Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Existence and uniqueness for a kind of nonlocal fractional evolution equations on the unbounded interval

Zhanmei Lv[1], Yanping Gong[1*] and Yi Chen[2]

*Correspondence:
gongyp2012@sina.com
[1]School of Business, Central South University, Changsha, P.R. China
Full list of author information is available at the end of the article

Abstract

By means of a Laplace transform and its inverse transform, we obtain a correct equivalent integral equation for some kind of nonlocal abstract differential equations (fractional order) on the right half-axis. Based on it, the existence result is established by Knaster's theorem, and the uniqueness of the mild solution is obtained using the Banach contraction principle.

MSC: Primary 26A33; 34A08; secondary 34G20

Keywords: Fractional evolution equations; Mild solutions; Nonlocal conditions

1 Introduction

Today, fractional order calculus has a great many of uses in engineering, science, economy, biology, physics and other scientific disciplines (see [1–3]). Numerous phenomena and processes in the real world are described by differential equations of fractional order. Given their wide applications, there are numerous scholars focusing on the study of the differential equations of fractional order recently. To acquire more general knowledge of fractional order calculus and equations with fractional derivatives, the reader may refer to [4–28].

Now, nonlocal evolution equations of fractional order have become one of the hot research topics in the field of differential equations with fractional derivatives, which play a role in modeling physics phenomena. For the first time, the existence of mild solutions for the nonlocal problems was discussed in [29]. In [30], Byszewski and Lakshmikantham indicated that as regards describing physical phenomena nonlocal conditions could be more applicable than standard conditions.

In [19], El-Borai obtained the existence results for fractional abstract equations of the following kind:

$$\begin{cases} \frac{d^\beta z}{d^\beta s} = B(s)z(t) + Az(s), & s \in [0,a], \\ z(0) = z_0, \end{cases}$$

where $0 < \beta \leq 1$, $a > 0$.

In this paper, the equivalent integral equation in regard to the abstract equations is firstly described by means of some probability densities using Laplace transform and its inverse transform. Since then, many researchers have drawn on El-Borai's results to investigate fractional evolution equations, such as in [20–28].

To the best of our knowledge, fractional evolution equations have attracted more and more attention recently [18–28]. For instance, in [20], Zhou investigated a class of nonlocal evolution equations of fractional order

$$\begin{cases} {}^{C}D_{0+}^{q} z(s) = f(s, z(s)) + Az(s), & s \in (0, a], \\ z(0) + g(z) = z_0, \end{cases}$$

Zhou and Jiao obtained many good conclusions about the existence and uniqueness of mild solution for this equation.

Yet despite all that, a fair number of those papers investigating the mild solutions of evolution equations with fractional derivatives are concerned with bounded intervals, and conclusions on the right half-axis are still rare.

Motivated by [19, 20, 31], we discuss a class of nonlocal functional differential equations of fractional order in a Banach space E

$$\begin{cases} {}^{C}D_{0+}^{q} u(t) = f(t, u(m(t))) + Au(t), & t \in (0, +\infty), \\ k(u) + u(0) = u_0, \end{cases} \tag{1.1}$$

where $0 < q < 1$, ${}^{C}D_{0+}^{q}$ denotes the fractional derivative in the Caputo sense, $u_0 \in E$, $\{T(t)\}_{t\geq 0}$ is a C_0 semigroup on Banach E and A is the semigroup's infinitesimal generator, $m \in C[0, \infty)$, $m(t) \geq 0$ is an increasing function, $f : [0, +\infty) \times E \to E$, and $k : E \to E$ satisfy certain conditions.

Here, entirely different from those already obtained in the previous literature, we obtain a correct equivalent integral equation for the main equation. Applying Knaster's theorem, the existence of positive mild solutions of the main problem (1.1) is given. Then, employing the Banach contraction theorem, the uniqueness of the mild solution is given.

The layout of the rest of the article is listed now. In Sect. 2, fractional derivative, fractional integral and several useful preliminaries are introduced. In Sect. 3, our main conclusions are presented by using Knaster's theorem and Banach contraction theorem, respectively. In Sect. 4, one example is provided to show the application of our main conclusion.

2 Preliminaries

In the following, we assume that E is ordered Banach space. Let $P \subset E$ be a cone, which defines a partial order by $y \leq z$ if and only if $z - y \in P$ on E. We refer the reader to [32, 33] for more details as regards the cone.

Throughout the paper, we set a normal positive cone $P = \{z \in E \mid z \geq \theta\}$ with N as its normal constant. Let

$$BC(J, E) = \{ v(s) \mid v(s) \text{ is bounded and continuous on } J \}, \quad J = [0, +\infty).$$

Evidently, it is a Banach space equipped with the norm $\|v\|_b = \sup_{t\in J} \|v(t)\|$. Set

$$P_{BC} = \{ v \mid v(t) \geq \theta, t \in J, v \in BC(J, E) \}.$$

Obviously, P_{BC} is a normal cone in $BC(J,E)$ with the normal constant N, the same normal to the cone P. Meanwhile, an ordered Banach space $BC(J,E)$ is induced by P_{BC} (for convenience, we also denote the partial order by "$\leq$" both on $BC(J,E)$ and on E without confusion).

