Niu et al. *Boundary Value Problems* (2018) 2018:58
https://doi.org/10.1186/s13661-018-0976-0

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Pattern formation for a nonlinear diffusion chemotaxis model with logistic source

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Abstract

This paper deals with a Neumann boundary value problem in a d-dimensional box $\mathbb{T}^d = (0,\pi)^d$ ($d = 1,2,3$) for a nonlinear diffusion chemotaxis model with logistic source. By using the embedding theorem, the higher-order energy estimates and bootstrap arguments, the condition of chemotaxis-driven instability and the nonlinear evolution near an unstable positive constant equilibrium for this chemotaxis model are proved. Our result provides a quantitative characterization for early spatial pattern formation on the positive constant equilibrium. Finally, numerical simulations are carried out to support our theoretical nonlinear instability results.

MSC: 35K57; 92C17; 92C15

Keywords: Chemotaxis; Nonlinear diffusion; Pattern formation; Nonlinear instability

1 Introduction

Chemotaxis models take into account a diffusion term in the cell dynamics to model an undirected or random component to movement. In general, a constant diffusion coefficient is assumed in many applications. However, using a nonlinear dependence on the cell density in research of cell movement has also proliferated in recent years, it can be utilized in ecological applications to describe "population-induced" movement for insect populations. In this case when possibly also cell proliferation is included, this leads to models of form

$$\begin{cases} U_t = d_1\nabla\cdot(D(U)\nabla U) - \chi\nabla\cdot(U\nabla V) + f(U), & x\in\Omega, t>0,\\ \tau V_t = d_2\Delta V + \alpha U - \beta V, & x\in\Omega, t>0,\\ \frac{\partial U}{\partial x_i} = \frac{\partial V}{\partial x_i} = 0, & x\in\partial\Omega, t>0,\\ U(x,0) = U_0(x)\geq 0, \qquad V(x,0) = V_0(x)\geq 0, & x\in\Omega, \end{cases} \tag{1.1}$$

where $\chi > 0$ is the chemotactic sensitivity, and $U(x,t)$, $V(x,t)$ denote the density of the cells population and the concentration of the chemoattractant, respectively. $d_1, d_2 > 0$ are the cell and chemical diffusion coefficients, respectively. The function $D(U)$ describes the density-dependent motility of cells, and $f(U)$ represents the proliferation rate of the cells. The term $\alpha U - \beta V$ asserts that the chemical has a linear production and degradation.

For model (1.1) without the growth term $f(U)$, when $\tau = 1$, Höfer et al. in [1] introduced a phenomenological description of cell-cell adhesion in a model for *Dictyostelium*

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discoideum aggregation by considering $D(U) = \mu_1 + \frac{\mu_2 N^4}{N^4+U^4}$ where N is a critical cell density. For the case $D(U) = U^n$, $n \geq 0$, Kowalczyk [2] obtained blow-up control relies on the presence of a pressure function, which increases faster than a logarithm for high enough cells densities: for such a pressure function the solutions cannot blow-up in a finite time, and proved the global boundedness for (1.1) under some other conditions. However, Hillen and Painter in [3] also reviewed its formulation from a biological perspective, and showed necessary conditions for instability by a standard linear analysis at the homogeneous steady state of the model (1.1). Senba and Suzuki [4] described that if the positive function $D(U)$ rapidly increases with respect to U, then solutions to the system (1.1) exist globally in time. When the diffusion function $D(U)$ takes values sufficiently large, i.e. takes values greater than the values of a power function with sufficiently high power, in [5], Kowalczyk and Szymańska proved that global-in-time existence of weak solutions. In addition, the uniqueness of solutions was given provided that some higher regularity condition on solutions is known a priori.

When $\tau = 0$, for the model (1.1) with logistic source, in [6] it is shown that under some suitable assumptions, then there exist initial data such that the smooth local-in-time solution of a higher-dimensional chemotaxis system blows up in finite time. Cao and Zheng [7] considered the boundedness for simplified the model (1.1) with the special case $D(U) \geq c(U+1)^p$, $p \in \mathbb{R}$. For the case $D(U) \geq C_D U^{m-1}$, $m \geq 1$, Wang et al. [8] proved that, for the case of positive diffusion function, the model (1.1) possesses a unique global classical solution which is uniformly bounded, and that the weak solutions are global existence if the diffusion function is zero at some point, or a positive diffusion function and the logistic damping effect is rather mild. Moreover, they asserted that the solutions approach constant equilibria in the large time for a specific case of the logistic source. In [9], for the case of $D(U) = (U+1)^p$, $p \geq 0$, Zheng et al. studied the global boundedness and finite-time blow-up of solutions for a chemotaxis system with generalized volume-filling effect and logistic source $f(U) = \lambda U - \mu U^k$, with $\lambda \geq 0$, $\mu > 0$ and $k > 1$. When $D(U) = 1$, Zheng et al. [10] considered the global boundedness of solutions in a chemotaxis system with nonlinear sensitivity and logistic source.

If $D(U) \equiv 1$ and $f(U) = 0$, then the model (1.1) reduces to the classical Keller–Segel system. This model attracted a lot of attention in the mathematical literature (refer to [11–14]). The first nonlinear instability result is due to Guo et al. in [15] who investigated nonlinear dynamics near an unstable constant equilibrium in the classical Keller–Segel model. Their results can be interpreted as a rigorous quantitative characterization for the early-stage pattern formation in the Keller–Segel model. Recently, Fu et al. in [16] and [17] studied nonlinear instability in the Keller–Segel model with a logistic source and cubic source term, respectively. Their results indicated that chemotaxis-driven nonlinear instability occurs in these models.

