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The existence and Hyers–Ulam stability of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$

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Abstract
This paper deals with the existence of solution for an impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$ and its Hyers–Ulam stability. We prove the mild solutions for the equation using basic theorems of fractional differential equation. The existence result of the equation is obtained by Mönch's fixed point theorem. Finally, we prove the Hyers–Ulam stability of the solution.

Keywords: Impulsive Riemann–Liouville fractional differential equation; Stochastic differential equation; Mönch's fixed point theorem; Hausdorff measure of noncompactness; Hyers–Ulam stability

1 Introduction

Fractional differential equation [1], due to its applications in describing memory and hereditary properties of various materials and processes in natural sciences and engineering, has been widely investigated. However, because of disturbance, the behaviors of many real-word systems are not a settled process. Thus, it is crucial to study stochastic differential functions with impulse [2]. A variety of results on the theory of Caputo fractional functional stochastic equations and the Hyers–Ulam stability [3–8] of such function have been obtained. Cui and Yan [9] studied the existence of mild solutions for neutral fractional stochastic integral differential equations with infinite delay using Sadovekii's fixed point theorem. Sakthivel [10] studied the existence of solution to nonlinear fractional stochastic differential equations. Riemann–Liouville fractional derivatives or integrals are strong tools for resolving some fractional differential problems in the real world. However, only a few results on such derivatives or integrals have been reported in the literature. It is possible to attribute physical meaning to the initial conditions expressed in terms of Riemann–Liouville fractional derivatives or integrals which have been verified by Heymans and Podlubny [11]. Such initial conditions are more appropriate in modeling a real-word system.

Due to the lack of results on the existence and stability of solutions to Riemann–Liouville fractional stochastic differential equations in the literature, it is of great significance to perform some investigations in this field. Weera Yukunthorn et al. [12] studied the existence and uniqueness of solutions to the impulsive multiorder Riemann–Liouville fractional differential equations:

$$\begin{cases} D_{t_k}^{\alpha_k} x(t) = f(t, x(t)), \quad t \in J, t \neq t_k, \\ \tilde{\Delta} x(t_k) = \varphi_k(x(t_k)), \qquad \Delta^* x(t_k) = \varphi_k^*(x(t_k)), \quad k = 1, 2, \ldots, m, \\ x(0) = 0, \qquad D^{\alpha_0 - 1} x(0) = \beta, \end{cases}$$

where $\beta \in R$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times R \to R$ is a continuous function, φ_k, $\varphi_k^* \in C(R, R)$ for $k = 1, 2, \ldots, m$, and $D_{t_k}^{\alpha_k}$ is the Riemann–Liouville fractional derivative of order $1 < \alpha_k < 2$ on intervals J_k for $k = 0, 1, 2, \ldots, m$. The notation $\tilde{\Delta} x(t_k)$ is defined by

$$\tilde{\Delta} x(t_k) = I_{t_k}^{1-\alpha_k} x(t_k^+) - I_{t_{k-1}}^{1-\alpha_{k-1}} x(t_k), \quad k = 1, 2, \ldots, m,$$

and $\Delta^* x(t_k)$ is defined by

$$\Delta^* x(t_k) = I_{t_k}^{2-\alpha_k} x(t_k^+) - I_{t_{k-1}}^{2-\alpha_{k-1}} x(t_k), \quad k = 1, 2, \ldots, m,$$

where $I_{t_k}^{2-\alpha_k}$ is the Riemann–Liouville fractional integral of order $2 - \alpha_k > 0$ on J_k. By using Banach's fixed point theorem, the authors developed the existence theorem for such equations.

Motivated by this work, we design the following impulsive Riemann–Liouville fractional neutral functional stochastic differential equation with infinite delay:

$$\begin{cases} D_{0^+}^{\beta} [x(t) - g(t, x_t)] = f(t, x_t) + \sigma(t, x_t) \frac{d\omega(s)}{dt}, \quad t \in [0, T], t \neq t_k, \\ \Delta I_{0^+}^{2-\beta} x(t_k) = I_k(x(t_k^-)), \qquad \Delta I_{0^+}^{1-\beta} x(t_k) = J_k(x(t_k^-)), \\ I_{0^+}^{2-\beta} [x(0) - g(0, x_0)] = \varphi_1 \in B_v, \qquad I_{0^+}^{1-\beta} [x(0) - g(0, x_0)] = \varphi_2 \in B_v, \end{cases} \tag{1.1}$$

where $k = 1, 2, \ldots, m$ and $D_{0^+}^{\beta}$ is the Riemann–Liouville fractional derivative of order $1 < \beta < 2$. We have $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$. Let $T_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, $T_0 = [0, t_1]$. Here, $\omega(t) : t \in J$ is a standard Wiener process, $f : J \times B_v$, $g : J \times B_v$ and $\sigma : J \times B_v$ are given functions, where B_v is the phase space defined in Sect. 2. The impulsive functions $I_k, J_k : H \to H$ ($k = 1, 2, \ldots, m$) are appropriate functions. The notations $\Delta I_{0^+}^{2-\beta} x(t_k)$, $\Delta I_{0^+}^{1-\beta} x(t_k)$ are defined by

$$\begin{aligned} \Delta I_{0^+}^{2-\beta} x(t_k) &= I_{0^+}^{2-\beta} x(t_k^+) - I_{0^+}^{2-\beta} x(t_k^-), \\ \Delta I_{0^+}^{1-\beta} x(t_k) &= I_{0^+}^{1-\beta} x(t_k^+) - I_{0^+}^{1-\beta} x(t_k^-), \quad k = 1, 2, \ldots, m, \end{aligned}$$

where $I_{0^+}^{2-\beta}$, $I_{0^+}^{1-\beta}$ is the Riemann–Liouville fractional integral of order $2 - \beta$, $1 - \beta$, respectively. The histories $x_t : (-\propto, 0] \to H$ defined by $x_t(s) = x(t + s)$, $s \leq 0$, belong to some abstract phase space B_v. By using Mönch's fixed point theorem via the measure of noncompactness as well as the basic theory of Hyers–Ulam stability, we investigate the existence and stability of the solution to the equation.

The rest of the paper is organized as follows. In Sect. 2, some basic definitions, notations, and preliminary facts that are used throughout the paper are presented. In Sect. 3 we prove the solution of the equation and present the main results for problem (1.1).

2 Preliminaries

In this section, we introduce some notations, definitions, lemmas, and preliminary facts that will be used to establish our main results.

We consider two separable Hilbert spaces K and H. Thus, $L(K,H)$ is a space of bounded linear operators from K into H. In this passage, we use the same notation $\|\cdot\|$ to denote the norms in K and H and $L(K,H)$, and we use $(\cdot,\cdot)$ to denote the inner product of H and K. We denote (Ω,F,P) to be a complete filtered probability space satisfying that F_0 contains all P-null sets of F. Let $W=(W_t)_{t\geq 0}$ be a Q-Winner process defined on (Ω,F,P) with the covariance operator Q such that $\operatorname{Tr} Q<\infty$. We suppose that there exists a complete orthonormal system $\{e_k\}_{k\leq 1}$ in K, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k=\lambda_k e_k$, where $k=1,2,\ldots$, and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$\bigl(\omega(t),e\bigr)_K=\sum_{k=1}^{\infty}\sqrt{\lambda_k}(e_k,e)_K\beta_k(t).$$

Let $J=[0,T]$, $J_0=[0,t_1]$, $J_k=(t_k,t_{k+1}]$ for $k=1,2,\ldots,m$. Let

$$\begin{aligned}\mathrm{PC}(J,H):=\bigl\{&x:J\to H,\text{is continuous everywhere except for some } t_k \text{ at which}\\ &x\bigl(t_k^+\bigr)\text{ and }x\bigl(t_k^-\bigr)\text{ exist, and }x\bigl(t_k^-\bigr)=x(t_k),k=1,2,\ldots,m\bigr\}.\end{aligned}$$