Let $\mathcal{S} \subset E$ be a subset such that

$$\overline{\mathcal{S}} = \{z \in E : y \leq z, \forall y \in \mathcal{S}\}.$$

The point $z^* \in E$ is called the infimum of $\overline{\mathcal{S}}$ if $z^* \in \overline{\mathcal{S}}$ and for each $z \in \overline{\mathcal{S}}$, $z^* \leq z$.

Similarly, we can define the supremum of $\overline{\mathcal{S}}$ by reversing the inequality in the above definition.

Next, we present the theorem established by Knaster [34].

Theorem 2.1 ([34]) *Let E be a Banach space ordered by partially ordering "$\preceq$". Assume that $\mathcal{S} \subset E$ is a subset of E, which has properties as follows*: (1) *Each nonempty subset of $\mathcal{S}$ has a supremum belonging to $\mathcal{S}$*; (2) *The infimum of $\mathcal{S}$ is in $\mathcal{S}$. Assume that the map $T : \mathcal{S} \to \mathcal{S}$ is increasing, i.e., $y \preceq z$ implies that $Ty \preceq Tz$ $(y, z \in \mathcal{S})$. Then T has a fixed point in $\mathcal{S}$.*

Next, we list two definitions of fractional integral and fractional derivative and one lemma, which will be used.

Definition 2.1 ([1–3]) The fractional integral of order $\alpha > 0$ in the sense of Riemann–Liouville for a function $g : (0,\infty) \to \mathbb{R}$ is defined by

$$I_{0+}^{\alpha} g(t) = D_{0+}^{-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds, \tag{2.1}$$

provided that the integral above is pointwise defined on $(0,\infty)$.

Definition 2.2 ([1–3]) The fractional derivative of order $\alpha > 0$ in the sense of Caputo for a function $g : (0,\infty) \to \mathbb{R}$ is defined by

$${}^{C}D_{0+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)\, ds, \tag{2.2}$$

where $n = [\alpha] + 1$, provided that the integral above is pointwise defined on $(0,\infty)$.

Lemma 2.1 ([1, 3]) *Let ${}^{C}D_{0+}^{\alpha} g(t) \in L^1(0,+\infty)$, $\alpha > 0$. Then one has*

$$I_{0+}^{\alpha}\, {}^{C}D_{0+}^{\alpha} g(t) = g(t) + C_1 + C_2 t + \cdots + C_M t^{M-1}, \quad t > 0, \tag{2.3}$$

for some constants C_i, $i = 1, 2, 3, \ldots, M$, where $M = \min\{m \mid m \geq \alpha, m$ is a integer$\}$.

Remark 2.1 If the function g appearing in the above lemma and two definitions takes values in the Banach space E, then the integrals here are all to be taken in the sense of Bochner. In addition, if the abstract function g is measurable and its norm is integrable in the sense of Lebesgue, it is Bochner integrable.

Next, we set out to present a series of concepts and results about the semigroups of linear operators. The reader can find more details in [35, 36].

Given a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ (i.e. C_0-semigroup), we can define the infinitesimal generator of $\{T(t)\}_{t\geq 0}$ as

$$Az = \lim_{s\to 0^+} \frac{T(s)z - z}{s}, \quad z \in E.$$

The domain of A is given by

$$D(A) = \left\{ z : \lim_{s\to 0^+} \frac{T(s)z - z}{s} \text{ exists}, z \in E \right\}.$$

Lemma 2.2 ([35, 36]) *Assume that* $\{T(t)\}_{t\geq 0}$ *is a strongly continuous semigroup, then* $\|T(t)\| \leq Ce^{\omega t}$, *where* ω *and* $C \geq 1$ *are two constants.*

Lemma 2.3 ([35, 36]) *Given a strongly continuous semigroup of contractions* $\{T(t)\}_{t\geq 0}$ *and a linear operator* A, *we have the following results*: A *is the infinitesimal generator of* $\{T(t)\}_{t\geq 0}$ *if and only if*

(1) $\overline{D(A)} = E$ *and* A *is closed.*

(2) $(0, +\infty) \subset \rho(A)$, *and* $\forall \mu > 0$, *one has*

$$\left\| R(\mu, A) \right\| \leq \frac{1}{\mu},$$

where $\rho(A)$ *is the resolvent set of* A, *and*

$$R(\mu, A)z := (\mu I - A)^{-1} z = \int_0^{+\infty} e^{-\mu t} T(t)z \, dt, \quad z \in E.$$

Definition 2.3 ([37]) For a given strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, it is positive on E, if $\theta \leq T(t)z$, $t \geq 0$, $z \in E$.

Definition 2.4 ([35, 36]) For a given strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, it is uniformly exponentially stable under the condition that $\omega_0 < 0$, where ω_0 denotes the growth bound of the semigroup, and it is given by

$$\omega_0 = \inf\left\{ \omega \mid \left\| T(t) \right\| \leq Ce^{\omega t}, t \geq 0, C \geq 1 \right\}.$$

