Xu et al. *Boundary Value Problems* (2018) 2018:174
https://doi.org/10.1186/s13661-018-1093-9

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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Low order nonconforming finite element method for time-dependent nonlinear Schrödinger equation

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Abstract

The main aim of this paper is to apply a low order nonconforming EQ_1^{rot} finite element to solve the nonlinear Schrödinger equation. Firstly, the superclose property in the broken H^1-norm for a backward Euler fully-discrete scheme is studied, and the global superconvergence results are deduced with the help of the special characters of this element and the interpolation postprocessing technique. Secondly, in order to reduce computing cost, a two-grid method is developed and the corresponding superconvergence error estimates are obtained. Finally, a numerical experiment is carried out to confirm the theoretical analysis.

MSC: 65N15; 65N30

Keywords: Nonlinear Schrödinger equation; Two-grid method; Low order nonconforming finite element; Superconvergence error estimates

1 Introduction

Consider the following nonlinear Schrödinger equation (NLSE):

$$\begin{cases} iu_t + \Delta u + \lambda f(|u|^2)u = 0, & (X,t) \in \Omega \times (0,T], \\ u(X,t) = 0, & (X,t) \in \partial\Omega \times (0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \end{cases} \tag{1.1}$$

where $X = (x,y)$, $\Omega \subset R^2$ is a bounded convex domain with Lipschitz boundary $\partial\Omega$. i is the imaginary unit, $u(X,t)$ is a complex-valued function, $T \in (0,+\infty)$ is a real parameter, $f(|u|^2) = |u|^{2r}$ ($r \geq 1$ is an integer) is a smooth real-valued function, and $u_0(X)$ is a known smooth function.

The Schrödinger equation may describe many physical phenomena in optics, mechanics, and plasma physics, and it plays a very important role in various areas of mathematical physics. Numerical methods for this problem have been investigated extensively, e.g., see [1–3] for finite difference methods, [4–8] for finite element methods (FEMs), and [9–11] for others. Especially, the superconvergence analysis of FEMs for the Schrödinger equation have been studied successfully. For example, [12] used the conforming bilinear element to solve the LSE and obtained the superclose and superconvergence results in H^1-norm for the semi-discrete scheme. [13] derived the same results as [12] for NLSE with the conforming linear triangular element by establishing the relationship between Ritz projection and the linear interpolation. Whereafter, a series of superconvergence results about backward Euler and Crank–Nicolson fully-discrete schemes for NLSE also were studied in [14–17].

A two-grid method was first introduced by Xu [18, 19] as a discretization technique for nonlinear and nonsymmetric indefinite partial differential equations. The main idea of this method is to use a coarse space (with mesh size H) to produce a rough approximation of the solution, and then use it as the initial guess for one Newton iteration on the fine grid (with mesh size h and $h \ll H$). Up to now, the two-grid method was deeply researched for different problems [20–25]. Especially, the two-grid method was used to solve the linear Schrödinger equation (LSE) and NLSE in [26–30]. However, this method is rarely considered for nonconforming elements.

As we know, EQ_1^{rot} element is an important quadrilateral nonconforming finite element and has been employed to deal with different problems successfully for its good theoretical and numerical behavior [31–37]. The purpose of this work is to use this element to deal with problem (1.1). By virtue of the special properties of this element, we obtain the superclose and superconvergence results in the broken H^1-norm for the backward Euler fully-discrete scheme. At the same time, in order to reduce the computing cost, we develop a new two-grid algorithm and deduce the corresponding superconvergence results.

The paper is organized as follows. In Sect. 2, EQ_1^{rot} element and some lemmas are briefly introduced. In Sect. 3, the backward Euler fully-discrete scheme for problem (1.1) is discussed and some important superconvergence results are derived. In Sect. 4, a two-grid scheme of (1.1) is established and the corresponding superconvergence results are obtained. Finally, a numerical experiment is carried out to confirm the theoretical results.

2 Nonconforming EQ_1^{rot} element and some lemmas

For simplicity, let $\Omega \subset \mathbb{R}^2$ be a convex polygon with edges parallel to the coordinate axes, T_h be a regular subdivision of Ω. For a given element K with the center point (x_K, y_K), its four vertices and edges are denoted as $a(x_i, y_i)$ $(i = 1,2,3,4)$ and $F_i = \overline{a_i a_{i+1}}$ $(i = 1,2,3,4 \mod 4)$, respectively. We assume that edges F_i $(i = 1,3)$ parallel to x-axis and F_i $(i = 2,4)$ parallel to y-axis, $h_{x,K}$ and $h_{y,K}$ denote the half length of element K along x and y-axis, respectively.

The EQ_1^{rot} finite element (K, P, Σ) on K is defined as follows:

$$\Sigma = \{v_1, v_2, v_3, v_4, v_5\}, \qquad P = \operatorname{span}\{1, x, y, \varphi(x), \varphi(y)\}, \tag{2.1}$$

where

$$v_i = \frac{1}{|F_i|}\int_{F_i} v\,ds \quad (i = 1,2,3,4), \qquad v_5 = \frac{1}{|K|}\int_K v\,dx\,dy, \varphi(t) = \frac{1}{2}\left(3t^2 - 1\right),$$

and $|F_i|$ and $|K|$ are the measures of F_i and K, respectively.

The associated finite element space V_h can be defined by

$$V_h = \left\{v_h : v_h|_K \in P, \forall K \in T_h, \int_F [v_h]\,ds = 0, F \subset \partial K\right\},$$

where $[v_h]$ denotes the jump value of v_h across the boundary F, and $[v_h] = v_h$ if $F \subset \partial\Omega$.

Obviously, $\|\cdot\|_h = (\sum_{K\in T_h} |\cdot|_{1,K}^2)^{\frac{1}{2}}$ is a norm over V_h.
Let Π_h be the associated interpolation operator over V_h, then we have

$$\|u - \Pi_h u\|_0 + h\|u - \Pi_h u\|_h \le Ch^2|u|_2, \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2.2}$$

Lemma 2.1 ([31, 32]) *For $v_h \in V_h$, we have*

$$\left|\sum_{K\in T_h}\int_{\partial K}\frac{\partial u}{\partial n}v_h\,ds\right| \le \begin{cases} Ch|u|_2\|v_h\|_h, & \forall u \in H^2(\Omega), \\ Ch^2|u|_3\|v_h\|_h, & \forall u \in H^3(\Omega). \end{cases} \tag{2.3}$$

Lemma 2.2 *For $v_h \in V_h$, we have*

$$\left|\sum_{K\in T_h}\int_{\partial K}\frac{\partial u}{\partial n}v_h\,ds\right| \le Ch^2|u|_4\|v_h\|_0, \quad \forall u \in H^4(\Omega). \tag{2.4}$$