Motivated by the arguments in [15–17], it is the goal of the present paper to investigate nonlinear evolution of pattern formation for the following nonlinear diffusion chemotaxis model with logistic source under homogeneous Neumann boundary conditions:

$$\begin{cases} U_t = d_1 \nabla \cdot (U^n \nabla U) - \chi \nabla \cdot (U \nabla V) + rU(1 - \frac{U}{k}), & x \in \mathbb{T}^d, t > 0, \\ V_t = d_2 \Delta V + \alpha U - \beta V, & x \in \mathbb{T}^d, t > 0, \\ \frac{\partial U}{\partial x_i} = \frac{\partial V}{\partial x_i} = 0, & \text{at } x_i = 0, \pi, 1 \leq i \leq d, t > 0, \\ U(x,0) = U_0(x) \geq 0, \qquad V(x,0) = V_0(x) \geq 0, & x \in \mathbb{T}^d, \end{cases} \tag{1.2}$$

where $\mathbb{T}^d = (0,\pi)^d$ ($d = 1,2,3$) is a d-dimensional box. The nonlinear diffusion is of the form $D(U) = U^n$ for $n \geq 0$, as studied by Kowalczyk [2], which means that the rate of diffusion increases with increasing cell density. Eberl [18] also used this formulation in a model for biofilm growth.

Yang et al. in [19] studied the global boundedness of solutions to (1.2) in the higher-dimension ($d \geq 2$). To the best of our knowledge, however, it is still open mathematically whether there exists an unstable solutions to the linearized problem of (1.2) and the nonlinear problem (1.2). Particularly, nonlinear instability remained open. The proof of the nonlinear instability based on unstable eigenvalues is nontrivial for several reasons. The main difficulty is that the nonlinear term $d_1\nabla\cdot[(\sum_{i=1}^{n} C_n^i k^{n-i}u^i)\nabla u]$. The technical key of our work is controlling the nonlinear growth of higher-order energy norm for the perturbation by the linear growth rate.

To avoid excessive technicalities, let $n \in \mathbb{Z}^+$ throughout this paper. Our main result (see Theorem 5.1) indicates that chemotaxis-driven nonlinear instability occurs in the model (1.2), that is, nonlinear patterns are created by chemotaxis for the model (1.2) with nonlinear diffusion.

The organization of this paper is as follows: In Sect. 2, we study local stability of positive constant equilibrium point for the model (1.2) without chemotaxis. In Sect. 3, we consider the growing modes of (1.2). In Sect. 4, the bootstrap lemma is established. In Sect. 5, we show that, given any general initial perturbation, its nonlinear evolution is dominated by the corresponding linear dynamics along a fixed finite number of fastest growing modes. In the last section we draw some conclusions and carry out simple numerical simulations for this model.

2 Analysis of local stability

We consider the following PDE system (1.2) without chemotaxis:

$$\begin{cases} U_t = d_1\nabla\cdot(U^n\nabla U) + rU(1-\frac{U}{k}), & x\in\mathbb{T}^d, t>0,\\ V_t = d_2\Delta V + \alpha U - \beta V, & x\in\mathbb{T}^d, t>0,\\ \frac{\partial U}{\partial x_i} = \frac{\partial V}{\partial x_i} = 0, & \text{at } x_i = 0,\pi, 1\leq i\leq d, t>0,\\ U(x,0) = U_0(x)\geq 0, \qquad V(x,0) = V_0(x)\geq 0, & x\in\mathbb{T}^d. \end{cases} \tag{2.1}$$

We use $[\cdot,\cdot]$ to denote a column vector. It is clear to see that the trivial equilibrium point $\mathbf{E}_0 = [0,0]$ is unconditionally unstable, and that $\bar{\mathbf{W}} = [\bar{U},\bar{V}] = [k,\frac{\alpha}{\beta}k]$ is the unique positive equilibrium solution.

Let $0 = \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ on $\mathbb{T}^d$ ($d = 1,2,3$) with the homogeneous Neumann boundary condition, and $E(\mu_i)$ be the eigenspace corresponding to μ_i in $H^1(\mathbb{T}^d)$. Let $X = [H^1(\mathbb{T}^d)]^2$, $\{\phi_{ij} : j = 1,\ldots,\dim E(\mu_i)\}$ be an orthonormal basis of $E(\mu_i)$, and $X_{ij} = \{\mathbf{c}\cdot\phi_{ij} \mid \mathbf{c}\in\mathbb{R}^2\}$. Then $X = \bigoplus_{i=1}^{\infty} X_i$, $X_i = \bigoplus_{j=1}^{\dim E(\mu_i)} X_{ij}$.

Denote $\mathcal{D} = \operatorname{diag}(d_1k^n, d_2)$, $\mathbf{F}(\mathbf{W}) = [rU(1-\frac{U}{k}), \alpha U - \beta V]$ and $\mathcal{L} = \mathcal{D}\Delta + \mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}})$, where

$$\mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}}) = \begin{pmatrix} -r & 0\\ \alpha & -\beta \end{pmatrix}.$$

The linearization of (2.1) at $[\bar{U},\bar{V}]$ is $\mathbf{W}_t = \mathcal{L}(\mathbf{W}-\bar{\mathbf{W}})$. For each $i\geq 1$, X_i is invariant under the operator $\mathcal{L}$, and λ is an eigenvalue of $\mathcal{L}$ on X_i if and only if it is an eigenvalue of the matrix

$$-\mu_i\mathcal{D} + \mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}}) = \begin{pmatrix} -d_1k^n\mu_i - r & 0 \\ \alpha & -d_2\mu_i - \beta \end{pmatrix}.$$

Then $-\mu_i\mathcal{D} + \mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}})$ has two negative eigenvalues $-d_1k^n\mu_i$ and $-d_2\mu_i - 1$. Hence, the positive equilibrium point $[\bar{U}, \bar{V}]$ of (2.1) is locally asymptotically stable (see [20]).

Remark 2.1 The above result indicates that the Turing instability does not occur in the absence of chemotactic effect.