By Definition 2.4 and Lemma 2.2, for a uniformly exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$, $\|T(t)\| \leq Ce^{\omega t}$, $t \geq 0, C \geq 1$ and $\omega \in (0, |\omega_0|]$ (ω_0 is the growth bound of the above semigroup). Next, define a norm on E as follows:

$$\|z\|_\omega = \sup_{s\geq 0} \left\| e^{\omega s} T(s)z \right\|.$$

Obviously, one has $\|z\| \leq \|z\|_\omega \leq C\|z\|$, which implies that $\|\cdot\|$ and $\|\cdot\|_\omega$ are equivalent norms. Denote by $\|T(t)\|_\omega$ the norm of $T(t)$ which is induced by $\|\cdot\|_\omega$, then, for each $t \geq 0$, we obtain

$$\left\| T(t) \right\|_\omega \leq e^{-\omega t}. \tag{2.4}$$

Meanwhile, it is easy to verify that the norm

$$\|v\|_{b\omega} = \sup_{s\in J} \|v(s)\|_{\omega}, \quad v \in BC(J,E),$$

is an equivalent one on $BC(J,E)$. Evidently, if $v(t) \equiv v_0, t \in J, v_0 \in E$, then we have

$$\|v\|_{b\omega} = \|v_0\|_{b\omega} = \|v\|_{\omega}.$$

Given the stable probability density of one-sided [20, 21, 38]

$$\psi_q(\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty} \sin(k\pi q)\frac{\Gamma(kq+1)}{k!}\theta^{-qk-1}, \quad \theta \in (0,+\infty),$$

where $0 < q < 1$. Noting Remark 2.8 in [21], if $\beta \in [0,1]$, we have

$$\int_0^{+\infty} \psi_q(\theta)\theta^{-q\beta}\,d\theta = \frac{\Gamma(1+\beta)}{\Gamma(1+q\beta)}. \tag{2.5}$$

From [20, 21, 38], the Laplace transform of $\psi_q(\theta)$ is

$$\mathcal{L}\big[\psi_q(\theta)\big] = \int_0^{\infty} e^{-\lambda\theta}\psi_q(\theta)\,d\theta = e^{-\lambda^q}, \quad 0 < q < 1. \tag{2.6}$$

In the remaining part, we show that the semigroup $\{T(t)\}_{t\geq 0}$ is strongly continuous of contractions and is uniformly exponentially stable with the growth bound ω_0, and $\omega \in (0, |\omega_0|]$.

Lemma 2.4 *Define an operator*

$$(\mathcal{J}h)(t) := \frac{q}{\Gamma(1-q)}\int_0^1\int_0^{\infty} \tau^{-q}(1-\tau)^{q-1}\frac{\psi_q(\theta)}{\theta^q}T\left(\frac{t^q(1-\tau)^q}{\theta^q}\right)h(s)\,d\theta\,d\tau, \\ h \in BC(J,E). \tag{2.7}$$

Thus, $\mathcal{J}$ maps $BC(J,E)$ into $BC(J,E)$ and

$$\|\mathcal{J}h\|_{b\omega} \leq \|h\|_{b\omega}.$$

In particular, if $h(t) \equiv x$, $t \in J$, $x \in E$, then

$$\|\mathcal{J}x\|_{b\omega} \leq \|x\|_{\omega}.$$

Proof Since

$$\begin{aligned}&\big\|(\mathcal{J}h)(t)\big\|_{\omega} \\ &\quad\leq \frac{q}{\Gamma(1-q)}\int_0^1\int_0^{\infty} (1-\tau)^{q-1}\tau^{-q}\frac{\psi_q(\theta)}{\theta^q}\left\|T\left(\frac{t^q(1-\tau)^q}{\theta^q}\right)h(s)\right\|_{\omega}\,d\theta\,d\tau\end{aligned}$$

$$
\begin{aligned}
&\le \frac{q}{\Gamma(1-q)} \int_0^1 \int_0^\infty (1-\tau)^{q-1}\tau^{-q} \frac{\psi_q(\theta)}{\theta^q} \left\| T\left(\frac{t^q(1-\tau)^q}{\theta^q}\right)\right\|_\omega \left\|h(s)\right\|_\omega \,d\theta\,d\tau \\
&\le \frac{q}{\Gamma(1-q)} \int_0^1 \int_0^\infty (1-\tau)^{q-1}\tau^{-q} \frac{\psi_q(\theta)}{\theta^q} e^{-\omega(\frac{t^q(1-\tau)^q}{\theta^q})} \|h\|_{b\omega}\,d\theta\,d\tau \\
&\le \frac{q}{\Gamma(1-q)} \|h\|_{b\omega} \int_0^1 (1-\tau)^{q-1}\tau^{-q} \left(\int_0^\infty \frac{\psi_q(\theta)}{\theta^q}\,d\theta\right) d\tau \\
&\le \|h\|_{b\omega},
\end{aligned}
$$

the proof is finished. □

Lemma 2.5 *Set*

$$
(\mathcal{K}h)(t) = \int_0^t \int_0^\infty q \frac{\psi_q(\theta)}{\theta^q}(t-s)^{q-1} T\left(\frac{(t-s)^q}{\theta^q}\right) h(s)\,d\theta\,ds, \quad h \in BC(J,E).
$$

Thus, $\mathcal{K}$ maps $BC(J,E)$ into $BC(J,E)$ and

$$
\left\|(\mathcal{K}h)(t)\right\|_\omega \le \frac{1}{\Gamma(q+1)} \frac{1}{\omega} \|h\|_{b\omega}; \qquad \left\|(\mathcal{K}h)\right\|_{b\omega} \le \frac{1}{\Gamma(q+1)} \frac{1}{\omega} \|h\|_{b\omega}.
$$