Proof By introducing two functions

$$E(x) = \frac{1}{2}\big((x - x_K)^2 - h_{x,K}^2\big), \qquad F(y) = \frac{1}{2}\big((y - y_K)^2 - h_{y,K}^2\big),$$

and notation $P_0 v_h|_{F_i} = \frac{1}{|F_i|}\int_{F_i} v_h\,ds$, which has continuity between elements and vanishes on $\partial\Omega$, and hence the summation

$$\sum_{K\in T_h}\left(\int_{F_2} - \int_{F_4}\right)u_x P_0 v_h\,dy = 0, \qquad \sum_{K\in T_h}\left(\int_{F_3} - \int_{F_1}\right)u_y P_0 v_h\,dx = 0.$$

So we can obtain that

$$\begin{aligned}
&\sum_{K\in T_h}\int_{\partial K}\frac{\partial u}{\partial n}v_h\,ds \\
&\quad = \sum_{K\in T_h}\left(\int_{F_2} - \int_{F_4}\right)u_x v_h\,dy + \sum_{K\in T_h}\left(\int_{F_3} - \int_{F_1}\right)u_y v_h\,dx \\
&\quad = \sum_{K\in T_h}\left(\int_{F_2} - \int_{F_4}\right)u_x(v_h - P_0 v_h)\,dy + \sum_{K\in T_h}\left(\int_{F_3} - \int_{F_1}\right)u_y(v_h - P_0 v_h)\,dx \\
&\quad = \sum_{K\in T_h}\int_K\left[u_{xx}\left((y - y_K)v_{hy} + \left((y - y_K)^2 - \frac{h_{y,K}^2}{3}\right)\frac{v_{hyy}}{2}\right)\right]dx\,dy \\
&\qquad + \sum_{K\in T_h}\int_K\left[u_{yy}\left((x - x_K)v_{hx} + \left((x - x_K)^2 - \frac{h_{x,K}^2}{3}\right)\frac{v_{hxx}}{2}\right)\right]dx\,dy \\
&\quad \triangleq A_1 + A_2,
\end{aligned} \tag{2.5}$$

where we use the expressions

$$(v_h - P_0 v_h)|_{F_i} = (y - y_K)v_{hy} + \left((y - y_K)^2 - \frac{h_{y,K}^2}{3}\right)\frac{v_{hyy}}{2}, \quad i = 2, 4,$$

$$(v_h - P_0 v_h)|_{F_i} = (x - x_K)v_{hx} + \left((x - x_K)^2 - \frac{h_{x,K}^2}{3}\right)\frac{v_{hxx}}{2}, \quad i = 1, 3.$$

Now we begin to estimate A_1, which can be rewritten as

$$\begin{aligned} A_1 &= \sum_{K\in T_h}\int_K u_{xx}(y-y_K)v_{hy}\,dx\,dy + \frac{1}{2}\sum_{K\in T_h}\int_K u_{xx}(y-y_K)^2 v_{hyy}\,dx\,dy \\ &\quad - \sum_{K\in T_h}\frac{h_{y,K}^2}{6}\int_K u_{xx}v_{hyy}\,dx\,dy \triangleq B_1 + B_2 + B_3. \end{aligned} \tag{2.6}$$

Firstly, for all $v_{hy}\in \operatorname{span}\{1,y\}$, we know that

$$v_{hy}(x,y) = v_{hy}(x,y_K) + (y-y_K)v_{hyy}.$$

Noting that $F(y)|_{F_1,F_3} = 0$, $F'(y) = y - y_K$, $F(y) = \frac{1}{6}(F^2(y))'' - \frac{h_{y,K}^2}{3}$, and by Green's formula, we have

$$\begin{aligned} B_1 &= \sum_{K\in T_h}\int_K u_{xx}F'(y)v_{hy}\,dx\,dy \\ &= -\sum_{K\in T_h}\int_K u_{xxy}F(y)v_{hy}\,dx\,dy \\ &= -\sum_{K\in T_h}\int_K u_{xxy}\left[\frac{1}{6}\big(F^2(y)\big)'' - \frac{h_{y,K}^2}{3}\right]v_{hy}\,dx\,dy \\ &= \sum_{K\in T_h}\left[\frac{1}{6}\int_K u_{xxyy}\big(F^2(y)\big)' v_{hy}\,dx\,dy + \int_K \frac{h_{y,K}^2}{3}v_{hy}\,dx\,dy\right] \\ &= \sum_{K\in T_h}\frac{1}{6}\int_K u_{xxyy}\big(F^2(y)\big)' v_{hy}\,dx\,dy + \sum_{K\in T_h}\frac{h_{y,K}^2}{3}\int_K u_{xxy}v_{hy}\,dx\,dy \\ &\quad - \sum_{K\in T_h}\frac{h_{y,K}^2}{3}\int_K u_{xxy}(y-y_K)v_{hyy}\,dx\,dy \\ &\triangleq B_{11} + B_{12} + B_{13}. \end{aligned} \tag{2.7}$$

By the inverse inequality, term B_{11} can be estimated as

$$\begin{aligned} B_{11} &= \sum_{K\in T_h}\frac{1}{6}\int_K u_{xxyy}\big(F^2(y)\big)' v_{hy}\,dx\,dy \\ &\le \sum_{K\in T_h} Ch_{y,K}^3\|u_{xxyy}\|_{0,K}\|v_{hy}\|_{0,K} \le Ch^2|u|_4\|v_h\|_0. \end{aligned} \tag{2.8}$$

For the term B_{12}, it can be written as

$$\begin{aligned} B_{12} &= \sum_{K\in T_h}\frac{h_{y,K}^2}{3}\int_K u_{xxy}v_{hy}\,dx\,dy \\ &= \sum_{K\in T_h}\frac{h_{y,K}^2}{3}\left[\left(\int_{F_3} - \int_{F_1}\right)u_{xxy}v_h\,dx - \int_K u_{xxyy}v_h\,dx\,dy\right]. \end{aligned} \tag{2.9}$$

Noting that

$$\begin{aligned}\sum_{K\in T_h}\left(\int_{F_3}-\int_{F_1}\right)u_{xxy}v_h\,dx &= \sum_{K\in T_h}\left(\int_{F_3}-\int_{F_1}\right)u_{xxy}(v_h-P_0v_h)\,dx\\ &= \sum_{K\in T_h}\int_K\left[u_{xxyy}\left((x-x_K)v_{hx}+\left((x-x_K)^2+\frac{h_{x,K}^2}{3}\right)\frac{v_{hxx}}{2}\right)\right]dx\,dy\\ &\le \sum_{K\in T_h}\left(Ch_{x,K}|u|_4\|v_{hx}\|_{0,K}+Ch_{x,K}^2|u|_4\|v_{hxx}\|_{0,K}\right)\le C|u|_4\|v_h\|_0, \end{aligned} \tag{2.10}$$

and substituting (2.10) into (2.9), we can derive

$$B_{12}\le Ch^2|u|_4\|v_h\|_0. \tag{2.11}$$

As to the term B_{13}, noting that $F(y)|_{F_1,F_3}=0$ and $F'(y)=y-y_K$, we have

$$\begin{aligned}B_{13} &= -\sum_{K\in T_h}\frac{h_{y,K}^2}{3}\int_K u_{xxy}F'(y)v_{hyy}\,dx\,dy\\ &= \sum_{K\in T_h}\frac{h_{y,K}^2}{3}\int_K u_{xxyy}F(y)v_{hyy}\,dx\,dy\le Ch^2|u|_4\|v_h\|_0. \end{aligned} \tag{2.12}$$