3 L^2-Estimate

Let $u(\mathbf{x},t) = U(\mathbf{x},t) - \bar{U}$, $v(\mathbf{x},t) = V(\mathbf{x},t) - \bar{V}$. Then

$$\begin{cases} u_t = d_1k^n\nabla^2u - \chi k\nabla^2 v + d_1\nabla[(\sum_{i=1}^n C_n^i k^{n-i}u^i)\nabla u] \\ \qquad - \chi\nabla(u\nabla v) - ru - \frac{r}{k}u^2, \\ v_t = d_2\nabla^2 v + \alpha u - \beta v, \end{cases} \tag{3.1}$$

where $C_n^i = \frac{n!}{i!(n-i)!}$. The corresponding linearized system is as follows:

$$\begin{cases} u_t = d_1k^n\nabla^2u - \chi k\nabla^2 v - ru, \\ v_t = d_2\nabla^2 v + \alpha u - \beta v. \end{cases} \tag{3.2}$$

Let $\mathbf{w}(\mathbf{x},t) \equiv [u(\mathbf{x},t), v(\mathbf{x},t)]$, $\mathbf{q} = [q_1,\dots,q_d] \in \mathbb{N}^d \equiv \Omega$, and $e_{\mathbf{q}}(\mathbf{x}) = \prod_{i=1}^d \cos(q_ix_i)$. Then $\{e_{\mathbf{q}}(\mathbf{x})\}_{\mathbf{q}\in\Omega}$ forms a basis of the space of functions in $\mathbb{T}^d$ that satisfy Neumann boundary conditions.

We will find a normal mode to the linear reaction–diffusion system (3.2) of the following form:

$$\mathbf{w}(\mathbf{x},t) = \mathbf{r}_{\mathbf{q}}e^{\lambda_{\mathbf{q}}t}e_{\mathbf{q}}(\mathbf{x}), \tag{3.3}$$

where $\mathbf{r}_{\mathbf{q}}$ is a vector depending on $\mathbf{q}$. Substituting (3.3) into (3.2), it is easy to see the following dispersion formula for $\lambda_{\mathbf{q}}$:

$$\lambda_{\mathbf{q}}^2 + \left[(d_1k^n + d_2)q^2 + r + \beta\right]\lambda_{\mathbf{q}} + q^2\left(d_1d_2k^nq^2 + d_1k^n\beta + rd_2 - \alpha\chi k\right) + r\beta = 0, \tag{3.4}$$

where $q^2 = \sum_{i=1}^d q_i^2$. Thus, we derive a linear instability criterion by requiring that there exists a $\mathbf{q}$ such that

$$d_1d_2k^nq^4 + \left(d_1k^n\beta + rd_2 - \alpha\chi k\right)q^2 + r\beta < 0 \tag{3.5}$$

to ensure that (3.4) has at least one positive root $\lambda_{\mathbf{q}}$. It follows from (3.5) that

$$\left[(d_1k^n - d_2)q^2 + r - \beta\right]^2 + 4\alpha\chi kq^2 > 0. \tag{3.6}$$

There exist two distinct real roots

$$\lambda_{\mathbf{q}}^{\pm} = \frac{-[q^2(d_1k^n + d_2) + r + \beta] \pm \sqrt{[(d_1k^n - d_2)q^2 + r - \beta]^2 + 4\alpha\chi kq^2}}{2}. \tag{3.7}$$

The corresponding (linearly independent) eigenvectors by $\mathbf{r}_-(\mathbf{q})$ and $\mathbf{r}_+(\mathbf{q})$ are given by

$$\mathbf{r}_\pm(\mathbf{q}) = \left[\frac{\lambda_{\mathbf{q}}^{\pm} + d_2q^2 + \beta}{\alpha}, 1\right]. \tag{3.8}$$

Hence, there are only finitely many $\mathbf{q}$ such that $\lambda_{\mathbf{q}}^{+} > 0$. We denote the largest eigenvalue by $\lambda_{\max} > 0$ and define $\Omega_{\max} \equiv \{\mathbf{q} \in \Omega \mid \lambda_{\mathbf{q}}^{+} = \lambda_{\max}\}$. From (3.7) we can regard $\lambda_{\mathbf{q}}^{+}$ as a function of q^2. Moreover, there is one q^2 (possible two) having $\lambda^+(q^2) = \lambda_{\max}$.

Given any initial perturbation $\mathbf{w}(\mathbf{x}, 0)$, we can expand it as

$$\mathbf{w}(\mathbf{x}, 0) = \sum_{\mathbf{q}\in\Omega} \mathbf{w}_{\mathbf{q}} e_{\mathbf{q}}(\mathbf{x}) = \sum_{\mathbf{q}\in\Omega} \bigl\{w_{\mathbf{q}}^{-}\mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^{+}\mathbf{r}_+(\mathbf{q})\bigr\} e_{\mathbf{q}}(\mathbf{x}). \tag{3.9}$$

We denote by $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ the inner product of $[L^2(\mathbb{T}^d)]^2$ and the scalar product of $\mathbb{R}^2$, respectively. For any $\mathbf{g}(\cdot, t) \in [L^2(\mathbb{T}^d)]^2$, we denote $\|\mathbf{g}(\cdot, t)\| \equiv \|\mathbf{g}(\cdot, t)\|_{L^2}$. Throughout this paper, we always denote universal constants depending on d_1, d_2, χ, k, r, α, β by C_i ($i = 1, 2, \ldots$) and choose an appropriate constant

$$K = \frac{\chi^2}{d_1d_2k^{n-2}}. \tag{3.10}$$

The unique solution $\mathbf{w}(\mathbf{x}, t) = [u(\mathbf{x}, t), v(\mathbf{x}, t)]$ of (3.2) takes the form

$$\mathbf{w}(\mathbf{x}, t) = \sum_{\mathbf{q}\in\Omega} \bigl\{w_{\mathbf{q}}^{-}\mathbf{r}_-(\mathbf{q})e^{\lambda_{\mathbf{q}}^{-}t} + w_{\mathbf{q}}^{+}\mathbf{r}_+(\mathbf{q})e^{\lambda_{\mathbf{q}}^{+}t}\bigr\} e_{\mathbf{q}}(\mathbf{x}) \equiv e^{\mathfrak{L}t}\mathbf{w}(\mathbf{x}, 0). \tag{3.11}$$

Our main result in this section is the following lemma.