Proof Since

$$
\begin{aligned}
(\mathcal{K}h)(t) &= \int_0^t \int_0^\infty q \frac{\psi_q(\theta)}{\theta^q}(t-s)^{q-1} T\left(\frac{(t-s)^q}{\theta^q}\right) h(s)\,d\theta\,ds \\
&= \int_0^1 \int_0^\infty q \frac{\psi_q(\theta)}{\theta^q}(1-\tau)^{q-1} t^q T\left(\frac{(1-\tau)^q t^q}{\theta^q}\right) h(t\tau)\,d\theta\,d\tau,
\end{aligned}
$$

we have

$$
\begin{aligned}
\left\|(\mathcal{K}h)(t)\right\|_\omega &\le \int_0^1 \int_0^\infty q \frac{\psi_q(\theta)}{\theta^q}(1-\tau)^{q-1} t^q \left\| T\left(\frac{(1-\tau)^q t^q}{\theta^q}\right)\right\|_\omega \left\|h(t\tau)\right\|_\omega \,d\theta\,d\tau \\
&\le \int_0^1 \int_0^\infty q \frac{\psi_q(\theta)}{\theta^q} t^q (1-\tau)^{q-1} e^{-\omega \frac{t^q(1-\tau)^q}{\theta^q}} \left\|h(t\tau)\right\|_\omega \,d\theta\,d\tau \\
&\le \frac{1}{\omega}\|h\|_{b\omega} \int_0^\infty \left[\left(\int_0^1 e^{-\omega(\frac{t^q(1-\tau)^q}{\theta^q})}\,d\left(-\omega \frac{t^q(1-\tau)^q}{\theta^q}\right)\right) \frac{\psi_q(\theta)}{\theta^q}\right] d\theta \\
&= \frac{1}{\omega}\|h\|_{b\omega} \int_0^\infty \left(1 - e^{-\omega \frac{t^q}{\theta^q}}\right) \frac{\psi_q(\theta)}{\theta^q}\,d\theta \\
&\le \frac{1}{\omega} \frac{1}{\Gamma(q+1)} \|h\|_{b\omega}.
\end{aligned}
$$

Therefore,

$$
\left\|(\mathcal{K}h)\right\|_{b\omega} \le \frac{1}{\omega} \frac{1}{\Gamma(q+1)} \|h\|_{b\omega}.
$$

□

Lemma 2.6 *The evolution equation of fractional order*

$$
\begin{cases} {}^C D_{0+}^q u(t) = h(t) + Au(t), & t \in (0,+\infty), \\ u(0) = u_0, \end{cases} \tag{2.8}
$$

where $h \in BC(J,E)$ and $u_0 \in D(A)$, has a unique solution in $BC(J,E)$

$$\begin{aligned} u(t) &= (\mathcal{J}u_0)(t) + (\mathcal{K}h)(t) \\ &= \int_0^1 \int_0^\infty \frac{q}{\Gamma(1-q)} \frac{\psi_q(\theta)}{\theta^q} (1-\tau)^{q-1} \tau^{-q} T\biggl(\frac{(1-\tau)^q t^q}{\theta^q}\biggr) u_0 \, d\theta \, d\tau \\ &\quad + \int_0^t \int_0^\infty q \frac{\psi_q(\theta)}{\theta^q} (t-s)^{q-1} T\biggl(\frac{(t-s)^q}{\theta^q}\biggr) h(s) \, d\theta \, ds. \end{aligned} \tag{2.9}$$

Proof By Lemma 2.1, Definition 2.1, and Definition 2.2, (2.8) is equivalent with the following integral equation:

$$u(t) = u_0 + \int_0^t \frac{1}{\Gamma(q)} (t-s)^{q-1} \bigl[Au(s) + h(s)\bigr] \, ds. \tag{2.10}$$

By a similar method used in [18, 19], after Laplace transformation of the above equation, we can get

$$U(\lambda) = \frac{1}{\lambda} u_0 + \frac{1}{\lambda^q} AU(\lambda) + \frac{1}{\lambda^q} H(\lambda), \quad \lambda > 0, \tag{2.11}$$

where $H(\lambda)$ and $U(\lambda)$ are the Laplace transform of $h(t)$ and $u(t)$, respectively.

Then one has

$$\bigl(\lambda^q I - A\bigr) U(\lambda) = \lambda^{q-1} u_0 + H(\lambda).$$

From Lemma 2.3 and (2.6), we obtain

$$\begin{aligned} U(\lambda) &= \bigl(\lambda^q I - A\bigr)^{-1} \lambda^{q-1} u_0 + \bigl(\lambda^q I - A\bigr)^{-1} H(\lambda) \\ &= \lambda^{q-1} \int_0^\infty e^{-\lambda^q s} T(s) u_0 \, ds + \int_0^\infty e^{-\lambda^q s} T(s) H(\lambda) \, ds \\ &= \lambda^{q-1} \int_0^\infty \int_0^\infty e^{-\lambda s^{1/q} \theta} \psi_q(\theta) T(s) u_0 \, d\theta \, ds \\ &\quad + \int_0^\infty \int_0^\infty e^{-\lambda s^{1/q} \theta} \psi_q(\theta) T(s) H(\lambda) \, d\theta \, ds \\ &= \lambda^{q-1} \int_0^\infty \biggl[\int_0^\infty q \frac{t^{q-1}}{\theta^q} \psi_q(\theta) T\biggl(\frac{t^q}{\theta^q}\biggr) u_0 \, d\theta\biggr] e^{-\lambda t} \, dt \\ &\quad + \int_0^\infty \biggl[\int_0^t \int_0^\infty \psi_q(\theta) \frac{(t-s)^{q-1}}{\theta^q} T\biggl(\frac{(t-s)^q}{\theta^q}\biggr) h(s) \, d\theta \, ds\biggr] q e^{-\lambda t} \, dt. \end{aligned}$$