Combining with (2.7)–(2.9) and (2.12), we can obtain

$$B_1\le Ch^2|u|_4\|v_h\|_0. \tag{2.13}$$

Secondly, noting that $(y-y_K)^2=\frac{1}{3}[(F^2(y))''+h_{y,K}^2]$, we have

$$\begin{aligned}&B_2+B_3\\ &\quad=\frac{1}{6}\sum_{K\in T_h}\int_K u_{xx}\left[\left(F^2(y)\right)''+h_{y,K}^2\right]v_{hyy}\,dx\,dy-\sum_{K\in T_h}\frac{h_{y,K}^2}{6}\int_K u_{xx}v_{hyy}\,dx\,dy\\ &\quad=\frac{1}{6}\sum_{K\in T_h}\int_K u_{xx}\left(F^2(y)\right)''v_{hyy}\,dx\,dy\frac{1}{6}\sum_{K\in T_h}\int_K u_{xxyy}F^2(y)v_{hyy}\,dx\,dy\\ &\quad\le Ch^4|u|_4\|v_{hyy}\|_0\le Ch^2|u|_4\|v_h\|_0. \end{aligned} \tag{2.14}$$

Finally, substituting estimates (2.13) and (2.14) into (2.6), we obtain

$$A_1\le Ch^2|u|_4\|v_h\|_0. \tag{2.15}$$

And similarly, we can derive the result

$$A_2\le Ch^2|u|_4\|v_h\|_0. \tag{2.16}$$

Combining with (2.15) and (2.16), the desired result is obtained. □

Remark 2.1 In [38], the authors derived the following result:

$$\left|\sum_{K\in T_h}\int_{\partial K}\frac{\partial u}{\partial n}v_h\,ds\right|\le Ch^2|u|_5\|v_h\|_0,\quad \forall u\in H^5(\Omega),v_h\in V_h. \tag{2.17}$$

Obviously, the regularity requirement of u is stronger than our result.

3 Backward Euler fully-discrete scheme and superconvergence results

The variational form of (1.1) is to find $u\in H_0^1(\Omega)$ such that

$$\begin{cases} i(u_t,v)-(\nabla u,\nabla v)+\lambda(f(|u|^2)u,v)=0, & v\in H_0^1(\Omega),\\ u(X,0)=u_0(X), & X\in\Omega,\end{cases} \tag{3.1}$$

where $(u,v)=\int_\Omega u\bar{v}\,dx\,dy$ denotes the inner product, $\bar{v}$ is the conjugate of v.

Given a time step $\tau=T/N$, where N is a positive integer, we shall approximate the solution at times $t_n=n\tau$, $n=0,1,\ldots,N$. For a given smooth function ϕ^n on $[0,T]$, define $\phi^n=\phi(X,t^n)$, $\partial_t\phi^n=\frac{\phi^n-\phi^{n-1}}{\tau}$, and $\partial_t\nabla\phi^n=\frac{\nabla\phi^n-\nabla\phi^{n-1}}{\tau}$.

Equation (3.1) has the following equivalent formulation:

$$\begin{cases} i(\partial_t u^n,v)-(\nabla u^n,\nabla v)+\lambda(f(|u^n|^2)u^n,v)=i(R_1^n,v), & v\in H_0^1(\Omega),\\ u(X,0)=u_0(X), & X\in\Omega,\end{cases} \tag{3.2}$$

where $R_1^n=\partial_t u^n-u_t^n$. Furthermore, we have

$$\begin{aligned}\|R_1^n\|_0^2&=\left\|\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(t_n-t)u_{tt}\,dt\right\|_0^2\le\frac{1}{\tau^2}\int_\Omega\left(\int_{t_{n-1}}^{t_n}(t_n-t)u_{tt}\,ds\right)^2dx\,dy\\ &\le\frac{C}{\tau^2}\int_\Omega\left(\int_{t_{n-1}}^{t_n}(t_n-t)\,dt\right)^2\left(\int_{t_{n-1}}^{t_n}u_{tt}\,ds\right)^2dx\,dy\le C\tau\int_{t_{n-1}}^{t_n}\|u_{tt}\|_0^2\,dt.\end{aligned} \tag{3.3}$$

The backward Euler fully-discrete scheme of (3.1) is to find $U^n\in V_h$ such that

$$\begin{cases} i(\partial_t U^n,v_h)-(\nabla U^n,\nabla v_h)_h+\lambda(f(|U^n|^2)U^n,v_h)=0, & v_h\in V_h,\\ U^0=\Pi_h u_0(X), & X\in\Omega,\end{cases} \tag{3.4}$$

where $(\cdot,\cdot)_h=\sum_{K\in T_h}(\cdot,\cdot)_K$.

In order to carry out the error estimate and superclose analysis, we introduce the following assumption.

Assumption 3.1 Let u^n and U^n be the solutions of (1.1) and (3.4), respectively, for $n=1,2,\ldots,N$, then there exists $0<h_0<1$ such that, for $0<h<h_0$, $n=1,2,\ldots,N$, it holds

$$\|u^n-U^n\|_{0,\infty}<1, \tag{3.5}$$

which means $\|U^n\|_{0,\infty}<C$.

Regarding the proof of Assumption 3.1, one can refer to [15] for details.

For simplicity, we write

$$u^n - U^n = (u^n - \Pi_h u^n) + (\Pi_h u^n - U^n) \triangleq \rho^n + \theta^n.$$

Then we have the following results.

Theorem 3.1 *Assume that u^n and U^n are the solutions of* (1.1) *and* (3.4), *respectively. If $u \in H^4(\Omega) \cap H_0^1(\Omega)$, $u_t \in H^4(\Omega)$, $u_{tt} \in H^2(\Omega)$, $u_{ttt} \in L^2(\Omega)$, we have*

$$\|\theta^n\|_0 + \|\theta^n\|_h = O(h^2 + \tau). \tag{3.6}$$

Proof From (1.1) and (3.4), we have the result

$$\begin{aligned} &i(\partial_t(u^n - U^n), v_h) - (\nabla(u^n - U^n), v_h)_h + \lambda(f(|u^n|^2)u^n - f(|U^n|^2)U^n, v_h) \\ &\quad = i(R_1^n, v_h) - \sum_{K \in T_h} \int_{\partial K} \frac{\partial u^n}{\partial n} \cdot \overline{v_h}\, ds, \end{aligned} \tag{3.7}$$

which can be rewritten as

$$\begin{aligned} i(\partial_t \theta^n, v_h) - (\nabla\theta^n, \nabla v_h)_h &= -\lambda(f(|u^n|^2)u^n - f(|U^n|^2)U^n, v_h) \\ &\quad - i(\partial_t \rho^n, v_h) + i(R_1^n, v_h) - \sum_{K \in T_h} \int_{\partial K} \frac{\partial u^n}{\partial n} \cdot \overline{v_h}\, ds. \end{aligned} \tag{3.8}$$