Lemma 3.1 *Let the instability criterion* (3.5) *hold. Let* $\mathbf{w}(\mathbf{x}, t) \equiv e^{\mathfrak{L}t}\mathbf{w}(\mathbf{x}, 0)$ *be a solution to the linearized system* (3.2) *with initial condition* $\mathbf{w}(\mathbf{x}, 0)$. *Then there exists a constant* $\hat{C}_1 \geq 1$ *depending on* d_1, d_2, k, χ, r, α, β, *such that*

$$\bigl\|\mathbf{w}(\cdot, t)\bigr\| \leq \hat{C}_1 e^{\lambda_{\max}t}\bigl\|\mathbf{w}(\cdot, 0)\bigr\|, \quad \forall t \geq 0. \tag{3.12}$$

Proof We prove the lemma in the following two cases.

(1) $t \geq 1$. From (3.7), for q large, it follows that

$$\lim_{q\to\infty} \frac{\lambda_{\mathbf{q}}^{\pm}}{q^2} = -d_1k^n, -d_2, \tag{3.13}$$

which leads to

$$\lambda_{\mathbf{q}}^{\pm} \leq -\min\bigl\{d_1k^n, d_2\bigr\}q^2. \tag{3.14}$$

Moreover, there exists positive constant C_1 for all $q > 0$ such that

$$\left|\frac{\lambda_{\mathbf{q}}^{\pm}}{q^2}\right| \le C_1. \tag{3.15}$$

By the quadratic formula of (3.4), we can obtain

$$\left|\lambda_{\mathbf{q}}^{+} - \lambda_{\mathbf{q}}^{-}\right| \ge 2q\sqrt{\alpha\chi k}. \tag{3.16}$$

It follows from $\mathbf{w_q} = w_{\mathbf{q}}^{-}\mathbf{r}_{-}(\mathbf{q}) + w_{\mathbf{q}}^{+}\mathbf{r}_{+}(\mathbf{q})$ that

$$\left|w_{\mathbf{q}}^{\pm}\right| \le \frac{|\mathbf{r}_{\pm}(\mathbf{q})| \times |\mathbf{w_q}|}{|\det[\mathbf{r}_{-}(\mathbf{q}), \mathbf{r}_{+}(\mathbf{q})]|}. \tag{3.17}$$

By (3.8) and (3.15), for all $q > 0$, there exists a positive constant C_2, such that

$$\left|\mathbf{r}_{\pm}(\mathbf{q})\right| \le \left(\frac{\lambda_{\mathbf{q}}^{\pm}}{\alpha q^2} + \frac{d_2}{\alpha}\right)q^2 + \frac{\beta}{\alpha} + 1 \le C_2 q^2, \tag{3.18}$$

where $C_2 = 2\max\{\frac{1}{\alpha}(C_1 + d_2), \frac{\beta}{\alpha} + 1\}$. By (3.16), we deduce that

$$\frac{1}{|\det[\mathbf{r}_{-}(\mathbf{q}), \mathbf{r}_{+}(\mathbf{q})]|} = \frac{1}{|\lambda_{\mathbf{q}}^{+} - \lambda_{\mathbf{q}}^{-}|} \le \frac{1}{2q\sqrt{\alpha\chi k}}. \tag{3.19}$$

Plugging (3.18), (3.19) into (3.17) yields

$$\left|w_{\mathbf{q}}^{\pm}\right| \le C_2 q^2 \frac{1}{2q\sqrt{\alpha\chi k}}|\mathbf{w_q}| = C_3 q|\mathbf{w_q}|. \tag{3.20}$$

It follows from (3.11), (3.14) and (3.20) that for $t \ge 1$ and q large

$$\begin{aligned}\left\|\mathbf{w}(\mathbf{x},t)\right\|^2 &\le 4\left(\frac{\pi}{2}\right)^d C_2^2 C_3^2 \sum_{\mathbf{q}\in\Omega} q^6 \exp\left(-2\min\left\{d_1 k^n, d_2\right\}q^2\right)|\mathbf{w_q}|^2 \\ &\le 4C_4^2\left(\frac{\pi}{2}\right)^d \sum_{\mathbf{q}\in\Omega} |\mathbf{w_q}|^2 = 4C_4^2\left\|\mathbf{w}(\mathbf{x},0)\right\|^2,\end{aligned}$$

thus,

$$\left\|\mathbf{w}(\mathbf{x},t)\right\| \le 2C_4\left\|\mathbf{w}(\mathbf{x},0)\right\| \exp(\lambda_{\max} t).$$

(2) $t \le 1$. It follows from (3.2) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\left\{|u|^2 + K|v|^2\right\}d\mathbf{x} + \int_{\mathbb{T}^d}\left\{d_1 k^n|\nabla u|^2 + Kd_2|\nabla v|^2 - \chi k\nabla u\nabla v\right\}d\mathbf{x} \\ &\quad = -K\beta\int_{\mathbb{T}^d} v^2\,d\mathbf{x} - r\int_{\mathbb{T}^d} u^2\,d\mathbf{x} + K\alpha\int_{\mathbb{T}^d} uv\,d\mathbf{x}.\end{aligned} \tag{3.21}$$

By Young's inequality,

$$-\chi k\nabla u\nabla v \ge -\frac{d_1 k^n}{2}|\nabla u|^2 - \frac{\chi^2}{2d_1 k^{n-2}}|\nabla v|^2. \tag{3.22}$$