Taking the inverse Laplace transforms on the above equations, according to the convolution theorem and Lemma 2.5, we have

$$\begin{aligned} u(t) &= \mathcal{L}^{-1}\bigl[\lambda^{q-1}\bigr] * \mathcal{L}^{-1}\biggl[\int_0^\infty \biggl[\int_0^\infty q \psi_q(\theta) \frac{t^{q-1}}{\theta^q} T\biggl(\frac{t^q}{\theta^q}\biggr) u_0 \, d\theta\biggr] e^{-\lambda t} \, dt\biggr] \\ &\quad + \mathcal{L}^{-1}\biggl[\int_0^\infty e^{-\lambda t} \biggl(\int_0^t \int_0^\infty q \frac{(t-s)^{q-1}}{\theta^q} \psi_q(\theta) T\biggl(\frac{(t-s)^q}{\theta^q}\biggr) h(s) \, d\theta \, ds\biggr) \, dt\biggr] \\ &= \frac{t^{-q}}{\Gamma(1-q)} * \biggl[\int_0^\infty q \frac{t^{q-1}}{\theta^q} \psi_q(\theta) T\biggl(\frac{t^q}{\theta^q}\biggr) u_0 \, d\theta\biggr] \end{aligned}$$

$$+\int_0^t\int_0^\infty q\psi_q(\theta)\frac{(t-s)^{q-1}}{\theta^q}T\left(\frac{(t-s)^q}{\theta^q}\right)h(s)\,d\theta\,ds$$
$$=\frac{q}{\Gamma(1-q)}\int_0^t\int_0^\infty s^{-q}\psi_q(\theta)\frac{(t-s)^{q-1}}{\theta^q}T\left(\frac{(t-s)^q}{\theta^q}\right)u_0\,d\theta\,ds+(\mathcal{K}h)(t)$$
$$=(\mathcal{J}u_0)(t)+(\mathcal{K}h)(t).$$

Since

$$\big\|(\mathcal{K}h)(t)+(\mathcal{J}u_0)(t)\big\|_\omega\le\big\|(\mathcal{K}h)(t)\big\|_\omega+\big\|(\mathcal{J}u_0)(t)\big\|_\omega\le\|u_0\|_\omega+\frac{1}{\omega}\frac{1}{\Gamma(q+1)}\|h\|_{b\omega},$$

we have

$$\|\mathcal{J}u_0+\mathcal{K}h\|_{b\omega}\le\|u_0\|_\omega+\frac{1}{\omega}\frac{1}{\Gamma(q+1)}\|h\|_{b\omega}.$$

Hence, $u\in BC(J,E)$. □

In consequence, it leads to the conclusion.

Lemma 2.7 *The main equation* (1.1) *can be written in the equivalent form*

$$u(t)=Tu(t):=\big[\mathcal{J}\big(u_0-k(u)\big)\big](t)+(\mathcal{K}f)(t). \tag{2.12}$$

It is obvious that problem (1.1) has a mild solution if T has a fixed point.

3 Main results

In the part, we give the main conclusions about the existence of mild solutions for the main equation on the right half-axis.

First, we give the result about the existence of positive mild solutions for problem (1.1).

Theorem 3.1 *Assume that the normal cone P is positive in the Banach space E with the normal constant N. We assume that semigroup $\{T(t)\}_{t\ge0}$ is strongly continuous and is uniformly exponentially stable with the growth bound $\omega_0(\omega_0<0)$, and the infinitesimal generator of $\{T(t)\}_{t\ge0}$ is the operator A. Let $m\in C[0,\infty)$ and $m(t)$ is increasing and non-negative. Provided that $u_0\in P, u_0\ne\theta, f(t,u):J\times E\longrightarrow E, f_0(t):=f(t,\theta)\ge\theta$ is bounded on J, $k(u)$ maps E into E, both $f(t,u)$ and $k(u)$ are continuous. If $k(u)$ and $f(t,u)$ satisfy*:

(F1) *for any $t\in J$,*

$$f(t,z)\ge f(t,y),\quad\forall\theta\le y\le z.$$

(F2) *For a given $\omega_1\in(0,|\omega_0|]$, there exists a positive number R such that*

$$\|u_0\|_b+\frac{1}{\omega_1}\frac{1}{\Gamma(q+1)}R_f\le R,$$

where

$$R_f=\sup_{s\in J,0\le\|z\|_b\le R}\big\|f(s,z)\big\|_b<+\infty.$$

(F3)

$$\theta \le k(y) \le k(x) \le u_0, \quad \forall \theta \le x \le y.$$

Thus, there exists a positive mild solution for Eq. (1.1) *in* $BC(J,E)$.