We introduce the space $C_{2-\beta,k}(J_k,H):=\{x:J_k\to H:t^{2-\beta}x(t)\in C(J_k,H)\}$ with the norm $\|x\|_{C_{2-\beta,k}}=\sup_{t\in J_k}E|t^{2-\beta}\|x(t)\||$ and $\mathrm{PC}_{2-\beta}=\{x:J\to H:\text{for each } t\in J_k \text{ and } t^{2-\beta}x(t)\in C(J_k,H),k=0,1,2,\ldots,m\}$ with the norm

$$\|x\|_{\mathrm{PC}_{2-\beta}}=\sup_{t\in J_k}E\bigl|t^{2-\beta}\bigl\|x(t)\bigr\|\bigr|:\quad k=0,1,2,\ldots,m.$$

Before introducing the fractional-order functional differential equation with infinite delay, we define the abstract phase space B_ν. Let $\nu:(\infty,0]\to(0,\infty)$ be a continuous function that satisfies $l=\int_{-\infty}^0\nu(t)\,dt<+\infty$. The Hilbert space $(B_\nu,\|\cdot\|_{B_\nu})$ induced by ν is then given by

$$\begin{aligned}B_\nu:=\biggl\{&\varphi:(-\infty,0)\to H:\text{for any } c>0,\varphi(\theta)\text{ is a bounded and measurable}\\ &\text{function on } [-c,0],\text{ and }\int_{-\infty}^0\nu(s)\sup_{s\leq\theta\leq 0}\bigl(E\bigl|\varphi(\theta)\bigr|^2\bigr)^{\frac12}\,ds<+\infty\biggr\},\end{aligned}$$

endowed with the norm $\|\varphi\|_{B_\nu}:=\int_{-\infty}^0\nu(s)\sup_{s\leq\theta\leq 0}(E|\varphi(\theta)|^2)^{\frac12}\,ds$.

Define the following space:

$$\begin{aligned}B_\nu':=\bigl\{&\varphi:(-\infty,T]\to X:\varphi_k\in C^1(J_k,X),k=0,1,2,\ldots,m,\text{and there exist}\\ &\varphi\bigl(t_k^-\bigr)\text{ and }\varphi\bigl(t_k^+\bigr)\text{ with }\varphi(t_k)=\varphi\bigl(t_k^-\bigr),\varphi_0=\phi\in B_\nu\bigr\},\end{aligned}$$

where φ_k is the restriction of φ to J_k, $J_0=[0,t_1]$, $J_k=(t_k,t_{k+1}]$, $k=1,2,\ldots,m$.

We use $\|\cdot\|_{B'_\nu}$ to denote a seminorm in the space B'_ν defined by

$$\|\varphi\|_{B'_\nu} := \|\phi\|_{B_\nu} + \max\bigl\{\|\varphi_k\|_{J_k(2-\beta)}, k = 0, 1, \ldots, m\bigr\},$$

where

$$\phi = \varphi_0, \qquad \|\varphi_k\|_{J_k(2-\beta)} = \sup_{s\in J_k}\bigl(E\bigl|s^{2-\beta}\|\varphi(s)\|\bigr|^2\bigr)^{\frac{1}{2}}.$$

Now we consider some definitions about fractional differential equations.

Definition 2.1 The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f; (a,b) \to H$ is defined by

$$D^\alpha_{a^+} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds, \quad n-1 < \alpha < n, t \in (a,b),$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right-hand side is pointwise defined on (a,b), Γ is the gamma function.

Definition 2.2 The Riemann–Liouville fractional integral of order $\alpha > 0$ of a continuous function $f : (a,b) \to H$ is defined by

$$I^\alpha_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \quad t \in (a,b),$$

provided that the right-hand side is pointwise defined on (a,b).

Lemma 2.1 (see [13]) *Let $\alpha > 0$. Then, for $x \in C(a,b) \cap L(a,b)$, we have*

$$D^\alpha_{a^+} I^\alpha_{a^+} x(t) = x(t),$$

$$I^\alpha_a D^\alpha_a x(t) = x(t) - \sum_{j=1}^{n} \frac{(I^{n-\alpha}_a)^{(n-j)} x(a)}{\Gamma(\alpha - j + 1)} (t-a)^{\alpha-j},$$

where $n - 1 < \alpha < n$.

Lemma 2.2 (see [13]) *If $\alpha \geq 0$ and $\beta > 0$, then*

$$I^\alpha_{a^+} (t-s)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1},$$

$$D^\alpha_{a^+} (t-s)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}.$$

Before investigating the solutions to Eq. (1.1), we consider a simplified version of (1.1), given by

$$\begin{cases} D^\beta_{0^+} x(t) = f(t), \quad t \in [0,T], t \neq t_k, \\ \Delta I^{2-\beta}_{0^+} x(t_k) = y_k, \qquad \Delta I^{1-\beta}_{0^+} x(t_k) = \overline{y_k}, \\ I^{2-\beta}_{0^+} x(0^+) = x_0, \qquad I^{1-\beta}_{0^+} x(0^+) = x_1, \end{cases} \tag{2.1}$$

where $k = 1, 2, \ldots, m$, $x_0, x_1, y_k, \overline{y_k} \in H$ and $D^{\beta}_{0^+}$ is the Riemann–Liouville fractional derivative of order $1 < \beta < 2$.

Theorem 2.1 *Let* $1 < \beta < 2$ *and* $f : J \to H$ *be continuous. Then* $x \in \mathrm{PC}_{2-\beta}(J, H)$ *is a solution of* (2.1) *if and only if* x *is a solution of the following fractional integral equation*:

$$x(t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\,ds + x_1 \frac{t^{\beta-1}}{\Gamma(\beta)} + x_0 \frac{t^{\beta-2}}{\Gamma(\beta-1)}, & t \in [0, t_1], \\ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\,ds + x_1 \frac{t^{\beta-1}}{\Gamma(\beta)} + x_0 \frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ \quad + \sum_{i=1}^k \overline{y}_i \frac{t^{\beta-2}}{\Gamma(\beta-1)} (\frac{t}{\beta-1} - t_i) + \sum_{i=1}^k y_i \frac{t^{\beta-2}}{\Gamma(\beta-1)}, & t \in (t_k, t_{k+1}], \end{cases} \tag{2.2}$$

where $k = 1, 2, \ldots, m$.

Proof For $t \in (0, t_1]$, by Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned} I_0^{\beta} D_0^{\beta} x(t) &= x(t) - \sum_{j=1}^{2} \frac{(I_{0^+}^{2-\beta})^{(2-j)} x(0^+)}{\Gamma(\beta-j+1)} (t-0)^{\beta-j} \\ &= x(t) - \frac{(I_{0^+}^{2-\beta})^{(1)} x(0^+)}{\Gamma(\beta)} (t-0)^{\beta-1} + \frac{I_{0^+}^{2-\beta} x(0^+)}{\Gamma(\beta-1)} (t-0)^{\beta-2} \\ &= x(t) - \frac{I_{0^+}^{1-\beta} x(0^+)}{\Gamma(\beta)} t^{\beta-1} + \frac{I_{0^+}^{2-\beta} x(0^+)}{\Gamma(\beta-1)} t^{\beta-2} \\ &= x(t) - x_1 \frac{t^{\beta-1}}{\Gamma(\beta)} - x_0 \frac{t^{\beta-2}}{\Gamma(\beta-1)}. \end{aligned}$$

Similarly, for the interval $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} x(t) = {}& \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\,ds + x_1 \frac{t^{\beta-1}}{\Gamma(\beta)} + x_0 \frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ & + \sum_{i=1}^k \overline{y}_i \frac{t^{\beta-2}}{\Gamma(\beta-1)} \left(\frac{t}{\beta-1} - t_i \right) + \sum_{i=1}^k y_i \frac{t^{\beta-2}}{\Gamma(\beta-1)}. \end{aligned}$$

Hence, (2.2) is a solution to problem (2.1). □

Next, based on the theorem, we consider the solutions of Eq. (1.1).