Then using Young's inequality, (3.21), (3.22) and (3.10), we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}\left\{|u|^2+K|v^2|\right\}d\mathbf{x}\le\frac{\sqrt{K}}{2}\int_{\mathbb{T}^d}\left\{|u|^2+K|v|^2\right\}d\mathbf{x}. \tag{3.23}$$

By the Grownwall inequality, we deduce that

$$\left\|\mathbf{w}(\mathbf{x},t)\right\|\le\hat{C}_1e^{\lambda_{\max}t}\left\|\mathbf{w}(\mathbf{x},0)\right\|,$$

where $\hat{C}_1=\max\{2C_4,\sqrt{e/K}\}\ge 1$ if $0<K<1$, and let $\hat{C}_1=\max\{2C_4,\sqrt{Ke^K}\}\ge 1$ if $K\ge 1$. This completes the proof of Lemma 3.1. □

4 Bootstrap lemma and H^2-estimate

By standard PDE theory [21], we can establish the existence of local solutions for (3.1).

Lemma 4.1 (Local existence) *For $s\ge 1$ ($d=1$) and $s\ge 2$ ($d=2,3$), there exists a $T_0>0$ such that* (3.1) *with $u(\cdot,0),v(\cdot,0)\in H^s$ has a unique solution $\mathbf{w}(\cdot,t)$ on $(0,T_0)$ which satisfies*

$$\left\|\mathbf{w}(t)\right\|_{H^s}\le C\left\|\mathbf{w}(0)\right\|_{H^s},\quad 0<t<T_0,$$

where C is a positive constant depending on d_1, d_2, k, α, β, r, χ.

Lemma 4.2 *Let $[u(\mathbf{x},t),v(\mathbf{x},t)]$ be a solution of* (3.1). *Then*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\sigma|=2}\int_{\mathbb{T}^d}\left\{\left|D^\sigma u\right|^2+K\left|D^\sigma v\right|^2\right\}d\mathbf{x}\\
&\quad+\sum_{|\sigma|=2}\int_{\mathbb{T}^d}\left\{\frac{d_1k^n}{4}\left|\nabla D^\sigma u\right|^2+\frac{d_2K}{2}\left|\nabla D^\sigma v\right|^2\right\}d\mathbf{x}\\
&\quad+r\sum_{|\sigma|=2}\int_{\mathbb{T}^d}\left|D^\sigma u\right|^2d\mathbf{x}+\frac{K}{2}\sum_{|\sigma|=2}\int_{\mathbb{T}^d}\left|D^\sigma v\right|^2d\mathbf{x}\\
&\le\hat{C}_2\left(\chi+d_1\sum_{i=1}^nC_n^ik^{n-i}+\frac{2r}{k}\right)\left(\sum_{i=1}^n\|\mathbf{w}\|_{H^2}^i\right)\left\|\nabla^3\mathbf{w}\right\|^2+\hat{C}_3\|u\|^2,
\end{aligned}$$

where $\hat{C}_3=\frac{C_0^3\chi^6\alpha^6}{2d_1^5d_2^3k^{5n-6}}$.

Proof Notice that if $\mathbf{w}(\mathbf{x},t)$ is a solution of (3.1) on $\mathbb{T}^d$, then the even extension of $\mathbf{w}(\mathbf{x},t)$ on $2\mathbb{T}^d=(-\pi,\pi)^d$ ($d=1,2,3$) is also the solution of (3.1) which satisfies homogeneous Neumann boundary conditions and periodical boundary conditions on $2\mathbb{T}^d=(-\pi,\pi)^d$ ($d=1,2,3$). From this, we have

$$\begin{cases}
\tilde{u}_t=d_1k^n\nabla^2\tilde{u}-\chi k\nabla^2\tilde{v}+d_1\nabla[(\sum_{i=1}^nC_n^ik^{n-i}\tilde{u}^i)\nabla\tilde{u}]\\
\qquad-\chi\nabla(\tilde{u}\nabla\tilde{v})-r\tilde{u}-\frac{r}{k}\tilde{u}^2,\\
\tilde{v}_t=d_2\nabla^2\tilde{v}+\alpha\tilde{u}-\beta\tilde{v},\\
\frac{\partial\tilde{u}}{\partial x_i}=\frac{\partial\tilde{v}}{\partial x_i}=0,\quad\text{at } x_i=-\pi,0,\pi,\text{for } 1\le i\le d,
\end{cases} \tag{4.1}$$