Taking $v_h = \theta^n$ in (3.8), we have

$$\begin{aligned} i(\partial_t \theta^n, \theta^n) - (\nabla\theta^n, \nabla\theta^n)_h &= -\lambda(f(|u^n|^2)u^n - f(|U^n|^2)U^n, \theta^n) \\ &\quad - i(\partial_t \rho^n, \theta^n) + i(R_1^n, \theta^n) - \sum_{K \in T_h} \int_{\partial K} \frac{\partial u^n}{\partial n} \cdot \overline{\theta^n}\, ds. \end{aligned} \tag{3.9}$$

Comparing the imaginary parts of (3.9), we get

$$\begin{aligned} &\frac{1}{\tau} \operatorname{Re}(\theta^n - \theta^{n-1}, \theta^n) \\ &\quad = \frac{1}{2\tau}\left(\|\theta^n\|_0^2 - \|\theta^{n-1}\|_0^2 + \|\theta^n - \theta^{n-1}\|_0^2\right) \\ &\quad = -\operatorname{Re}(\partial_t \rho^n, \theta^n) - \operatorname{Im} \lambda(f(|u^n|^2)u^n - f(|U^n|^2)U^n, \theta^n) \\ &\qquad + \operatorname{Re}(R_1^n, \theta^n) - \operatorname{Im} \sum_{K \in T_h} \int_{\partial K} \frac{\partial u^n}{\partial n} \cdot \overline{\theta^n}\, ds, \end{aligned} \tag{3.10}$$

which implies

$$\begin{aligned} &\|\theta^n\|_0^2 - \|\theta^{n-1}\|_0^2 \\ &\quad \le -2\tau \operatorname{Re}(\partial_t \rho^n, \theta^n) - 2\tau \operatorname{Im} \lambda(f(|u^n|^2)u^n - f(|U^n|^2)U^n, \theta^n) \\ &\qquad + 2\tau \operatorname{Re}(R_1^n, \theta^n) - 2\tau \operatorname{Im} \sum_{K \in T_h} \int_{\partial K} \frac{\partial u^n}{\partial n} \cdot \overline{\theta^n}\, ds \triangleq \sum_{i=1}^4 D_i. \end{aligned} \tag{3.11}$$

Now, we start to estimate each term D_i ($i = 1,2,3,4$) one by one.

Applying ε-Young's inequality, we obtain

$$|D_1| \le \left|2\tau\left(\partial_t\rho^n,\theta^n\right)\right| \le Ch^4\int_{t_{n-1}}^{t_n}|u_t|_2^2\,dt + C\tau\left\|\theta^n\right\|_0^2. \tag{3.12}$$

By using the continuity of $f(s)$ and Assumption 3.1, we have

$$\begin{aligned}
|D_2| &\le 2\tau|\lambda|\left|\left(f\left(\left|u^n\right|^2\right)u^n - f\left(\left|U^n\right|^2\right)U^n,\theta^n\right)\right| \\
&= 2\tau|\lambda|\left|\left(f\left(\left|u^n\right|^2\right)u^n - f\left(\left|u^n\right|^2\right)U^n + f\left(\left|u^n\right|^2\right)U^n - f\left(\left|U^n\right|^2\right)U^n,\theta^n\right)\right| \\
&= C\tau\left\|f\left(\left|u^n\right|^2\right)\right\|_{0,\infty}\left|\left(\rho^n+\theta^n,\theta^n\right)\right| \\
&\quad + C\tau\left\|U^n\right\|_{0,\infty}\|f'(\xi_1)\|_{0,\infty}\left(\left\|u^n\right\|_{0,\infty} + \left\|U^n\right\|_{0,\infty}\right)\left|\left(\rho^n+\theta^n,\theta^n\right)\right| \\
&\le C\tau\left|\left(\rho^n+\theta^n,\theta^n\right)\right| \le C\tau h^4\left|u^n\right|_2^2 + C\tau\left\|\theta^n\right\|_0^2,
\end{aligned} \tag{3.13}$$

where ξ_1 lies between $|u^n|^2$ and $|U^n|^2$.

With the help of the result (3.3) and Lemma 2.2, terms D_3 and D_4 can be estimated as

$$|D_3| \le 2\tau\left|\left(R_1^n,\theta^n\right)\right| \le 2\tau\left\|R_1^n\right\|_0\left\|\theta^n\right\|_0 \le C\tau^2\int_{t_{n-1}}^{t_n}\|u_{tt}\|_0^2\,dt + C\tau\left\|\theta^n\right\|_0^2, \tag{3.14}$$

$$|D_4| \le C\tau h^4\left|u^n\right|_4\left\|\theta^n\right\|_0 \le C\tau h^4\left|u^n\right|_4^2 + C\tau\left\|\theta^n\right\|_0^2. \tag{3.15}$$

Combining the above estimates (3.11)–(3.15) yields

$$\begin{aligned}
\left\|\theta^n\right\|_0^2 - \left\|\theta^{n-1}\right\|_0^2 &\le Ch^4\left(\tau\left\|u^n\right\|_4^2 + \int_{t_{n-1}}^{t_n}|u_t|_2^2\,dt\right) \\
&\quad + C\tau^2\int_{t_{n-1}}^{t_n}\|u_{tt}\|_0^2\,dt + C\tau\left\|\theta^n\right\|_0^2.
\end{aligned} \tag{3.16}$$

Summing (3.16) up with respect to n and noting $\theta^0 = 0$, we have

$$\begin{aligned}
\left\|\theta^n\right\|_0^2 &\le Ch^4\int_0^T|u_t|_2^2\,dt + C\tau^2\int_0^T\|u_{tt}\|_0^2\,dt \\
&\quad + C\tau h^4\sum_{i=1}^n\left\|u^i\right\|_4^2 + C\tau\sum_{i=1}^n\left\|\theta^i\right\|_0^2.
\end{aligned} \tag{3.17}$$

By Gronwall's lemma, we can derive

$$\left\|\theta^n\right\|_0^2 \le Ch^4\int_0^T|u_t|_2^2\,dt + C\tau^2\int_0^T\|u_{tt}\|_0^2\,dt + C\tau h^4\sum_{i=1}^n\left\|u^i\right\|_4^2, \tag{3.18}$$

which implies

$$\left\|\theta^n\right\|_0 = O\left(h^2+\tau\right). \tag{3.19}$$

On the other hand, taking $v_h = \partial_t\theta^n$ in (3.8), we obtain

$$
\begin{aligned}
& i\big(\partial_t\theta^n, \partial_t\theta^n\big) - \big(\nabla\theta^n, \nabla\partial_t\theta^n\big)_h \\
&\quad = -\lambda\big(f\big(\big|u^n\big|^2\big)u^n - f\big(\big|U^n\big|^2\big)U^n, \partial_t\theta^n\big) \\
&\qquad - i\big(\partial_t\rho^n, \partial_t\theta^n\big) + i\big(R_1^n, \partial_t\theta^n\big) - \sum_{K\in T_h}\int_{\partial K}\frac{\partial u}{\partial n}\cdot\overline{\partial_t\theta^n}\,ds.
\end{aligned}
\tag{3.20}
$$