Definition 2.3 Suppose that function $x : (-\propto, T] \to H$. The solution of the fractional differential equation, given by

$$x(t) = \begin{cases} x_0 = \phi \in B_v, \quad t \in (-\propto, 0], \\ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x_s)\,ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sigma(s, x_s)\,d\omega(s) + g(t, x_t) \\ \quad + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in [0, t_1], \\ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x_s)\,ds + g(t, x_t) + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ \quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sigma(s, x_s)\,d\omega(s) + \sum_{i=1}^k J_i(x(t_i^-)) \frac{t^{\beta-2}}{\Gamma(\beta-1)} (\frac{t}{\beta-1} - t_i) \\ \quad + \sum_{i=1}^k I_i(x(t_i^-)) \frac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in (t_k, t_{k+1}], k = 1, 2, \ldots, m, \end{cases}$$

will be called a fundamental solution of problem (1.1).

Lemma 2.3 (see [14]) *Assume $x \in B'_\nu$. Then, for $t \in J$, $x_t \in B_\nu$. Moreover,*

$$l\big(E\|x(t)\|^2\big)^{\frac{1}{2}} \le \|x_t\|_{B_\nu} \le \|\phi\|_{B_\nu} + l \sup_{s\in[0,t]} \big(E\|s^{2-\beta}x(s)\|^2\big)^{\frac{1}{2}},$$

where $l = \int_{-\infty}^{0} \nu(t)\,dt < +\infty$, $\phi = x_0$.

Next, we consider some definitions and properties of the measure of noncompactness.

The Hausdorff measure of noncompactness $\beta(\cdot)$ defined on each bounded subset Ω of the Banach space B is given by

$$\beta(\Omega) = \inf\{\varepsilon > 0; \Omega \text{ has a finite } \varepsilon \text{ net in } B\}.$$

Some basic properties of $\beta(\cdot)$ are given in the following lemma.

Lemma 2.4 (see [15, 16]) *If B is a real Banach space and $\Omega, \Lambda \subset B$ are bounded, then the following properties are satisfied:*

(1) *Monotone: if for all bounded subsets Ω, Λ of B, $\Omega \subseteq \Lambda$ implies $\beta(\Omega) \le \beta(\Lambda)$;*
(2) *Nonsingular: $\beta(\{x\} \cup \Omega) = \beta(\Omega)$ for every $x \in H$ and every nonempty subset $\Omega \subset H$;*
(3) *Regular: Ω is precompact if and only if $\beta(\Omega) = 0$;*
(4) *$\beta(\Omega + \Lambda) \le \beta(\Omega) + \beta(\Lambda)$, where $\Omega + \Lambda = \{x + y; x \in \Omega, y \in \Lambda\}$;*
(5) *$\beta(\Omega \cup \Lambda) \le \max\{\beta(\Omega), \beta(\Lambda)\}$;*
(6) *$\beta(\lambda\Omega) \le |\lambda|\beta(\Omega)$;*
(7) *If $W \subset C(J; B)$ is bounded and equicontinuous, then $t \to \beta(W(t))$ is continuous on J, and*

$$\beta(W) \le \max_{t\in J} \beta\big(W(t)\big),$$

$$\beta\left(\int_0^t W(s)\,ds\right) \le \int_0^t \beta\big(W(s)\big)\,ds, \quad \textit{for all } t \in J,$$

where

$$\int_0^t W(s)\,ds = \left\{\int_0^t u(s)\,ds : \textit{for all } u \in W, t \in J\right\};$$

(8) *If $\{u_n\}_1^\infty$ is a sequence of Bochner integrable functions from J into B with $\|u_n(t)\| \le \widehat{m}(t)$ for almost all $t \in J$ and every $n \ge 1$, where $\widehat{m}(t) \in L(J; R^+)$, then the function $\psi(t) = \beta(\{u_n\}_{n=1}^\infty)$ belongs to $L(J; R^+)$ and satisfies*

$$\beta\left(\left\{\int_0^t u_n(s)\,ds : n \ge 1\right\}\right) \le 2\int_0^t \psi(s)\,ds;$$

(9) *If W is bounded, then for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^\infty \subset W$ such that*

$$\beta(W) \le 2\beta\big(\{u_n\}_{n=1}^\infty\big) + \varepsilon.$$

The above lemmas about the Hausdorff measure of noncompactness will be used in proving our main results.

Lemma 2.5 (see [17]) *Let D be a closed convex subset of the Banach space B and $0 \in D$. Assume that $F : D \to B$ is a continuous map which satisfies Mönch's condition, that is, $M \subseteq D$ is countable, $M \subseteq \overline{\text{co}}(0 \cup F(M)) \Rightarrow \overline{M}$ is compact. Then F has a fixed point in D.*

Lemma 2.6 (see [18]) *If $W \subset C([0,T]; L_2^0(V,Y))$, ω is a standard Winer process, then*

$$\beta\left(\int_0^t W(s)\,d\omega(s)\right) \le \sqrt{T \cdot \mathrm{Tr}(Q)}\,\beta\big(W(s)\big),$$

where

$$\int_0^t W(s)\,d\omega(s) = \left\{\int_0^t u(s)\,d\omega(s) : \textit{for all } u \in W, t \in [0,T]\right\}.$$

Next, we consider the Hyers–Ulam stability for the equation.
Consider the following inequality:

$$E\left\| D_{0^+}^{\beta}\big[y(t) - g(t,y_t)\big] - f(t,y_t) - \sigma(t,y_t)\frac{d\omega(s)}{dt}\right\|^2 < \varepsilon. \tag{2.3}$$

Definition 2.4 (see [19]) Equation (1.1) is Hyers–Ulam stable if, for any $\varepsilon > 0$, there exists a solution $y(t)$ which satisfies the above inequality and has the same initial value as $x(t)$, where $x(t)$ is a solution to (1.1). Then $y(t)$ satisfies

$$E\left|t^{2-\beta}\left\|y(t) - x(t)\right\|\right|^2 < K\varepsilon,$$

in which K is a constant.

3 Main result

In this section, we list the following basic assumptions of this paper and prove our main results.

3.1 Existence

(H_1): The function $f : J \times B_\nu \to H$ satisfies the following conditions:

(i) $f(\cdot,\phi)$ is measurable for all $\phi \in B_\nu$ and $f(t,\cdot)$ is continuous for a.e. $t \in J$.

(ii) There exist a constant $\alpha_1 \in (0,\alpha)$, $m_1 \in L^{\frac{1}{\alpha_1}}(J,R^+)$, and a positive integrable function $\Omega : R^+ \to R^+$ such that

$$E\left\|f(t,\phi)\right\|^2 \le m_1(t)\Omega\left(\|\phi\|_{B_\nu}\right),$$

for all $(t,\phi) \in J \times B_\nu$, where Ω satisfies

$$\liminf_{n\to\propto} \frac{\Omega(n)}{n} = 0.$$

(iii) There exist a constant $\alpha_2 \in (0,\alpha)$ and a function $\eta \in L^{\frac{1}{\alpha_2}}(J,R^+)$ such that, for any bounded subset $F_1 \subset B_\nu$,

$$\beta\big(f(t,F_1)\big) \le \eta_1(t)\Big[\sup_{\theta\in(-\propto,0]} \beta\big(F_1(\theta)\big)\Big],$$

for a.e. $t \in J$, where $F_1(\theta) = \{\nu(\theta) : \nu \in F_1\}$ and β is the Hausdorff MNC.