Proof Define a nonempty subset of $BC(J,E)$

$$\mathcal{D} := \{u \in P_{BC} : \|u\|_{b\omega} \le R\}.$$

By (F1), $\forall u \in \mathcal{D}$, one has

$$\begin{aligned} \theta &\le Tu(t) \qquad (3.1)\\ &= (\mathcal{K}f)(t) + \big[\mathcal{J}\big(u_0 - k(u)\big)\big](t)\\ &\le (\mathcal{K}f)(t) + \big[\mathcal{J}(u_0)\big](t)\\ &= q\int_0^t \int_0^\infty \frac{\psi_q(\theta)}{\theta^q}(t-s)^{q-1} T\left(\frac{(t-s)^q}{\theta^q}\right) f\big(s, u\big(m(s)\big)\, d\theta\, ds\\ &\quad + \int_0^1 \int_0^\infty \frac{q}{\Gamma(1-q)} \frac{\psi_q(\theta)}{\theta^q}(1-\tau)^{q-1}\tau^{-q} T\left(\frac{(1-\tau)^q t^q}{\theta^q}\right) u_0\, d\theta\, d\tau. \qquad (3.2) \end{aligned}$$

Then, combined with Lemma 2.4 and Lemma 2.5, one has

$$\big\|Tu(t)\big\|_b \le \frac{1}{\omega}\frac{1}{\Gamma(q+1)} R_f + \|u_0\|_b.$$

Thus,

$$\|Tu\|_{b\omega} \le \|u_0\|_b + \frac{1}{\omega}\frac{1}{\Gamma(q+1)} R_f.$$

From condition (F2), one can get

$$\|Tu\|_{b\omega} \le \|u_0\|_b + \frac{1}{\omega_1}\frac{1}{\Gamma(q+1)} R_f \le R,$$

which shows that $T(\mathcal{D}) \subseteq \mathcal{D}$.

From condition (F1) and (F3), it is obvious that T maps $\mathcal{D}$ into $\mathcal{D}$ and it is also an increasing mapping. Then, applying Theorem 2.1, the map T possesses a fixed point in the nonempty set $\mathcal{D}$. Thus, Eq. (1.1) has a positive mild solution in $BC(J,E)$. □

Corollary 3.1 *Assume that the normal cone P is positive in the Banach space E with the normal constant N. Provided that semigroup $\{T(t)\}_{t\ge 0}$ is strongly continuous and is uniformly exponentially stable with the growth bound ω_0 ($\omega_0 < 0$), and the infinitesimal generator of $\{T(t)\}_{t\ge 0}$ is the operator A. Let $m \in C[0,\infty)$ and $m(t)$ be increasing and nonnegative. Provided that $u_0 \in P, u_0 \ne \theta$, $f(t,u) : J \times E \longrightarrow E$ is continuous and bounded, $f_0(t) := f(t,\theta) \ge \theta$, and $k(u) : E \longrightarrow E$ is continuous. If $f(t,u)$ and $k(u)$ satisfy the conditions* (F1) *and* (F3), *respectively, there exists a positive mild solution for Eq.* (1.1) *in* $BC(J,E)$.

Proof Since $f(t,u)$ is bounded and continuous on $J \times E$, condition (F2) is fulfilled, then Theorem 3.1 implies that there is a positive mild solution of Eq. (1.1) in $BC(J,E)$. □

Theorem 3.2 *Provided that* $\{T(t)\}_{t\geq 0}$ (*the semigroup of operators on Banach space* E) *is strongly continuous and is uniformly exponentially stable with the growth bound* ω_0 ($\omega_0 < 0$), *and the infinitesimal generator of* $\{T(t)\}_{t\geq 0}$ *is the operator* A. *Assume that* $m(t) \geq 0$ *and* $m \in C[0,\infty)$. *If* $f(t,u)$ *and* $k(u)$ *satisfy*:

(H1) *For a given* $\omega^* \in (0,|\omega_0|]$, *we have two positive numbers* $\mathcal{C}_f$ *and* $\mathcal{C}_k$ *satisfying* $\mathcal{M} < \omega^*/\chi$ *such that*

$$\mathcal{C}_f \|z_1 - z_2\|_{\omega^*} \geq \big\|f(t,z_1) - f(t,z_2)\big\|_{\omega^*}, \quad \forall z_1, z_2 \in E,$$

and

$$\mathcal{C}_k \|z_1 - z_2\|_{\omega^*} \geq \big\|k(z_1) - k(z_2)\big\|_{\omega^*}, \quad \forall z_1, z_2 \in E.$$

(H2)

$$\mathcal{L}_{kf} := \mathcal{C}_k + \frac{1}{\omega^*}\frac{\mathcal{C}_f}{\Gamma(q+1)} < 1.$$

Thus, one has a unique mild solution for Eq. (1.1) *in* $BC(J,E)$.

Proof By condition (H1), Lemma 2.4, Lemma 2.5, and Lemma 2.7, $\forall v$, $u \in BC(J,E)$, one has

