (3.10), has a solution $z(\breve{x},\breve{y})$ it can be written as

$$z^q(\breve{x},\breve{y}) = a_1(x) + a_2(y) + \int_0^{\breve{x}}\int_0^{\breve{y}} \breve{F}\left(\breve{s},\breve{s}\breve{t},u(\breve{s},\breve{t}),\int_0^{\breve{s}} \breve{k}\big(\breve{\tau},\breve{t},u(\breve{\tau},\breve{t})\big)\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \tag{3.15}$$

for any $(\breve{x},\breve{y}) \in \Omega$. Using the conditions (3.11), (3.12) and (3.13) in (3.15), we get

$$\begin{aligned}\left|z^q(\breve{x},\breve{y})\right| &\le a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left|z^r(s,t)\right|\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\ &\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\left|z^r(\breve{\tau},\breve{t})\right|\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \qquad (3.16)\end{aligned}$$

from (3.16), we get

$$\begin{aligned}\left|z^q(\breve{x},\breve{y})\right| &\le a(\breve{x},\breve{y}) + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left|z^r(\breve{t},\breve{t})\right|\hat{\Delta}\breve{t}\hat{\Delta}\breve{s} \\ &\quad + \int_0^{\breve{x}}\int_0^{\breve{y}} f(\breve{s},\breve{t})\left(\int_0^{\breve{s}} g(\breve{\tau},\breve{t})\left|z^r(\breve{\tau},\breve{t})\right|\hat{\Delta}\breve{\tau}\right)\hat{\Delta}\breve{t}\hat{\Delta}\breve{s}, \qquad (3.17)\end{aligned}$$

for any $(\breve{x},\breve{y}) \in \Omega$. A suitable application of Theorem 2.1 to (3.17) with $\bar{\psi}(u) = u^q$, $\bar{\varphi}(u) = u^r$ and $p(\breve{x},\breve{y}) = 0$ gives the required inequality in (3.14). □

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Author details
[1]Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo, Egypt. [2]Department of Mathematics, College of Science, Princess Noura bint Abdulrahman University, Riyadh, Saudi Arabia.

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