where $[\tilde{u}(\mathbf{x},t),\tilde{v}(\mathbf{x},t)]$ is the even extension of $[u(\mathbf{x},t),v(\mathbf{x},t)]$ on $2\mathbb{T}^d$. By (4.1), we can easily deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{2\mathbb{T}^d}|\partial_{x_ix_j}\tilde{u}|^2\,d\mathbf{x}+d_1k^n\int_{2\mathbb{T}^d}\bigl|\nabla(\partial_{x_ix_j}\tilde{u})\bigr|^2\,d\mathbf{x}\\
&\qquad-\chi k\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\nabla(\partial_{x_ix_j}\tilde{v})\,d\mathbf{x}+r\int_{2\mathbb{T}^d}|\partial_{x_ix_j}\tilde{u}|^2\,d\mathbf{x}\\
&\quad=\chi\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}(\tilde{u}\nabla\tilde{v})\,d\mathbf{x}-d_1nk^{n-1}\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}(\tilde{u}\nabla\tilde{u})\,d\mathbf{x}\\
&\qquad-\frac{1}{2}d_1n(n-1)k^{n-2}\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}\bigl(\tilde{u}^2\nabla\tilde{u}\bigr)\,d\mathbf{x}\\
&\qquad-\frac{1}{6}d_1n(n-1)(n-2)k^{n-3}\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}\bigl(\tilde{u}^3\nabla\tilde{u}\bigr)\,d\mathbf{x}-\cdots\\
&\qquad-d_1nk\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}\bigl(\tilde{u}^{n-1}\nabla\tilde{u}\bigr)\,d\mathbf{x}\\
&\qquad-d_1\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}\bigl(\tilde{u}^n\nabla\tilde{u}\bigr)\,d\mathbf{x}\\
&\qquad-\frac{2r}{k}\int_{2\mathbb{T}^d}\tilde{u}|\partial_{x_ix_j}\tilde{u}|^2\,d\mathbf{x}-\frac{2r}{k}\int_{2\mathbb{T}^d}\partial_{x_i}\tilde{u}\cdot\partial_{x_j}\tilde{u}\cdot\partial_{x_ix_j}\tilde{u}\,d\mathbf{x}
\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{2\mathbb{T}^d}K|\partial_{x_ix_j}\tilde{v}|^2\,d\mathbf{x}+Kd_2\int_{2\mathbb{T}^d}\bigl|\nabla(\partial_{x_ix_j}\tilde{v})\bigr|^2\,d\mathbf{x}+K\beta\int_{2\mathbb{T}^d}|\partial_{x_ix_j}\tilde{v}|^2\,d\mathbf{x}\\
&\quad=K\alpha\int_{2\mathbb{T}^d}\partial_{x_ix_j}\tilde{u}\cdot\partial_{x_ix_j}\tilde{v}\,d\mathbf{x}.
\end{aligned}\tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{2\mathbb{T}^d}\bigl(|\partial_{x_ix_j}\tilde{u}|^2+K|\partial_{x_ix_j}\tilde{v}|^2\bigr)\,d\mathbf{x}\\
&\qquad+\int_{2\mathbb{T}^d}\bigl(d_1k^n\bigl|\nabla(\partial_{x_ix_j}\tilde{u})\bigr|^2+Kd_2\bigl|\nabla(\partial_{x_ix_j}\tilde{v})\bigr|^2-\chi k\nabla(\partial_{x_ix_j}\tilde{u})\cdot\nabla(\partial_{x_ix_j}\tilde{v})\bigr)\,d\mathbf{x}\\
&\qquad+r\int_{2\mathbb{T}^d}|\partial_{x_ix_j}\tilde{u}|^2\,d\mathbf{x}+\beta K\int_{2\mathbb{T}^d}|\partial_{x_ix_j}\tilde{v}|^2\,d\mathbf{x}\\
&\quad=\chi\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}(\tilde{u}\nabla\tilde{v})\,d\mathbf{x}-d_1nk^{n-1}\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}(\tilde{u}\nabla\tilde{u})\,d\mathbf{x}\\
&\qquad-\frac{1}{2}d_1n(n-1)k^{n-2}\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}\bigl(\tilde{u}^2\nabla\tilde{u}\bigr)\,d\mathbf{x}\\
&\qquad-\frac{1}{6}d_1n(n-1)(n-2)k^{n-3}\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}\bigl(\tilde{u}^3\nabla\tilde{u}\bigr)\,d\mathbf{x}\\
&\qquad-\cdots-d_1nk\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}\bigl(\tilde{u}^{n-1}\nabla\tilde{u}\bigr)\,d\mathbf{x}\\
&\qquad-d_1\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}\bigl(\tilde{u}^n\nabla\tilde{u}\bigr)\,d\mathbf{x}+\alpha K\int_{2\mathbb{T}^d}\partial_{x_ix_j}\tilde{u}\cdot\partial_{x_ix_j}\tilde{v}\,d\mathbf{x}\\
&\qquad-\frac{2r}{k}\int_{2\mathbb{T}^d}\bigl(\tilde{u}|\partial_{x_ix_j}\tilde{u}|^2+\partial_{x_i}\tilde{u}\cdot\partial_{x_j}\tilde{u}\cdot\partial_{x_ix_j}\tilde{u}\bigr)\,d\mathbf{x}\\
&\quad:=J_\chi+J_1+J_2+J_3+\cdots+J_{n-1}+J_n+J_\alpha+J_r.
\end{aligned}\tag{4.4}$$

Thanks to Young's inequality and (3.10),

$$-\chi k\nabla(\partial_{x_ix_j}\tilde{u})\cdot\nabla(\partial_{x_ix_j}\tilde{v})\geq -\frac{d_1k^n}{2}\left|\nabla(\partial_{x_ix_j}\tilde{u})\right|^2-\frac{d_2K}{2}\left|\nabla(\partial_{x_ix_j}\tilde{v})\right|^2. \tag{4.5}$$

The nonlinear term J_χ is bounded by

$$\begin{aligned} J_\chi &:= \chi\int_{2\mathbb{T}^d}\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}(\tilde{u}\nabla\tilde{v})\,d\mathbf{x} \\ &\leq \chi\left\{\int_{2\mathbb{T}^d}\left|\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_ix_j}\tilde{u}\cdot\nabla\tilde{v}\right|d\mathbf{x}+\int_{2\mathbb{T}^d}\left|\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_j}\tilde{u}\cdot\nabla(\partial_{x_i}\tilde{v})\right|d\mathbf{x}\right. \\ &\quad \left.+\int_{2\mathbb{T}^d}\left|\nabla(\partial_{x_ix_j}\tilde{u})\cdot\partial_{x_i}\tilde{u}\cdot\nabla(\partial_{x_j}\tilde{v})\right|d\mathbf{x}+\int_{2\mathbb{T}^d}\left|\tilde{u}\cdot\nabla(\partial_{x_ix_j}\tilde{u})\cdot\nabla(\partial_{x_ix_j}\tilde{v})\right|d\mathbf{x}\right\} \\ &\leq \chi\|\nabla\tilde{v}\|_{L^\infty}\left\|\nabla(\partial_{x_ix_j}\tilde{u})\right\|\|\partial_{x_ix_j}\tilde{u}\|+2\chi\sum_{i=1}^d\|\nabla\tilde{u}\|_{L^\infty}\|\partial_{x_ix_j}\tilde{v}\|\left\|\nabla(\partial_{x_ix_j}\tilde{u})\right\| \\ &\quad +\chi\|\tilde{u}\|_{L^\infty}\left\|\nabla(\partial_{x_ix_j}\tilde{u})\right\|\left\|\nabla(\partial_{x_ix_j}\tilde{v})\right\|. \end{aligned} \tag{4.6}$$