(H_2): The function $g : J \times B_\nu \to H$ satisfies the following conditions:

(i) g is continuous, there exist a constant $H_1 > 0$ and

$$\left(E\left\|t^{2-\beta} g(t,x)\right\|^2\right) \le H_1\left(1 + \|x\|_{B_\nu}^2\right).$$

(ii) There exist a constant $\alpha_3 \in (0,\alpha)$ and $g^* \in L^{\frac{1}{\alpha_3}}(J,R^+)$ such that, for any bounded subset $F_2 \subset B_\nu$,

$$\beta\left(g(t,F_2)\right) \le g^*(t) \sup_{\theta\in(-\propto,0]} \beta\left(F_2(\theta)\right), \qquad G = \sup_{t\in J} g^*(t).$$

(H_3): $I_k, J_k : H \to H$, $k = 1,2,\ldots,m$, are continuous functions and they satisfy

$$\begin{aligned}
&\left\|I_k(x)\right\|_H \le c_k \|x\|_{B_\nu'}, \qquad \left\|J_k(x)\right\|_H \le f_k \|x\|_{B_\nu'}, \\
&\beta\left(t^{\beta-2} I_k(F_3)\right) \le K_k \sup_{\theta\in(-\propto,T]} \beta\left(F_3(\theta)\right), \\
&\beta\left(t^{\beta-2} J_k(F_4)\right) \le M_k \sup_{\theta\in(-\propto,T]} \beta\left(F_4(\theta)\right),
\end{aligned}$$

where $c_k, f_k, K_k, M_k > 0$. $F_3, F_4 \subset B_\nu'$.

(H_4): The function $\sigma(t,x_t)$ satisfies the following conditions:

(i) There exist a constant $\alpha_4 \in (0,\alpha)$, $m_2 \in L^{\frac{1}{\alpha_4}}(J,R^+)$, and a positive integrable function $\Psi : R^+ \to R^+$ such that

$$E\left\|\sigma(t,\phi)\right\|^2 \le m_2(t)\Psi\left(\|\phi\|_{B_\nu}\right)$$

for all $(t,\phi) \in J \times B_\nu$, where Ψ satisfies

$$\liminf_{n\to\propto} \frac{\Psi(n)}{n} = 0.$$

(ii) There exists a constant $\nu_1 > 0$ such that $E\|\sigma(t,x) - \sigma(t,y)\|^2 \le \nu_1 E\|x - y\|^2$.

(iii) There exist a constant $\alpha_5 \in (0,\alpha)$ and a function $\eta_2 \in L^{\frac{1}{\alpha_5}}(J,R^+)$ such that, for any bounded subset $F_5 \subset B_\nu$,

$$\beta\left(\sigma(t,F_5)\right) \le \eta_2(t)\left[\sup_{\theta\in(-\propto,0]} \beta\left(F_5(\theta)\right)\right]$$

for a.e. $t \in J$, where $F_5(\theta) = \{\nu(\theta) : \nu \in F_5\}$ and β is the Hausdorff MNC.

(H_5):

$$\begin{aligned}
&H_1 l^2 + \frac{(T^*)^2}{(\Gamma^*)^2} \sum_{i=1}^{m} \left(f_i^2 + c_i^2\right) < 1, \\
&M^* = \frac{2T^\beta}{\Gamma(\beta+1)} \|\eta\|_{L^{\frac{1}{\alpha_2}}(J,R^+)} + G + \frac{T^*}{\Gamma^*} \sum_{i=1}^{m} (M_i + K_i) \\
&\quad + \frac{2T^{\beta+\frac{1}{2}}\sqrt{\operatorname{Tr}(Q)}}{\Gamma(\beta+1)} \|\eta_2\|_{L^{\frac{1}{\alpha_4}}(J,R^+)} < 1,
\end{aligned}$$

where $T^* = \max\{1, T, T^2\}$, $\Gamma^* = \min\{\Gamma(\beta+1), \Gamma(\beta), \Gamma(\beta-1)\}$.

Theorem 3.1 *Suppose that conditions* (H_1)–(H_5) *are satisfied. Then system* (1.1) *has at least one solution on* J.

Proof We define the operator $\Gamma : B'_\nu \to B'_\nu$ by

$$\Gamma x(t) = \begin{cases} x_0 = \phi \in B_\nu, \quad t \in (-\propto, 0] \\ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x_s)\, ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sigma(s, x_s)\, d\omega(s) \\ \quad + g(t, x_t) + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in [0, t_1], \\ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x_s)\, ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sigma(s, x_s)\, d\omega(s) \\ \quad + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)} + \sum_{i=1}^k J_i(x(t_i^-)) \frac{t^{\beta-2}}{\Gamma(\beta-1)} (\frac{t}{\beta-1} - t_i) \\ \quad + g(t, x_t) + \sum_{i=1}^k I_i(x(t_i^-)) \frac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in (t_k, t_{k+1}], k = 1, 2, \dots, m. \end{cases}$$

The operator Γ has a fixed point if and only if system (1.1) has a solution. For $\phi \in B_\nu$, denote

$$\hat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\propto, 0], \\ 0, & t \in J. \end{cases}$$

Then $\hat{\phi}(t) \in B'_\nu$.

Let $x(t) = y(t) + \hat{\phi}(t)$, $-\propto < t \leq T$. It is easy to see that y satisfies $y_0 = 0$, $t \in (-\propto, 0]$ and

$$y(t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, y_s + \hat{\phi}_s)\, ds + \hat{\phi}_t) + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ \quad + g(t, y_t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sigma(x, x_s)\, d\omega(s), \quad t \in [0, t_1], \\ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, y_s + \hat{\phi}_s)\, ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sigma(s, x_s)\, d\omega(s) \\ \quad + \hat{\phi}_t) + \sum_{i=1}^k J_i(y(t_i^-) + \phi(\hat{t}_i^-)) \frac{t^{\beta-2}}{\Gamma(\beta-1)} (\frac{t}{\beta-1} - t_i) \\ \quad + g(t, y_t) + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ \quad + \sum_{i=1}^k I_i(y(t_i^-) + \phi(\hat{t}_i^-)) \frac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in (t_k, t_{k+1}], k = 1, 2, \dots, m, \end{cases}$$

if and only if $x(t)$ satisfies $x(t) = \phi(t)$, $t \in (-\propto, 0]$, and

$$x(t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x_s)\, ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sigma(s, x_s)\, d\omega(s) \\ \quad + g(t, x_t) + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in [0, t_1], \\ \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, x_s)\, ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \sigma(s, x_s)\, d\omega(s) \\ \quad + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)} + \sum_{i=1}^k J_i(x(t_i^-)) \frac{t^{\beta-2}}{\Gamma(\beta-1)} (\frac{t}{\beta-1} - t_i) \\ \quad + g(t, x_t) + \sum_{i=1}^k I_i(x(t_i^-)) \frac{t^{\beta-2}}{\Gamma(\beta-1)}, \quad t \in (t_k, t_{k+1}], k = 1, 2, \dots, m. \end{cases}$$

Define the space $(B''_\nu, \|\cdot\|_{B''_\nu})$ induced by B'_ν

$$B''_\nu = \{ y : y \in B'_\nu, y_0 = 0 \},$$

with the norm

$$\| y(t) \|_{B''_\nu} = \sup \{ (E | s^{2-\beta} \| y(s) \| |^2)^{\frac{1}{2}}, s \in [0, T] \}.$$

Let $B_r = \{y \in B''_\nu : \|y\|_{B'_\nu} \le r\}$. Then, for each r, B_r is a bounded, close, and convex subset. For any $y \in B_r$, it follows from Lemma 2.3 that

$$\begin{aligned}\|y_t + \hat{\phi}_t\|_{B_\nu} &\le \|y_t\|_{B_\nu} + \|\hat{\phi}_t\|_{B_\nu} \\ &\le l \sup_{s\in[0,t]} \left(E\left|s^{2-\beta}\left\|x(s)\right\|\right|^2\right)^{\frac{1}{2}} + \|\phi\|_{B_\nu} \\ &\le lr + \|\phi\|_{B_\nu} = r'.\end{aligned}$$