$$\begin{aligned}
&\big\|Tv(t) - Tu(t)\big\|_{\omega^*} \\
&\quad = \big\|\big[\mathcal{J}\big(u_0 - k(v)\big)\big](t) + \mathcal{K}f\big(t,v\big(m(t)\big)\big) - \big[\mathcal{J}\big(u_0 - k(u)\big)\big](t) - \mathcal{K}f\big(t,u\big(m(t)\big)\big)\big\|_{\omega^*} \\
&\quad = \big\|\mathcal{K}\big[f\big(t,v\big(m(t)\big)\big) - f\big(t,u\big(m(t)\big)\big)\big]\big\|_{\omega^*} + \big(\mathcal{J}\big[k(u) - k(v)\big]\big)(t) \\
&\quad \leq \big\|\mathcal{K}\big[f\big(t,v\big(m(t)\big)\big) - f\big(t,u\big(m(t)\big)\big)\big]\big\|_{\omega^*} + \big\|\big(\mathcal{J}\big[k(u) - k(v)\big]\big)(t)\big\|_{\omega^*} \\
&\quad \leq \frac{1}{\Gamma(q+1)}\frac{1}{\omega^*}\big\|f\big(t,v\big(m(t)\big)\big) - f\big(t,u\big(m(t)\big)\big)\big\|_{\omega^*} + \big\|k\big(u(t)\big) - k\big(v(t)\big)\big\|_{\omega^*} \\
&\quad \leq \frac{\mathcal{C}_f}{\Gamma(q+1)}\frac{1}{\omega^*}\big\|v\big(m(t)\big) - u\big(m(t)\big)\big\|_{\omega^*} + \mathcal{C}_k\big\|u(t) - v(t)\big\|_{\omega^*} \\
&\quad \leq \left(\mathcal{C}_k + \frac{1}{\omega^*}\frac{\mathcal{C}_f}{\Gamma(q+1)}\right)\|v - u\|_{b\omega^*}.
\end{aligned}$$

Therefore,

$$\mathcal{L}_{kf}\|v - u\|_{b\omega^*} \geq \|Tv - Tu\|_{b\omega^*}.$$

Then, according to condition (H2), Eq. (1.1) has a unique mild solution. □

4 Examples

In order to certify the effectiveness of the main conclusion, an example is given in the following. Consider the following partial differential equation of fractional order.

Example 4.1

$$\begin{cases} \partial_t^q z(t,x) = F(t,m(z(t,x))) + \partial_x^2 z(t,x), & t \in [0,+\infty), \\ z(0,x) + k(z(t,x)) = z_0, & x \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0, & t \in [0,+\infty), \end{cases} \tag{4.1}$$

where ∂_t^q ($q \in (0,1)$) is fractional partial derivative of fractional order in the Caputo sense.

Set $E = L^2([0,\pi],\mathbb{R})$ and $Az = \partial_x^2 z$, in view of [39], we can conclude that A is a linear operator mapping $D(A)$ into E, and the domain $D(A) = \{v \mid v(0) = v(\pi) = 0, v' \in E, v \in E\}$. In addition, the operator A generates a semigroup $\{T(t)\}_{t\geq 0}$, which is strongly continuous and uniformly exponentially stable. Denote ω_0 as the growth bound of $\{T(t)\}_{t\geq 0}$, then we can get $\omega_0 \leq -1$.

Set $v(t) = z(t,\cdot)$, $v_0 = z_0$, $g(t,m(v(t))) = F(t,z(t,\cdot))$. At this point, the above problem can be written as

$$\begin{cases} {}^C D_{0+}^q v(t) = f(t,m(v(t))) + Av(t), & t \in (0,+\infty), \\ v(0) + k(v) = v_0. \end{cases} \tag{4.2}$$

Consider the function

$$g(t,v) = \frac{2+t^2}{1+t^2}\left(2 - \frac{1}{1+v^2}\right)$$

and

$$k(v) = 1 + \frac{1}{4+\arctan v}.$$

Take $q = 1/2$, $v_0 = 3$ and $m(v) = v^2$.

Then the conditions of Theorem 3.1 are all fulfilled. Therefore, there one has a positive mild solution for Eq. (4.1).

5 Conclusion

Here, using Laplace transform and its inverse transform, we obtain a correct equivalent integral equation for some kind of nonlocal abstract differential equations (fractional order) on the right half-axis, which is different from those given in the existing literature. According to this equivalent integral equation as obtained above, we investigate a kind of abstract fractional order differential equations. By using Knaster's theorem, we get the existence of positive mild solution for the main equation, and the uniqueness of a mild solution by the Banach contraction theorem.

Acknowledgements

The authors are grateful to the referees for their careful reading of the manuscript. The authors thank the help from the editor.

Funding

This research is supported by the National Natural Science Foundation of China (No. 71672195, 11601508), and supported by Science Fund for Creative Research Groups of the National Natural Science Foundation of China (No. 71221061).

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All of the authors contributed equally in writing this paper. They both read and approved the final manuscript.

Author details
[1]School of Business, Central South University, Changsha, P.R. China. [2]School of Mathematics, China University of Mining and Technology, Xuzhou, P.R. China.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 26 July 2018 Accepted: 9 November 2018 Published online: 20 November 2018