By applying the Sobolev embedding to control the L^∞ norm for $d\leq 3$, there exists a constant $C_5>0$ such that

$$\|g\|_{L^\infty(2\mathbb{T}^d)}\leq C_5\|g\|_{H^2(2\mathbb{T}^d)}. \tag{4.7}$$

Moreover, notice that

$$\int_{2\mathbb{T}^d}\nabla\tilde{u}\,d\mathbf{x}=\int_{2\mathbb{T}^d}\nabla\tilde{v}\,d\mathbf{x}=0,\qquad \int_{2\mathbb{T}^d}\partial_{x_ix_j}\tilde{u}\,d\mathbf{x}=\int_{2\mathbb{T}^d}\partial_{x_ix_j}\tilde{v}\,d\mathbf{x}=0. \tag{4.8}$$

Using the Poincaré inequality, there exists a constant $C_6>0$ such that if $g\in H^1(2\mathbb{T}^d)$ and $\int_{2\mathbb{T}^d}g\,d\mathbf{x}=0$, then

$$\|g\|_{L^4(2\mathbb{T}^d)}\leq C_6\|\nabla g\|. \tag{4.9}$$

From (4.8) and (4.9), it follows that

$$\|\partial_{x_i}g\|\leq C_7\|\nabla\partial_{x_i}g\|,\qquad \|\partial_{x_ix_j}g\|\leq C_8\|\nabla\partial_{x_ix_j}g\|$$

and

$$\|\nabla g\|\leq C_9\left(\sum_{|\sigma|=2}\left\|\nabla D^\sigma g\right\|^2\right)^{\frac{1}{2}}\leq C_9^2\left(\sum_{|\alpha|=2}\left\|\nabla\left(D^\sigma\right)g\right\|^2\right)^{\frac{1}{2}}. \tag{4.10}$$

Using (4.7), we get

$$\|\nabla g\|_{L^\infty}\leq C_{10}\|\nabla g\|_{H^2}\leq C_{11}\left\|\nabla^3 g\right\|. \tag{4.11}$$

Thus from (4.7) and (4.11), one can derive that

$$J_\chi\leq\chi C_{12}\|\tilde{\mathbf{w}}\|_{H^2}\left\|\nabla^3\tilde{\mathbf{w}}\right\|^2. \tag{4.12}$$

Similarly, one knows that

$$J_1 \le d_1 n k^{n-1} C_{12} \|\tilde{\mathbf{w}}\|_{H^2} \|\nabla^3 \tilde{\mathbf{w}}\|^2. \tag{4.13}$$

Applying interpolation inequalities, one has

$$\|\partial_{x_i x_j} g\|^2 \le C_0 \left(a \|\nabla(\partial_{x_i x_j} g)\|^2 + \frac{\|g\|^2}{4a^2} \right), \quad \forall a > 0. \tag{4.14}$$

By the choice of $a = \frac{d_1^2 d_2 k^{2n-2}}{2\alpha^2 \chi^2 C_0}$ in (4.14),

$$\begin{aligned} J_\alpha &:= \alpha K \int_{2\mathbb{T}^d} \partial_{x_i x_j} \tilde{u} \cdot \partial_{x_i x_j} \tilde{v} \, d\mathbf{x} \\ &\le \frac{K}{2} \sum_{|\sigma|=2} \int_{2\mathbb{T}^d} |D^\sigma \tilde{v}|^2 \, d\mathbf{x} + \frac{d_1 k^n}{4} \sum_{|\sigma|=2} \int_{2\mathbb{T}^d} |\nabla(D^\sigma \tilde{u})|^2 \, d\mathbf{x} + \hat{C}_3 \|\tilde{u}\|^2, \end{aligned} \tag{4.15}$$

where $\hat{C}_3 = \frac{C_0^3 \chi^6 \alpha^6}{2 d_1^5 d_2^3 k^{5n-6}}$.

Now, according to Hölder's inequality, (4.7), (4.9) and (4.11), one can verify that

$$\begin{aligned} J_2 &:= -\frac{1}{2} d_1 n(n-1) k^{n-2} \int_{2\mathbb{T}^d} \nabla(\partial_{x_i x_j} \tilde{u}) \cdot \partial_{x_i x_j} (\tilde{u}^2 \nabla \tilde{u}) \, d\mathbf{x} \\ &\le d_1 n(n-1) k^{n-2} \|\nabla \tilde{u}\|_{L^\infty} \|\nabla \tilde{u}\|_{L^4}^2 \|\nabla(\partial_{x_i x_j} \tilde{u})\| \\ &\quad + d_1 n(n-1) k^{n-2} \|\nabla \tilde{u}\|_{L^\infty} \|\tilde{u}\|_{L^\infty} \|\partial_{x_i x_j} \tilde{u}\| \|\nabla(\partial_{x_i x_j} \tilde{u})\| \\ &\quad + 2 d_1 n(n-1) k^{n-2} \sum_{i=1}^d \|\tilde{u}\|_{L^\infty} \|\nabla \tilde{u}\|_{L^\infty} \|\partial_{x_i x_j} \tilde{u}\| \|\nabla(\partial_{x_i x_j} \tilde{u})\| \\ &\quad + \frac{1}{2} d_1 n(n-1) k^{n-2} \|\tilde{u}\|_{L^\infty}^2 \|\nabla(\partial_{x_i x_j} \tilde{u})\|^2 \\ &\le \frac{1}{2} d_1 n(n-1) k^{n-2} C_{13} \|\tilde{\mathbf{w}}\|_{H^2}^2 \|\nabla^3 \tilde{\mathbf{w}}\|^2, \end{aligned} \tag{4.16}$$