We define the operator $N : B''_\nu \to B''_\nu$ by

$$Ny(t) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, y_s + \hat{\phi}_s)\,ds + g(t, y_t + \hat{\phi}_t) + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ \quad + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\sigma(s, y_s + \hat{\phi}_s)\,d\omega(s), \quad t \in [0, t_1], \\ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f(s, y_s + \hat{\phi}_s)\,ds + g(t, y_t + \hat{\phi}_t) \\ \quad + \varphi_2 \frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1 \frac{t^{\beta-2}}{\Gamma(\beta-1)} + \sum_{i=1}^k I_i(y(t_i^-) + \phi(\hat{t_i^-}))\frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ \quad + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\sigma(s, y_s + \hat{\phi}_s)\,d\omega(s) \\ \quad + \sum_{i=1}^k J_i(y(t_i^-) + \phi(\hat{t_i^-}))\frac{t^{\beta-2}}{\Gamma(\beta-1)}(\frac{t}{\beta-1} - t_i), \quad t \in (t_k, t_{k+1}], \end{cases}$$

in which $k = 1, 2, \ldots, m$.

Step 1: We prove that there exists some $r > 0$ such that $N(B_r) \subset B_r$. If this is not true, then, for each positive integer r, there exist $y_r \in B_r$ and $t_r \in (-\propto, T]$ such that $\|(Ny_r)(t_r)\|^2_{B''_\nu} > r^2$. On the other hand, it follows from the assumption that

$$\begin{aligned}&E\left(\left|t_r^{2-\beta}\left\|N\left(y_r(t_r)\right)\right\|\right|^2\right) \\ &\le 7E\left\|\frac{1}{\Gamma(\beta)}\int_0^{t_r}(t_r-s)^{\beta-1}t_r^{2-\beta}f\left(s,(y_r)_s+\hat{\phi}_s\right)ds\right\|^2 \\ &\quad + 7E\left\|t_r^{2-\beta}g\left(t_r,(y_r)_{t_r}+\hat{\phi}_{t_r}\right)\right\|^2 \\ &\quad + 7E\left\|\varphi_2\frac{t_r^{2-\beta+\beta-1}}{\Gamma(\beta)}\right\|^2 + 7E\left\|\varphi_1\frac{t_r^{2-\beta+\beta-2}}{\Gamma(\beta-1)}\right\|^2 \\ &\quad + 7E\left\|\sum_{i=1}^k J_i\left(y_r\left(t_i^-\right)+\phi\left(\hat{t_i^-}\right)\right)\frac{t_r^{2-\beta+\beta-2}}{\Gamma(\beta-1)}\left(\frac{t_r}{\beta-1}-t_i\right)\right\|^2 \\ &\quad + 7E\left\|\sum_{i=1}^k I_i\left(y_r\left(t_i^-\right)+\phi\left(\hat{t_i^-}\right)\right)\frac{t_r^{2-\beta+\beta-2}}{\Gamma(\beta-1)}\right\|^2 \\ &\quad + 7E\left\|\frac{1}{\Gamma(\beta)}\int_0^{t_r}(t_r-s)^{\beta-1}t_r^{2-\beta}\sigma\left(s,(y_r)_s+\hat{\phi}_s\right)d\omega(s)\right\|^2 \\ &= 7\sum_{i=1}^7 I_i.\end{aligned}$$

From I_1 to I_7, it follows from (H_1)–(H_5) that

$$\begin{aligned}I_1 &\le \left\|\frac{T^{2-\beta}}{\Gamma(\beta)}\int_0^{t_r}(t_r-s)^{\beta-1}f\left(s,(y_r)_s+\hat{\phi}_s\right)ds\right\|^2 \\ &\le \frac{T^{4-2\beta}}{\Gamma^2(\beta)}\int_0^{t_r}(t_r-s)^{\beta-1}ds\int_0^{t_r}(t_r-s)^{\beta-1}E\left\|f\left(s,(y_r)_s+\hat{\phi}_s\right)\right\|^2ds\end{aligned}$$

$$
\begin{aligned}
&\le \frac{T^4}{\Gamma^2(\beta+1)} m_1(t)\Omega\bigl(r'\bigr), \\
I_2 &\le H_1\bigl(1 + \bigl\|(y_r)_{t_r} + \hat{\phi}_t\bigr\|_{B_v}\bigr) \\
&\le H_1 + H_1\bigl(r'\bigr)^2, \\
I_3 &\le \frac{T^2}{\Gamma^2(\beta)} \|\varphi_2\|^2 \\
&\le \frac{(T^*)^2}{(\Gamma^*)^2} \|\varphi_2\|^2, \\
I_4 &\le \frac{1}{\Gamma^2(\beta-1)} \|\varphi_1\|^2 \\
&\le \frac{(T^*)^2}{(\Gamma^*)^2} \|\varphi_1\|^2, \\
I_5 + I_6 &\le \frac{T^2}{\Gamma^2(\beta)} \Biggl\| \sum_{i=1}^{k} J_i\bigl(y_r\bigl(t_i^-\bigr) \\
&\qquad + \phi\bigl(\hat{t_i^-}\bigr)\bigr) \Biggr\| + \frac{1}{\Gamma^2(\beta-1)} \Biggl\| \sum_{i=1}^{k} I_i\bigl(y_r\bigl(t_i^-\bigr) + \phi\bigl(\hat{t_i^-}\bigr)\bigr) \Biggr\|^2 \\
&\quad \le \frac{(T^*)^2}{(\Gamma^*)^2} \sum_{i=1}^{m} \bigl(f_i^2 + c_i^2\bigr) r^2, \\
I_7 &\le \biggl\| \frac{T^{2-\beta}}{\Gamma(\beta)} \int_0^{t_r} (t_r - s)^{\beta-1} \sigma\bigl(s, (y_r)_s + \hat{\phi}_s\bigr)\, d\omega \biggr\|^2 \\
&\le \frac{T^{4-2\beta} \cdot \mathrm{Tr}(Q)}{\Gamma^2(\beta)} \int_0^{t_r} (t_r - s)^{2\beta-2} E\bigl\|\sigma\bigl(s, (y_r)_s + \hat{\phi}_s\bigr)\bigr\|^2\, ds \\
&\le \frac{T^4 \cdot \mathrm{Tr}(Q)}{\Gamma^2(\beta+1)} m_2(t)\Psi\bigl(r'\bigr).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
r &< \bigl\|Ny_r(t_r)\bigr\|_{B_v''} \\
&\le \frac{T^4}{\Gamma^2(\beta+1)} \bigl(m_1(t)\Omega\bigl(r'\bigr) + \mathrm{Tr}(Q) m_2(t)\Psi\bigl(r'\bigr)\bigr) + H_1 + H_1\bigl(r'\bigr)^2 \\
&\quad + \frac{(T^*)^2}{(\Gamma^*)^2} \|\varphi_2\|^2 + \frac{(T^*)^2}{(\Gamma^*)^2} \|\varphi_1\|^2 + \frac{(T^*)^2}{(\Gamma^*)^2} \sum_{i=1}^{m} \bigl(f_i^2 + c_i^2\bigr) r^2.
\end{aligned}
$$

Dividing both sides by r^2 and taking $r \to +\propto$ from

$$
\lim_{r\to\propto} \frac{r'}{r} = \lim_{r\to\propto} \frac{lr + \|\phi\|_{B_v}}{r} = l, \qquad \liminf_{n\to\propto} \frac{\Omega(n)}{n} = 0, \quad \text{and} \quad \liminf_{n\to\propto} \frac{\Psi(n)}{n} = 0
$$

yield

$$
H_1 l^2 + \frac{(T^*)^2}{(\Gamma^*)^2} \sum_{i=1}^{m} \bigl(f_i^2 + c_i^2\bigr) < 1.
$$

This contradicts (H_5). Thus, for some number r, $N(B_r) \subset B_r$.