Comparing the real parts of (3.20), we get

$$
\begin{aligned}
& \frac{1}{\tau}\operatorname{Re}\big(\nabla\theta^n, \nabla\theta^n - \nabla\theta^{n-1}\big)_h \\
&\quad = \frac{1}{2\tau}\big(\big\|\nabla\theta^n\big\|_0^2 - \big\|\nabla\theta^{n-1}\big\|_0^2 + \big\|\nabla\theta^n - \nabla\theta^{n-1}\big\|_0^2\big) \\
&\quad = -\operatorname{Im}\big(\partial_t\rho^n, \partial_t\theta^n\big) + \operatorname{Re}\lambda\big(f\big(\big|u^n\big|^2\big)u^n - f\big(\big|U^n\big|^2\big)U^n, \partial_t\theta^n\big) \\
&\qquad + \operatorname{Im}\big(R_1^n, \partial_t\theta^n\big) + \operatorname{Re}\sum_{K\in T_h}\int_{\partial K}\frac{\partial u^n}{\partial n}\cdot\overline{\partial_t\theta^n}\,ds,
\end{aligned}
\tag{3.21}
$$

which implies

$$
\begin{aligned}
& \big\|\theta^n\big\|_h^2 - \big\|\theta^{n-1}\big\|_h^2 \\
&\quad \le -2\tau\operatorname{Im}\big(\partial_t\rho^n, \partial_t\theta^n\big) + 2\tau\operatorname{Re}\lambda\big(f\big(\big|u^n\big|^2\big)u^n - f\big(\big|U^n\big|^2\big)U^n, \partial_t\theta^n\big) \\
&\qquad + 2\tau\operatorname{Im}\big(R_1^n, \partial_t\theta^n\big) + 2\tau\operatorname{Re}\sum_{K\in T_h}\int_{\partial K}\frac{\partial u^n}{\partial n}\cdot\overline{\partial_t\theta^n}\,ds \\
&\quad \le C\tau h^4\bigg(\big\|u^n\big\|_4^2 + \int_{t_{n-1}}^{t_n}|u_t|_2^2\,dt\bigg) + C\tau^2\int_{t_{n-1}}^{t_n}\|u_{tt}\|_0^2\,dt + C\tau\big\|\partial_t\theta^n\big\|_0^2.
\end{aligned}
\tag{3.22}
$$

Now we estimate the term $\|\partial_t\theta^n\|_0^2$. To do this, take difference between two time levels n and $n-1$ of (3.8) and multiply by $\frac{1}{\tau}$ on both sides, then set $v_h = \partial_t\theta^n$ to get

$$
\begin{aligned}
& i\big(\partial_t\big(\partial_t\theta^n\big), \partial_t\theta^n\big) - \big(\nabla\partial_t\theta^n, \nabla\partial_t\theta^n\big) \\
&\quad = -\lambda\big(\partial_t M^n, \partial_t\theta^n\big) - i\big(\partial_t\big(\partial_t\rho^n\big), \partial_t\theta^n\big) \\
&\qquad + i\big(\partial_t R_1^n, \partial_t\theta^n\big) - \sum_{K\in T_h}\int_{\partial K}\frac{\partial(\partial_t u^n)}{\partial n}\cdot\overline{\partial_t\theta^n}\,ds,
\end{aligned}
\tag{3.23}
$$

where $M^n = f(|u^n|^2)u^n - f(|U^n|^2)U^n$.

From [15], we know that

$$
\big\|\partial_t M^n\big\|_0^2 \le C\big(\big\|\partial_t\theta^n\big\|_0^2 + \big\|\partial_t\eta^n\big\|_0^2 + \big\|\eta^{n-1}\big\|_0^2 + \big\|\theta^{n-1}\big\|_0^2\big),
\tag{3.24}
$$

$$
\big\|\partial_t\big(\partial_t\rho^n\big)\big\|_0^2 \le \frac{Ch^4}{\tau}\int_{t_{n-2}}^{t_n}|u_{tt}|_2^2\,dt,
\tag{3.25}
$$

$$
\big\|\partial_t R_1^n\big\|_0^2 \le C\tau\int_{t_{n-2}}^{t_n}|u_{ttt}|_0^2\,dt.
\tag{3.26}
$$

Further, we have by Lemma 2.2 that

$$\left|\sum_{K\in T_h}\int_{\partial K}\frac{\partial(\partial_t u^n)}{\partial n}\cdot\overline{\partial_t\theta^n}\,ds\right|\le Ch^2\left|\partial_t u^n\right|_4\left\|\partial_t\theta^n\right\|_0 \\ \le Ch^4\left(\left|\partial_t u^n\right|_4^2+\left\|\partial_t\theta^n\right\|_0^2\right) \\ \le Ch^4\left(\frac{1}{\tau}\int_{t_{n-1}}^{t_n}\|u_t\|_4^2\,dt+\left\|\partial_t\theta^n\right\|_0^2\right). \tag{3.27}$$

Comparing the imaginary part of (3.23) with the above estimations, we obtain

$$\frac{\|\partial_t\theta^n\|_0^2-\|\partial_t\theta^{n-1}\|_0^2}{2\tau} \\ \le\frac{Ch^4}{\tau}\int_{t_{n-2}}^{t_n}|u_{tt}|_2^2\,dt+C\tau\int_{t_{n-2}}^{t_n}|u_{ttt}|_0^2\,dt+\frac{Ch^4}{\tau}\int_{t_{n-1}}^{t_n}\|u_t\|_4^2\,dt \\ +\left\|\eta^{n-1}\right\|_0^2+\left\|\theta^{n-1}\right\|_0^2+C\left\|\partial_t\theta^n\right\|_0^2. \tag{3.28}$$

Summing (3.28) up with respect to n leads to

$$\left\|\partial_t\theta^n\right\|_0^2\le Ch^4\int_0^T\left(|u_{tt}|_2^2+\|u_t\|_4^2\right)dt+C\tau^2\int_0^T|u_{ttt}|_0^2\,dt \\ +C\tau\sum_{i=2}^n\left(\left\|\eta^{i-1}\right\|_0^2+\left\|\theta^{i-1}\right\|_0^2\right)+C\sum_{i=2}^n\left\|\partial_t\theta^i\right\|_0^2+\left\|\partial_t\theta^1\right\|_0^2. \tag{3.29}$$