References

1. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
2. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
3. Lakshmikantham, V., Leela, S., Vasundhara Devi, J.: Theory of Fractional Dynamic Systems. Cambridge Academic Publishers, Cambridge (2009)
4. Zhou, Y., Ahmadb, B., Alsaedi, A.: Existence of nonoscillatory solutions for fractional neutral differential equations. Appl. Math. Lett. **72**, 70–74 (2017)
5. Zou, Y., He, G.: On the uniqueness of solutions for a class of fractional differential equations. Appl. Math. Lett. **74**, 68–73 (2017)
6. Zhou, Y.: Attractivity for fractional differential equations in Banach space. Appl. Math. Lett. **75**, 1–6 (2018)
7. Raheem, A., Maqbul, M.D.: Oscillation criteria for impulsive partial fractional differential equations. Comput. Math. Appl. **73**, 1781–1788 (2017)
8. Cabada, A., Kisela, T.: Existence of positive periodic solutions of some nonlinear fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **50**, 51–67 (2017)
9. Henderson, J., Luca, R.: Systems of Riemann–Liouville fractional equations with multi-point boundary conditions. Appl. Math. Comput. **309**, 303–323 (2017)
10. Ahmad, B., Alsaedi, A., Garout, D.: Existence results for Liouville–Caputo type fractional differential equations with nonlocal multi-point and sub-strips boundary conditions. Comput. Math. Appl. (2018, in press)
11. Dhifli, A., Khamessi, B.: Existence and boundary behavior of positive solution for a Sturm–Liouville fractional problem with p-Laplacian. J. Fixed Point Theory Appl. **19**(2), 1–22 (2017)
12. Ahmada, B., Ntouyas, S.K., Alsaedi, A.: On a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions. Chaos Solitons Fractals **83**, 234–241 (2016)
13. Becker, L.C., Burton, T.A., Purnaras, I.K.: Integral and fractional equations, positive solutions, and Schaefer's fixed point theorem. Opusc. Math. **36**(4), 431–458 (2016)
14. Bayour, B., Torres, D.F.M.: Existence of solution to a local fractional nonlinear differential equation. J. Comput. Appl. Math. **312**, 127–133 (2017)
15. Bachar, I., Mâagli, H., Radulescu, V.: Fractional Navier boundary value problems. Bound. Value Probl. **2016**, Article ID 79 (2016)
16. Frioui, A., Guezane-Lakoud, A., Khaldi, R.: Fractional boundary value problems on the half line. Opusc. Math. **37**(2), 265–280 (2017)
17. Kumar, S., Kumar, D., Singh, J.: Fractional modelling arising in unidirectional propagation of long waves in dispersive media. Adv. Nonlinear Anal. **5**(4), 383–394 (2016)
18. Mei, Z.D., Peng, J.G., Gao, J.H.: General fractional differential equations of order $\alpha \in (1,2)$ and type $\xi_i \in [0,1]$ in Banach spaces. Semigroup Forum **94**, 712–737 (2017)
19. El-Borai, M.M.: Some probability densities and fundamental solutions of fractional evolution equations. Chaos Solitons Fractals **14**, 433–440 (2002)
20. Zhou, Y., Jiao, F.: Nonlocal Cauchy problem for fractional evolution equations. Nonlinear Anal., Real World Appl. **11**, 4465–4475 (2010)
21. Wang, J., Zhou, Y.: A class of fractional evolution equations and optimal controls. Nonlinear Anal., Real World Appl. **12**, 262–272 (2011)
22. Chen, P., Zhang, X., Li, Y.: Study on fractional non-autonomous evolution equations with delay. Comput. Math. Appl. **73**, 794–803 (2017)
23. Chen, P., Li, Y., Li, Q.: Existence of mild solutions for fractional evolution equations with nonlocal initial conditions. Ann. Pol. Math. **110**, 13–24 (2014). https://doi.org/10.4064/ap110-1-2
24. Chen, P., Li, Y., Zhang, X.: On the initial value problem of fractional stochastic evolution equations in Hilbert spaces. Commun. Pure Appl. Anal. **14**, 1817–1840 (2015)
25. Wang, R., Ma, Q.: Some new results for multi-valued fractional evolution equations. Appl. Math. Comput. **257**, 285–294 (2015)
26. Zhao, J., Wang, R.: Mixed monotone iterative technique for fractional impulsive evolution equations. Miskolc Math. Notes **17**, 683–696 (2016)

27. Jabeena, T., Lupulescu, V.: Existence of mild solutions for a class of non-autonomous evolution equations with nonlocal initial conditions. J. Nonlinear Sci. Appl. **10**, 141–153 (2017)
28. Zhou, Y., Shen, X.H., Zhang, L.: Cauchy problem for fractional evolution equations with Caputo derivative. Eur. Phys. J. Spec. Top. **222**, 1749–1765 (2013)
29. Byszewski, L.: Theorems about existence and uniqueness of solutions of a semi-linear evolution nonlocal Cauchy problem. J. Math. Anal. Appl. **162**, 494–505 (1991)
30. Byszewski, L., Lakshmikantham, V.: Theorems about existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. Appl. Anal. **40**, 11–19 (1991)
31. Byszewski, L.: Existence and uniqueness of a classical solution to a functional-differential abstract nonlocal Cauchy problem. J. Appl. Math. Stoch. Anal. **12**, 91–97 (1999)
32. Deimling, K.: Nonlinear Functional Analysis. Springer, New York (1985)
33. Guo, D., Lakshmikantham, V.: Nonlinear Problems in Abstract Cone. Academic Press, Orlando (1988)
34. Agarwal, R.P., Bohner, M., Li, W.T.: Nonoscillation and Oscillation: Theory for Functional Differential Equations. Dekker, New York (2004)
35. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)
36. Engel, K., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Springer, New York (1995)
37. Chen, P., Li, Y., Zhang, X.: Existence and uniqueness of positive mild solutions for nonlocal evolution equations. Positivity **19**, 927–939 (2015)
38. Mainardi, F., Paradisi, P., Gorenflo, R.: Probability distributions generated by fractional diffusion equations. In: Kertesz, J., Kondor, I. (eds.) Econophysics: An Emerging Science. Kluwer, Dordrecht (2000)
39. Hernandez, E., Sakthivel, R., Tanaka Aki, S.: Existence results for impulisive evolution differential equations with state-depentent delay. Electron. J. Differ. Equ. **2008**, Article ID 28 (2008)