Step 2: N is continuous on B_r.

Let $\{y^n\}_{n=1}^{+\propto} \subset B_r$ with $y^n \to y$ in B_r as $n \to +\propto$. Then, by using hypotheses (H_1), (H_2), and (H_3), we have

$$\begin{aligned}
&\text{(i)}\quad f\big(s, y_s^n + \hat{\phi}_s\big) \to f(s, y_s + \hat{\phi}_s), \quad n \to \propto. \\
&\text{(ii)}\quad g\big(t, y_t^n + \hat{\phi}_t\big) \to g(t, y_t + \hat{\phi}_t), \quad n \to \propto. \\
&\text{(iii)}\quad \big\| I_i\big(y^n(t_i^-) + \hat{\phi}(t_i^-)\big) - I_i\big(y(t_i^-) + \hat{\phi}(t_i^-)\big)\big\| \to 0, \\
&\qquad \big\| J_i\big(y^n(t_i^-) + \hat{\phi}(t_i^-)\big) - J_i\big(y(t_i^-) + \hat{\phi}(t_i^-)\big)\big\| \to 0, \\
&\qquad n \to \propto, i = 1, 2, \ldots, m.
\end{aligned}$$

Now, for every $t \in [0, t_1]$, we have

$$\begin{aligned}
&E\big(\big|t^{2-\beta}\big\|N\big(y^n(t)\big)\big\| - t^{2-\beta}\big\|N\big(y(t)\big)\big\|\big|^2\big) \\
&\quad \le 3E\left\| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} t^{2-\beta} \big[f\big(s, y_s^n + \hat{\phi}_s\big) - f(s, y_s + \hat{\phi}_s)\big]\, ds \right\|^2 \\
&\qquad + 3E\big\| t^{2-\beta}\big[g\big(t, y_t^n + \hat{\phi}_t\big) - g(t, y_t + \hat{\phi}_t)\big]\big\|^2 \\
&\qquad + 3E\left\| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} t^{2-\beta} \big[\sigma\big(s, y_s^n + \hat{\phi}_s\big) - \sigma(s, y_s + \hat{\phi}_s)\big]\, d\omega \right\|^2 \\
&\quad \le 3\frac{T^{4-2\beta}}{\Gamma^2(\beta)} \int_0^t (t-s)^{\beta-1}\, ds \int_0^t (t-s)^{\beta-1} E\big\| f\big(s, y_s^n + \hat{\phi}_s\big) - f(s, y_s + \hat{\phi}_s)\big\|^2\, ds \\
&\qquad + 3T^{2-\beta} E\big\| g\big(t, y_t^n + \hat{\phi}_t\big) - g(t, y_t + \hat{\phi}_t)\big\|^2 \\
&\qquad + 3\frac{T^{4-2\beta} \cdot \operatorname{Tr}(Q)}{\Gamma^2(\beta)} \int_0^t (t-s)^{2\beta-2} \nu_1 E\big\| y_s^n + \hat{\phi}_s - y_s - \hat{\phi}_s\big\|^2\, ds \\
&\quad \to 0 \quad (n \to \propto).
\end{aligned}$$

Moreover, for all $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we have

$$\begin{aligned}
&E\big(\big|t^{2-\beta}\big\|N\big(y^n(t)\big)\big\| - t^{2-\beta}\big\|N\big(y(t)\big)\big\|\big|^2\big) \\
&\quad \le 5\frac{T^{4-2\beta}}{\Gamma^2(\beta)} \int_0^t (t-s)^{\beta-1}\, ds \int_0^t (t-s)^{\beta-1} E\big\| f\big(s, y_s^n + \hat{\phi}_s\big) - f(s, y_s + \hat{\phi}_s)\big\|^2\, ds \\
&\qquad + 5E\big\| t^{2-\beta}\big[g\big(t, y_t^n + \hat{\phi}_t\big) - g(t, y_t + \hat{\phi}_t)\big]\big\|^2 \\
&\qquad + 5E\left\| \sum_{i=1}^k \big[J_i\big(y^n(t_i^-) + \phi\big(\hat{t_i^-}\big)\big) - J_i\big(y(t_i^-) + \phi\big(\hat{t_i^-}\big)\big)\big] \frac{1}{\Gamma(\beta-1)} \left(\frac{t}{\beta-1} - t_i\right) \right\|^2 \\
&\qquad + 5E\left\| \sum_{i=1}^k \big[I_i\big(y^n(t_i^-) + \phi\big(\hat{t_i^-}\big)\big) - I_i\big(y(t_i^-) + \phi\big(\hat{t_i^-}\big)\big)\big] \frac{1}{\Gamma(\beta-1)} \right\|^2 \\
&\qquad + 5\frac{T^{4-2\beta} \cdot \operatorname{Tr}(Q)}{\Gamma^2(\beta)} \int_0^t (t-s)^{2\beta-2} \nu_1 E\big\| y_s^n + \hat{\phi}_s - y_s - \hat{\phi}_s)\big\|^2\, ds \\
&\quad \to 0 \quad (n \to \propto).
\end{aligned}$$

Thus, we obtain

$$\|Ny^n - Ny\|_{B_\nu''} \to 0 \quad \text{as } n \to \propto,$$

implying that N is continuous on B_r.

Step 3: The map $N(B_r)$ is equicontinuous on J.

The functions $\{Ny : y \in B_r\}$ are equicontinuous at $t = 0$. For $t_1, t_2 \in J_k$, $t_1 < t_2$, $k = 0, 1, 2, \dots, m$, and $y \in B_r$, we have

$$\begin{aligned}
E\big|t^{2-\beta}\|Ny(t_1) - Ny(t_2)\|\big|^2 &\le C_1^2(t_1)E\big|t_2^{2-\beta}\|Ny(t_1) - Ny(t_2)\|\big|^2 \\
&\le C_1^2(t_1)E\big\|t_1^{2-\beta}Ny(t_1) - t_2^{2-\beta}Ny(t_2)\big\|^2 \\
&\quad + C_1^2(t_1)E\big\|t_2^{2-\beta}Ny(t_2) - t_1^{2-\beta}Ny(t_2)\big\|^2 \\
&\le C_1^2(t_1)E\big\|t_1^{2-\beta}Ny(t_1) - t_2^{2-\beta}Ny(t_2)\big\|^2 \\
&\quad + C_1^2(t_1)E\big\|Ny(t_2)\big\|^2\big\|t_2^{2-\beta} - t_1^{2-\beta}\big\|^2,
\end{aligned}$$

where $C_1(t_1) > 0$. The right-hand side of the equation is independent of $y \in B_r$ and tends to zero as $t_1 \to t_2$ since $t^{2-\beta}Ny(t) \in C(J_{k,X})$ and $\|t_2^{2-\beta} - t_1^{2-\beta}\| \to 0$ as $t_1 \to t_2$. Therefore, $\|Ny(t_1) - Ny(t_2)\|_{B_\nu''} \to 0$ as $t_1 \to t_2$. Hence, $N(B_r)$ is equicontinuous on J.

Step 4: Mönch's condition holds.