Setting $n=1$ and taking $v_h=\partial_t\theta^1$ in (3.8), with an argument similar to (3.19), we can derive that

$$\left\|\partial_t\theta^1\right\|_0\le C\left(h^2+\tau\right), \tag{3.30}$$

which together with (3.19), (3.29), and (3.30) gives

$$\left\|\partial_t\theta^n\right\|_0\le C\left(h^2+\tau\right). \tag{3.31}$$

Substituting (3.31) into (3.22) and summing up from 1 to n yields

$$\left\|\theta^n\right\|_h=O\left(h^2+\tau\right). \tag{3.32}$$

Thus the proof is complete. □

Now we will introduce a proper interpolation postprocessing operator to get the global superconvergence result. For this purpose, we further assume that T_h has been obtained from T_{2h} by dividing each element into four congruent rectangles. Let $\mathcal{T}=\bigcup_{i=1}^4 K_i$, L_1, L_2, L_3, and L_4 be four edges. As in [31, 38], we define the interpolation operator Π_{2h} on the partition T_{2h}:

$$\begin{cases}\Pi_{2h}u|_{\mathcal{T}}\in P_2(\mathcal{T}), & \forall\mathcal{T}\in T_{2h},\\ \int_{L_i}(\Pi_{2h}u-u)\,ds=0, & i=1,2,3,4,\\ \int_{K_1\cup K_3}(\Pi_{2h}u-u)\,dx\,dy=0, & \int_{K_2\cup K_4}(\Pi_{2h}u-u)\,dx\,dy=0,\end{cases}$$

where $P_2(\mathcal{T})$ denotes the set of polynomials of degree 2.

It has been shown in [38] that the interpolation operator Π_{2h} defined above satisfies the following properties:

$$\Pi_{2h}\Pi_h u = \Pi_{2h}u, \qquad \|u - \Pi_{2h}u\|_h \le Ch^r|u|_{r+1,\Omega}, \quad 0 \le r \le 2, \tag{3.33}$$

$$\|\Pi_{2h}v\|_h \le C\|v\|_h, \quad \forall v \in V_h. \tag{3.34}$$

Theorem 3.2 *Under the same assumptions of Theorem 3.1, we have*

$$\|u^n - \Pi_{2h}U^n\|_h = O(h^2 + \tau). \tag{3.35}$$

Proof Noticing that

$$u^n - \Pi_{2h}U^n = u^n - \Pi_{2h}\Pi_h u^n + \Pi_{2h}\Pi_h u^n - \Pi_{2h}U^n, \tag{3.36}$$

by (3.33) and interpolation error estimates, we have

$$\|u^n - \Pi_{2h}\Pi_h U^n\|_h = \|u^n - \Pi_{2h}u^n\|_h \le Ch^2|u^n|_3. \tag{3.37}$$

Consequently, it follows from (3.34) and Theorem 3.1 that

$$\begin{aligned}\|\Pi_{2h}\Pi_h u^n - \Pi_{2h}U^n\|_h &= \|\Pi_{2h}(\Pi_h u^n - U^n)\|_h \\ &\le \|\Pi_h u^n - U^n\|_h = O(h^2 + \tau).\end{aligned} \tag{3.38}$$

From (3.36)–(3.38), we can derive the result (3.35) directly. □

Remark 3.1 Theorems 3.1–3.2 are also valid to the Q_1^{rot} element [39] on square meshes.

4 The two-grid finite element scheme and error analysis

In this section, we design a two-grid finite element algorithm (see Algorithm 4.1 below) for problem (1.1) to reduce the computing cost. The idea of the two-grid method is to reduce the nonlinear problem on a fine grid into a linear system through solving a nonlinear problem on a coarse grid. T_H and T_h are two regular subdivisions of Ω with two different mesh sizes H and h ($h \ll H$), and the corresponding EQ_1^{rot} finite element spaces V_H and V_h (which will be called the coarse-grid space and the fine-grid space), respectively.

Algorithm 4.1

Step 1. Find $u_H^n \in V_H$ ($n = 1, 2, \dots, N$) such that

$$\begin{cases} i(\partial_t u_H^n, v_H) - (\nabla u_H^n, \nabla v_H) + \lambda(f(|u_H^n|^2)u_H^n, v_H) = 0, & v_H \in V_H, \\ u_H^0 = \Pi_h u_0(X) \in V_H, & X \in \Omega. \end{cases} \tag{4.1}$$

Step 2. Find $u_h^n \in V_h$ ($n = 1, 2, \dots, N$) such that

$$\begin{cases} i(\partial_t u_h^n, v_h) - (\nabla u_h^n, \nabla v_h) + \lambda(\widetilde{f(|u_H^n|^2)}u_h^n, v_h) = 0, & v_h \in V_h, \\ u_h^0 = \Pi_h u_0(X) \in V_h, & X \in \Omega, \end{cases} \tag{4.2}$$

where $\widetilde{f(|u_H^n|^2)} = f(|u_H^n|^2) + f'(|u_H^n|^2)(|u_h^{n-1}|^2 - |u_H^n|^2)$.

Now we consider the error estimates in the broken H^1-norm for Algorithm 4.1.

Theorem 4.1 *Let u and u_h^n be the solutions of problem (1.1) and the two-grid Algorithm 4.1, respectively. If $u \in H^4(\Omega)\cap W^{2,\infty}(\Omega)\cap H_0^1(\Omega)$, $u_t \in H^4(\Omega)$, $u_{tt} \in H^2(\Omega)$, and $u_{ttt} \in L^2(\Omega)$, there holds*

$$\|\theta^n\|_0 + \|\theta^n\|_h = O\bigl[h^2 + \tau + H^4|\ln H|^{\frac{1}{2}}\bigr]. \tag{4.3}$$

Proof From (1.1) and (4.2), similar to (3.11), we have the result

$$\begin{aligned} &\|\theta^n\|_0^2 - \|\theta^{n-1}\|_0^2 \\ &\quad = -2\tau\,\mathrm{Re}\bigl(\partial_t\rho^n,\theta^n\bigr) - \mathrm{Im}\,\lambda\bigl(f\bigl(|u^n|^2\bigr)u_h^n - f\bigl(|u_H^n|^2\bigr)u_h^n,\theta^n\bigr) \\ &\qquad + 2\tau\,\mathrm{Re}\bigl(R_1^n,\theta^n\bigr) - 2\tau\,\mathrm{Im}\sum_{K\in T_h}\int_{\partial K}\frac{\partial u^n}{\partial n}\cdot\overline{\theta^n}\,ds \triangleq \sum_{i=1}^{4}M_i. \end{aligned} \tag{4.4}$$

We only need to estimate the term M_2. In fact, by using the continuity of $f(s)$ and Taylor expansions, we have

$$f\bigl(|u^n|^2\bigr) = f\bigl(|u_H^n|^2\bigr) + f'\bigl(|u_H^n|^2\bigr)\bigl(|u^n|^2 - |u_H^n|^2\bigr) + \frac{f''(\xi_2)}{2!}\bigl(|u^n|^2 - |u_H^n|^2\bigr)^2, \tag{4.5}$$

where ξ_1 lies between $|u^n|^2$ and $|u_H^n|^2$.