Let $N = N_1 + N_2 + N_3 + N_4$, where

$$\begin{aligned}
N_1y(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s, y_s + \hat{\phi}_s)\,ds, \\
N_2y(t) &= g(t, y_t + \hat{\phi}_t) + \varphi_2\frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1\frac{t^{\beta-2}}{\Gamma(\beta-1)}, \\
N_3y(t) &= \sum_{i=1}^{k} J_i\big(y(t_i^-) + \phi(\hat{t_i^-})\big)\frac{t^{\beta-2}}{\Gamma(\beta-1)}\left(\frac{t}{\beta-1} - t_i\right) \\
&\quad + \sum_{i=1}^{k} I_i\big(y(t_i^-) + \phi(\hat{t_i^-})\big)\frac{t^{\beta-2}}{\Gamma(\beta-1)}, \\
N_4y(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\sigma(s, y_s + \hat{\phi}_s)\,d\omega.
\end{aligned}$$

Assume that $W \subseteq B_r$ is countable and $W \subseteq \overline{\text{co}}(\{0\} \cup N(W))$. We show that $\beta(W) = 0$, where β is the Hausdorff MNC. Without loss of generality, we may suppose that $W = \{y^n\}_{n=1}^{\propto}$. Since $N(W)$ is equicontinuous on J_k, $W \subseteq \overline{\text{co}}(\{0\} \cup N(W))$ is equicontinuous on J_k as well.

Using Lemmas 2.4 and 2.6, (H_1)(iii), (H_2)(ii), (H_3), and (H_4)(iii), we have

$$\begin{aligned}
\beta\big(\{N_1y^n(t)\}_{n=1}^{\propto}\big) &\le \frac{2}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\eta_1(s)\Big[\sup_{-\propto<\theta\le 0}\beta\big(\{y_s^n(\theta)\}_{n=1}^{\propto}\big)\Big]\,ds \\
&\le \frac{2T^\beta}{\Gamma(\beta+1)}\|\eta_1\|_{L^{\frac{1}{\alpha_2}}(J,R^+)}\sup_{-\propto<\theta\le 0}\beta\big(\{y_s^n(\theta)\}_{n=1}^{\propto}\big),
\end{aligned}$$

$$\begin{aligned}
\beta\left(\left\{N_2y^n(t)\right\}_{n=1}^{\propto}\right) &\le \beta\left(g\left(t,y_t^n+\hat{\phi}_t\right)\right)\\
&\le G\sup_{-\propto<\theta\le 0}\beta\left(\left\{y_t^n(\theta)\right\}_{n=1}^{\propto}\right),\\
\beta\left(\left\{N_3y^n(t)\right\}_{n=1}^{\propto}\right) &\le \frac{T}{\Gamma(\beta)}\beta\left(\left\{\sum_{i=1}^{k}t^{\beta-2}J_i\left(y^n\left(t_i^-\right)+\phi\left(\hat{t_i^-}\right)\right)\right\}_{n=1}^{\propto}\right)\\
&\quad+\frac{1}{\Gamma(\beta-1)}\beta\left(\left\{\sum_{i=1}^{k}t^{\beta-2}I_i\left(y^n\left(t_i^-\right)+\phi\left(\hat{t_i^-}\right)\right)\right\}_{n=1}^{\propto}\right)\\
&\le \frac{T^*}{\Gamma^*}\left(\beta\left(\left\{\sum_{i=1}^{k}t^{\beta-2}J_i\left(y^n\left(t_i^-\right)+\phi\left(\hat{t_i^-}\right)\right)\right\}_{n=1}^{\propto}\right)\right.\\
&\quad\left.+\beta\left(\left\{\sum_{i=1}^{k}t^{\beta-2}I_i\left(y^n\left(t_i^-\right)+\phi\left(\hat{t_i^-}\right)\right)\right\}_{n=1}^{\propto}\right)\right)\\
&\le \frac{T^*}{\Gamma^*}\sum_{i=1}^{m}(M_i+K_i)\sup_{-\propto<\theta\le 0}\beta\left(\left\{y^n(\theta)\right\}_{n=1}^{\propto}\right),\\
\beta\left(\left\{N_4y^n(t)\right\}_{n=1}^{\propto}\right) &\le \frac{2\sqrt{T\cdot\operatorname{Tr}(Q)}}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\eta_2(s)\left[\sup_{-\propto<\theta\le 0}\beta\left(\left\{y_s^n(\theta)\right\}_{n=1}^{\propto}\right)\right]ds\\
&\le \frac{2T^{\beta+\frac{1}{2}}\sqrt{\operatorname{Tr}(Q)}}{\Gamma(\beta+1)}\|\eta_2\|_{L^{\frac{1}{\alpha_4}}(J,R^+)}\sup_{-\propto<\theta\le 0}\beta\left(\left\{y_s^n(\theta)\right\}_{n=1}^{\propto}\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\beta\left(\left\{Ny^n(t)\right\}_{n=1}^{\propto}\right)\\
&\quad\le \beta\left(\left\{N_1y^n(t)\right\}_{n=1}^{\propto}\right)+\beta\left(\left\{N_2y^n(t)\right\}_{n=1}^{\propto}\right)\\
&\qquad+\beta\left(\left\{N_3y^n(t)\right\}_{n=1}^{\propto}\right)+\beta\left(\left\{N_4y^n(t)\right\}_{n=1}^{\propto}\right)\\
&\quad\le \frac{2T^{\beta}}{\Gamma(\beta+1)}\|\eta_1\|_{L^{\frac{1}{\alpha_2}}(J,R^+)}\sup_{-\propto<\theta\le 0}\beta\left(\left\{y_s^n(\theta)\right\}_{n=1}^{\propto}\right)\\
&\qquad+G\sup_{-\propto<\theta\le 0}\beta\left(\left\{y_t^n(\theta)\right\}_{n=1}^{\propto}\right)\\
&\qquad+\frac{T^*}{\Gamma^*}\sum_{i=1}^{m}(M_i+K_i)\sup_{-\propto<\theta\le 0}\beta\left(\left\{y^n(\theta)\right\}_{n=1}^{\propto}\right)\\
&\qquad+\frac{2T^{\beta+\frac{1}{2}}\sqrt{\operatorname{Tr}(Q)}}{\Gamma(\beta+1)}\|\eta_2\|_{L^{\frac{1}{\alpha_4}}(J,R^+)}\sup_{-\propto<\theta\le 0}\beta\left(\left\{y_s^n(\theta)\right\}_{n=1}^{\propto}\right)\\
&\quad\le \left(\frac{2T^{\beta}}{\Gamma(\beta+1)}\|\eta_1\|_{L^{\frac{1}{\alpha_2}}(J,R^+)}+G+\frac{T^*}{\Gamma^*}\sum_{i=1}^{m}(M_i\right.\\
&\qquad\left.+K_i)+\frac{2T^{\beta+\frac{1}{2}}\sqrt{\operatorname{Tr}(Q)}}{\Gamma(\beta+1)}\|\eta_2\|_{L^{\frac{1}{\alpha_4}}(J,R^+)}\right)\beta\left(\left\{y^n(t)\right\}_{n=1}^{\propto}\right)\\
&\quad= M^*\beta\left(\left\{y^n(t)\right\}_{n=1}^{\propto}\right),
\end{aligned}$$

where M^* is defined in assumption (H_5). Since W and $N(W)$ are equicontinuous on every J_k, it follows from Lemma 2.4 that the inequality implies $\beta(NW)\le M^*\beta(W)$.

Thus, from Mönch's condition, we have

$$\beta(W) \le \beta\bigl(\overline{\mathrm{co}}\{0\} \cup N(W)\bigr) = \beta(NM) \le M^*\beta(W).$$

Since $M^* < 1$, we get $\beta(W) = 0$. It follows that W is relatively compact. Using Lemma 2.5, we know that N has a fixed point y in W. The proof is completed. □

3.2 Hyers–Ulam stability

We prove the Ulam stability of the solution using the following hypothesis.

(H_6): The function $g(t,x)$ satisfies the condition $E\|g(t,x) - g(t,y)\|^2 \le L\|x - y\|^2$, where L is a constant and $0 < (1 - 5Ll)\Gamma^2(\beta + 1) - 5\nu_2 T^{2\beta} \cdot \mathrm{Tr}(Q)$.

Theorem 3.2 *Suppose that conditions* (H_1), (H_3)–(H_6) *are satisfied. Then system* (1.1) *has at least one solution on* J, *and this solution is Hyers–Ulam stable.*

Proof It is easy to see that the solution satisfies condition (H_2) when the solution satisfies condition (H_6). Using Theorem 3.1, we can prove the existence of this solution. Now we consider the Ulam stability of this solution.

Consider the inequality

$$E\left\| D_{0^+}^{\beta}\bigl[y(t) - g(t,y_t)\bigr] - f(t,y_t) - \sigma(t,y_t)\frac{d\omega(s)}{dt}\right\|^2 < \varepsilon.$$

Suppose that there exists a function $f_1(t,y_t)$ such that $\|f(t,x_t) - f_1(t,y_t)\| < \varepsilon$.

Consider the following equation:

$$\begin{cases} D_{0^+}^{\beta}[y(t) - g(t,y_t)] = f_1(t,y_t) + \sigma(t,y_t)\frac{d\omega(s)}{dt}, & t \in [0,T], t \neq t_k, \\ \Delta I_{0^+}^{2-\beta} y(t_k) = I_k(y(t_k^-)), \qquad \Delta I_{0^+}^{1-\beta} y(t_k) = J_k(y(t_k^-)), \\ I_{0^+}^{2-\beta}[y(0) - g(0,y_0)] = \varphi_1 \in B_\nu, \qquad I_{0^+}^{1-\beta}[y(0) - g(0,y_0)] = \varphi_2 \in B_\nu, \end{cases} \tag{3.1}$$

in which $k = 1, 2, \ldots, m$. Using the fundamental solution to Eq. (3.1), we get

$$y(t) = \begin{cases} y_0 = \phi \in B_\nu, \quad t \in (-\propto, 0] \\ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f_1(s,y_s)\,ds + g(t,y_t) + \varphi_2\frac{t^{\beta-1}}{\Gamma(\beta)} \\ \quad + \varphi_1\frac{t^{\beta-2}}{\Gamma(\beta-1)} + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\sigma(s,y_s)\,d\omega(s), \quad t \in [0,t_1], \\ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} f_1(s,y_s)\,ds + g(t,y_t) + \varphi_2\frac{t^{\beta-1}}{\Gamma(\beta)} + \varphi_1\frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ \quad + \sum_{i=1}^k J_i(y(t_i^-))\frac{t^{\beta-2}}{\Gamma(\beta-1)}(\frac{t}{\beta-1} - t_i) + \sum_{i=1}^k I_i(y(t_i^-))\frac{t^{\beta-2}}{\Gamma(\beta-1)} \\ \quad + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\sigma(s,y_s)\,d\omega(s), \quad t \in (t_k, t_{k+1}], k = 1, 2, \ldots, m. \end{cases}$$

It is obvious that the solution is Ulam stable in the interval $(-\propto, 0]$. Now, we consider the interval $t \in (0, t_1]$ and suppose $\varepsilon < 1$. We have

$$\begin{aligned} &E\bigl|t^{2-\beta}\|x(t) - y(t)\|\bigr|^2 \\ &\quad\le 3E\left\|\frac{T^{2-\beta}}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\bigl(f(s,x_s) - f_1(s,y_s)\bigr)\,ds\right\|^2 \end{aligned}$$

$$\begin{aligned} &+ 3E\left|t^{2-\beta}\left\|g(t,x_t) - g(t,y_t)\right\|\right|^2 \\ &+ 3E\left\|\frac{t^{2-\beta}\operatorname{Tr}(Q)}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\big(\sigma(s,x_s) - \sigma(s,y_s)\big)\,d\omega(s)\right\|^2 \\ \leq{}& \frac{3T^4\varepsilon}{\Gamma^2(\beta+1)} + 3\left(L + \frac{\nu_2 T^{2\beta}\operatorname{Tr}(Q)}{\Gamma^2(\beta+1)}\right) lE\left|t^{2-\beta}\left\|x(t) - y(t)\right\|\right|^2. \end{aligned}$$

Thus,

$$E\left|t^{2-\beta}\left\|x(t) - y(t)\right\|\right|^2 \leq \frac{3T^4}{(1-3Ll)\Gamma^2(\beta+1) - 3\nu_2 T^{2\beta} l \cdot \operatorname{Tr}(Q)}\varepsilon.$$

Here, $K_0 = \frac{T^4}{(1-3Ll)\Gamma^2(\beta+1)-3\nu_2 T^{2\beta} l\cdot\operatorname{Tr}(Q)}$.

Secondly, we consider the interval $t \in (t_1, t_2]$. We have

$$\begin{aligned} &E\left|t^{2-\beta}\left\|x(t) - y(t)\right\|\right|^2 \\ &\leq \frac{5T^4\varepsilon}{\Gamma^2(\beta+1)} + 5\left(L + \frac{\nu_2 T^{2\beta}\cdot\operatorname{Tr}(Q)}{\Gamma^2(\beta+1)}\right) lE\left|t^{2-\beta}\left\|x(t) - y(t)\right\|\right|^2 \\ &\quad + 5E\left\|\sum_{i=1}^{k}\big(J_i\big(x(t_i^-)\big) - J_i\big(y(t_i^-)\big)\big)\frac{1}{\Gamma(\beta-1)}\left(\frac{t}{\beta-1} - t_i\right)\right\|^2 \\ &\quad + 5E\left\|\sum_{i=1}^{k}\big(I_i\big(x(t_i^-)\big) - I_i\big(y(t_i^-)\big)\big)\frac{1}{\Gamma(\beta-1)}\right\|^2. \end{aligned}$$

The conclusion $|y(t) - x(t)| < K_0\varepsilon$ for $t \in (0, t_1]$ implies that

$$\begin{aligned} &\left|I_i\big(x(t_i^-)\big) - I_i\big(y(t_i^-)\big)\right| < R_1\varepsilon, \\ &\left|J_i\big(x(t_i^-)\big) - J_i\big(y(t_i^-)\big)\right| < R_2\varepsilon, \end{aligned}$$

since I_k, J_k are continuous functions.

Therefore,

$$\begin{aligned} &E\left|t^{2-\beta}\left\|x(t) - y(t)\right\|\right|^2 \\ &\leq \frac{5T^4\varepsilon}{\Gamma^2(\beta+1)} + 5\left(L + \frac{\nu_2 T^{2\beta}\cdot\operatorname{Tr}(Q)}{\Gamma^2(\beta+1)}\right) lE\left|t^{2-\beta}\left\|x(t) - y(t)\right\|\right|^2 \\ &\quad + \sum_{i=1}^{k}\frac{5R_1^2\varepsilon}{\Gamma^2(\beta-1)}\left(\frac{T}{\beta-1} - t_1\right)^2 + \frac{5R_2^2 k^2\varepsilon}{\Gamma^2(\beta-1)}. \end{aligned}$$

Thus,

$$\begin{aligned} &E\left|t^{2-\beta}\left\|x(t) - y(t)\right\|\right|^2 \\ &\leq \frac{5\varepsilon}{(1-5Ll)\Gamma^2(\beta+1) - 5\nu_2 T^{2\beta} l\cdot\operatorname{Tr}(Q)}\left(T^4 + \sum_{i=1}^{k} R_1^2\left(\frac{T}{\beta-1} - t_1\right)^2 + R_2^2 k^2\right). \end{aligned}$$

Hence, in the interval $t \in (t_1, t_2]$, we have

$$K_1 = \frac{5}{(1-5Ll)\Gamma^2(\beta+1) - 5\nu_2 T^{2\beta} l \cdot \operatorname{Tr}(Q)} \left(T^4 + \sum_{i=1}^{k} R_1^2 \left(\frac{T}{\beta-1} - t_1 \right)^2 + R_2^2 k^2 \right).$$

In this way, we can prove for $t \in (t_i, t_{i+1}]$, $i = 1, \dots, m$.

Thus, there exists $K = \max\{K_0, K_1, \dots, K_m\}$ that satisfies Definition 2.4. □

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Authors' contributions

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