Then M_2 can be expressed as

$$\begin{aligned} |M_2| &\le 2\tau|\lambda|\bigl|\bigl(f\bigl(|u^n|^2\bigr)u^n - \widetilde{f\bigl(|u_H^n|^2\bigr)}u_h^n,\theta^n\bigr)\bigr| \\ &= 2\tau|\lambda|\bigl|\bigl(f\bigl(|u^n|^2\bigr)u^n - \widetilde{f\bigl(|u_H^n|^2\bigr)}u^n + \widetilde{f\bigl(|u_H^n|^2\bigr)}u^n - \widetilde{f\bigl(|u_H^n|^2\bigr)}u_h^n,\theta^n\bigr)\bigr| \\ &= 2\tau|\lambda|\bigl|\bigl(f\bigl(|u^n|^2\bigr)u^n - \widetilde{f\bigl(|u_H^n|^2\bigr)}u^n,\theta^n\bigr)\bigr| \\ &\quad + 2\tau|\lambda|\bigl|\bigl(\widetilde{f\bigl(|u_H^n|^2\bigr)}\bigl(u^n - u_h^n\bigr),\theta^n\bigr)\bigr| \triangleq E_1 + E_2. \end{aligned} \tag{4.6}$$

Firstly, for the term E_1, we have

$$\begin{aligned} E_1 &= 2\tau|\lambda|\Bigl|\Bigl(\Bigl[f'\bigl(|u_H^n|^2\bigr)\bigl(|u^n|^2 - |u_h^{n-1}|^2\bigr) + \frac{f''(|\xi_2|^2)}{2}\bigl(|u^n|^2 - |u_H^n|^2\bigr)^2\Bigr]u^n,\theta^n\Bigr)\Bigr| \\ &= 2\tau|\lambda|\bigl|\bigl(f'\bigl(|u_H^n|^2\bigr)\bigl(|u^n|^2 - |u^{n-1}|^2\bigr)u^n,\theta^n\bigr)\bigr| \\ &\quad + 2\tau|\lambda|\bigl|\bigl(f'\bigl(|u_H^n|^2\bigr)\bigl(|u^{n-1}|^2 - |u_h^{n-1}|^2\bigr)u^n,\theta^n\bigr)\bigr| \\ &\quad + \tau|\lambda|\bigl|\bigl(f''\bigl(|\xi_2|^2\bigr)\bigl(|u^n|^2 - |u_H^n|^2\bigr)^2u^n,\theta^n\bigr)\bigr| \\ &\triangleq E_{11} + E_{12} + E_{13}. \end{aligned} \tag{4.7}$$

Applying the boundedness of $u, f(s)$, and Theorem 3.1, we can derive that

$$E_{11} + E_{12} \le C\tau\bigl(h^4 + \tau^2\bigr). \tag{4.8}$$

Similar to (3.13), E_{13} can be estimated as

$$E_{13} \le C\tau\left(\left\|\left(\left|u^n\right|^2 - \left|u_H^n\right|^2\right)^2\right\|_0^2 + \left\|\theta^n\right\|_0^2\right). \tag{4.9}$$

Notice that the term $\|(|u^n|^2 - |u_H^n|^2)^2\|_0^2$ can be rewritten as

$$\begin{aligned}\left\|\left(\left|u^n\right|^2 - \left|u_H^n\right|^2\right)^2\right\|_0^2 &\le C\left\|\left(\left|u^n\right|^2 - \left|u_H^n\right|^2\right)^2\right\|_0^2 \le C\left\|u^n - u_H^n\right\|_{0,\infty}^2\left\|u^n - u_H^n\right\|_0^2 \\ &\le \left(\left\|u^n - \Pi_H u^n\right\|_{0,\infty}^2 + \left\|\Pi_H u^n - u_H^n\right\|_{0,\infty}^2\right)\left\|u^n - u_H^n\right\|_0^2.\end{aligned} \tag{4.10}$$

Since

$$\left\|u^n - \Pi_H u^n\right\|_{0,\infty}^2 \le CH^4\left\|u^n\right\|_{2,\infty}^2, \tag{4.11}$$

$$\left\|u^n - u_H^n\right\|_0^2 \le C\left(H^4 + \tau^2\right), \tag{4.12}$$

and $\|v_h\|_{0,\infty} \le C|\ln h|^{\frac{1}{2}}\|v_h\|_h$ [40], we have

$$\left\|\Pi_H u^n - u_H^n\right\|_{0,\infty}^2 \le C|\ln H|\left\|\Pi_H u^n - u_H^n\right\|_h^2 \le C|\ln H|\left(H^4 + \tau^2\right). \tag{4.13}$$

Then from (4.10)–(4.13), we know that

$$\left\|\left(\left|u^n\right|^2 - \left|u_H^n\right|^2\right)^2\right\|_0^2 \le C\left[H^4 + \tau^2 + |\ln H|\left(H^4 + \tau^2\right)\right]\left(H^4 + \tau^2\right). \tag{4.14}$$

Further, when τ is small enough, there holds

$$\left\|\left(\left|u^n\right|^2 - \left|u_H^n\right|^2\right)^2\right\|_0^2 \le CH^8|\ln H|, \tag{4.15}$$

which implies

$$E_1 \le C\tau\left(h^4 + \tau^2 + H^8|\ln H|\right). \tag{4.16}$$

Secondly, for the term E_2, we have

$$E_2 \le C\tau\left(\left\|u^n - u_h^n\right\|_0^2 + \left\|\theta^n\right\|_0^2\right) \le C\tau\left(h^4 + \tau^2\right). \tag{4.17}$$

Finally, substituting (4.16) and (4.17) into (4.6), we obtain

$$|M_2| \le C\tau\left(h^4 + \tau^2 + H^8|\ln H|\right), \tag{4.18}$$

and substituting (3.12), (3.14), (3.15), and (4.18) into (4.4) yields

$$\left\|\theta^n\right\|_0^2 - \left\|\theta^{n-1}\right\|_0^2 \le C\tau\left(h^4 + \tau^2 + H^8|\ln H|\right) + C\tau\left\|\theta^n\right\|_0^2. \tag{4.19}$$

Then summing (4.19) up with respect to n and noting $\theta^0 = 0$, we have

$$\left\|\theta^n\right\|_0^2 \le C\left(h^4 + \tau^2 + H^8|\ln H|\right) + C\tau\sum_{i=1}^{n}\left\|\theta^i\right\|_0^2. \tag{4.20}$$

An application of Gronwall's lemma yields

$$\|\theta^n\|_0^2 \le C\big(h^4 + \tau^2 + H^8|\ln H|\big), \tag{4.21}$$

which implies that

$$\|\theta^n\|_0 = O\big(h^2 + \tau + H^4|\ln H|^{\frac{1}{2}}\big). \tag{4.22}$$

Thus with the similar arguments to the estimates of (3.32) and (4.22), we can also derive

$$\|\theta^n\|_h = O\big(h^2 + \tau + H^4|\ln H|^{\frac{1}{2}}\big), \tag{4.23}$$

which is the desired result. □

Similar to the proof of Theorem 3.2, we can derive the following superconvergence results.

Theorem 4.2 *Under the same assumptions of Theorem 4.1, and setting* $h = H^2(|\ln H|)^{\frac{1}{4}}$, *we can derive*

$$\|u^n - \Pi_{2h}u_h^n\|_h = O\big(h^2 + \tau\big). \tag{4.24}$$

5 Numerical experiment

In this section, we present the following numerical example to confirm the theoretical analysis, which comes from [8, 16, 41].

Consider the cubic-quintic Schrödinger equation ($f(s) = -s + s^2$)

$$\begin{cases} iu_t + \Delta u - |u|^2u + |u|^4u = g, & (X,t) \in \Omega \times (0,T], \\ u(X,t) = 0, & (X,t) \in \partial\Omega \times (0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \end{cases} \tag{5.1}$$

on $\Omega = [0,1]^2$ with the exact solution

$$u = e^{it+(x+y)/2}\big(1+3t^2\big)x(1-x)y(1-y),$$

where g is given corresponding to the exact solution u.

The domain Ω is divided into families T_H and T_h of rectangular meshes, and V_H, V_h are EQ_1^{rot} finite element spaces defined on T_H, T_h, respectively. In such a way, to obtain enough accuracy, it suffices to take $h = O(H^2|\ln H|^{\frac{1}{4}})$ in both the broken H^1-norm and L^2-norm. Under the same condition of computing environment and control strategy, the numerical results and CPU times are shown in Tables 1–2, respectively. The exact solution u at time $t = 1$ and FEM solution u_h at time $t = 1$ with mesh size $h = 1/32$ are pictured in Figs. 1 and 2, respectively.

It can be seen from Tables 1–2 that $\|u - u_h\|_h$ is convergence at rate of $O(h)$, $\|u - u_h\|_0$, $\|\Pi_h u - u_h\|_h$ and $\|u - \Pi_{2h}u\|_h$ are convergence at rate of $O(h^2)$, which coincide with the theoretical analysis. Thus the two-grid FEM is more efficient in solving NLSE than the usual Galerkin FEM.

Table 1 Numerical results of two-grid Algorithm 4.1 at $t = 1$ ($\tau = h^2$, $h \approx H^2$)

	$h = \frac{1}{16}, H = \frac{1}{4}$	$h = \frac{1}{32}, H = \frac{1}{6}$	$h = \frac{1}{64}, H = \frac{1}{8}$	Ratio
$\|u - u_h\|_h$	0.058942538	0.029468965	0.014736033	0.99
$\|u - u_h\|_0$	0.001096657	0.000281836	0.000075823	1.93
$\|\Pi_h u - u_h\|_h$	0.003761590	0.000980134	0.000264975	1.91
$\|u - \Pi_{2h} u_h\|_h$	0.018173033	0.004582958	0.001179143	1.97
CPU time (s)	54.06	2973.62	237,992.52	

Table 2 Numerical results of the usual Galerkin FEM at $t = 1$ ($\tau = h^2$)

	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$	Ratio
$\|u - u_h\|_h$	0.058940220	0.029468644	0.014523013	1.01
$\|u - u_h\|_0$	0.001090311	0.000280127	0.000072122	1.96
$\|\Pi_h u - u_h\|_h$	0.003725084	0.000970437	0.000253940	1.94
$\|u - \Pi_{2h} u_h\|_h$	0.018165504	0.004580893	0.001067421	2.04
CPU time (s)	68.88	5178.21	416,552.35	

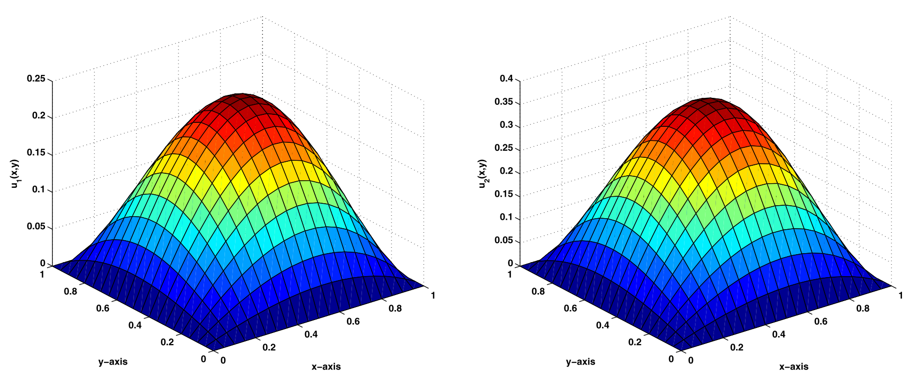

Figure 1 The exact solutions u_1 (Real part) and u_2 (Imaginary part) at $t = 1$

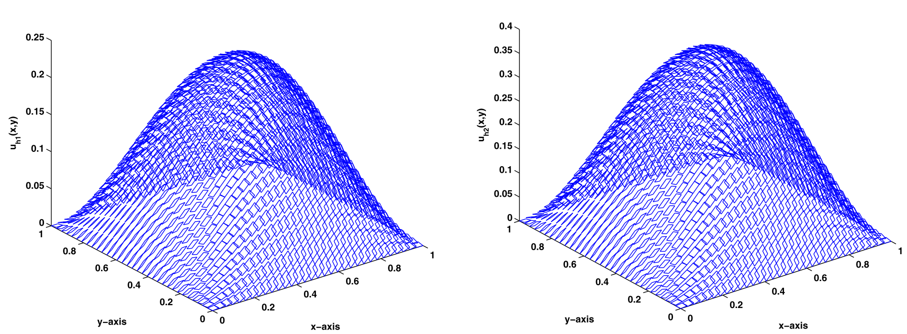

Figure 2 The FEM solutions u_{h1} (Real part) and u_{h2} (Imaginary part) at $t = 1$

6 Conclusions

In this paper, we applied low order nonconforming EQ_1^{rot} finite element to solve the nonlinear Schrödinger equation, and derived the global superconvergence results for the backward Euler fully-discrete scheme and a type of two-grid scheme, respectively. A numerical example is presented to demonstrate the theoretical results. The method presented in this paper is suitable for the standard Galerkin finite element and can be extended to dealing with other nonlinear problems.

Acknowledgements
We are thankful to the editor and the anonymous reviewers for many valuable suggestions to improve this paper.

Funding
The research is supported by the National Natural Science Foundation of China (Nos. 11271340, 11671369), the Educational Commission of Henan Province of China (No. 14B110025).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The study was carried out in collaboration among all authors. CX and JQZ carried out the main theorem and wrote the paper together; DYS revised and checked the paper; HCZ checked the article. All authors read and approved the final manuscript.

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Publisher's Note

Received: 7 April 2018 Accepted: 5 November 2018 Published online: 16 November 2018

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