$$\begin{aligned} J_3 &:= -\frac{1}{6} d_1 n(n-1)(n-2) k^{n-3} \int_{2\mathbb{T}^d} \nabla(\partial_{x_i x_j} \tilde{u}) \cdot \partial_{x_i x_j} (\tilde{u}^3 \nabla \tilde{u}) \, d\mathbf{x} \\ &\le d_1 n(n-1)(n-2) k^{n-3} \|\tilde{u}\|_{L^\infty} \|\nabla \tilde{u}\|_{L^\infty} \|\nabla \tilde{u}\|_{L^4}^2 \|\nabla(\partial_{x_i x_j} \tilde{u})\| \\ &\quad + \frac{3}{2} d_1 n(n-1)(n-2) k^{n-3} \|\tilde{u}\|_{L^\infty}^2 \|\nabla \tilde{u}\|_{L^\infty} \|\partial_{x_i x_j} \tilde{u}\| \|\nabla(\partial_{x_i x_j} \tilde{u})\| \\ &\quad + \frac{1}{6} d_1 n(n-1)(n-2) k^{n-3} \|\tilde{u}\|_{L^\infty}^3 \|\nabla(\partial_{x_i x_j} \tilde{u})\|^2 \\ &\le \frac{1}{6} d_1 n(n-1)(n-2) k^{n-3} C_{14} \|\tilde{\mathbf{w}}\|_{H^2}^3 \|\nabla^3 \tilde{\mathbf{w}}\|^2, \end{aligned} \tag{4.17}$$

$$\begin{aligned} J_{n-1} &:= -d_1 n k \int_{2\mathbb{T}^d} \nabla(\partial_{x_i x_j} \tilde{u}) \cdot \partial_{x_i x_j} (\tilde{u}^{n-1} \nabla \tilde{u}) \, d\mathbf{x} \\ &\le d_1 n k \bigl\{ (n-1)(n-2) \|\tilde{u}\|_{L^\infty}^{n-3} \|\nabla \tilde{u}\|_{L^\infty} \|\nabla \tilde{u}\|_{L^4}^2 \|\nabla(\partial_{x_i x_j} \tilde{u})\| \\ &\quad + 3(n-1) \|\tilde{u}\|_{L^\infty}^{n-2} \|\nabla \tilde{u}\|_{L^\infty} \|\partial_{x_i x_j} \tilde{u}\| \|\nabla(\partial_{x_i x_j} \tilde{u})\| + \|\tilde{u}\|_{L^\infty}^{n-1} \|\nabla(\partial_{x_i x_j} \tilde{u})\|^2 \bigr\} \\ &\le d_1 n k C_{1n} \|\tilde{\mathbf{w}}\|_{H^2}^{n-1} \|\nabla^3 \tilde{\mathbf{w}}\|^2, \end{aligned} \tag{4.18}$$

and

$$
\begin{aligned}
J_n &:= -d_1 \int_{2\mathbb{T}^d} \nabla(\partial_{x_ix_j}\tilde{u}) \cdot \partial_{x_ix_j}\bigl(\tilde{u}^n \nabla\tilde{u}\bigr)\, d\mathbf{x} \\
&\leq d_1 \bigl\{ n(n-1)\|\tilde{u}\|_{L^\infty}^{n-2} \|\nabla\tilde{u}\|_{L^\infty} \|\nabla\tilde{u}\|_{L^4}^2 \bigl\|\nabla(\partial_{x_ix_j}\tilde{u})\bigr\| \\
&\quad + 3n\|\tilde{u}\|_{L^\infty}^{n-1} \|\nabla\tilde{u}\|_{L^\infty} \|\partial_{x_ix_j}\tilde{u}\| \bigl\|\nabla(\partial_{x_ix_j}\tilde{u})\bigr\| + \|\tilde{u}\|_{L^\infty}^n \bigl\|\nabla(\partial_{x_ix_j}\tilde{u})\bigr\|^2 \bigr\} \\
&\leq d_1 C_{1n+1} \|\tilde{\mathbf{w}}\|_{H^2}^n \bigl\|\nabla^3\tilde{\mathbf{w}}\bigr\|^2,
\end{aligned}
\tag{4.19}
$$

where $C_{13} = 2C_6^2C_{11} + 6C_5C_{11} + C_5^2$, $C_{14} = 6C_5C_6^2C_{11} + 9C_5^2C_{11} + C_5^3$, $C_{1n} = (n-1)(n-2)C_5^{n-3}C_6^2C_{11} + 3(n-1)C_5^{n-2}C_{11} + C_5^{n-1}$ and $C_{1,n+1} = n(n-1)C_5^{n-2}C_6^2C_{11} + 3nC_5^{n-2}C_{11} + C_5^n$.

Again by (4.7), (4.10) and (4.11), we can estimate

$$
J_r \leq \frac{2r}{k} \bigl\{ \|\tilde{u}\|_{L^\infty} \|\partial_{x_ix_j}\tilde{u}\|^2 + \|\nabla\tilde{u}\|_{L^\infty} \|\nabla\tilde{u}\| \|\partial_{x_ix_j}\tilde{u}\| \bigr\},
$$

further,

$$
\sum_{|\alpha|=2} J_r \leq \frac{2r}{k} C_{1r} \|\tilde{\mathbf{w}}\|_{H^2} \bigl\|\nabla^3\tilde{\mathbf{w}}\bigr\|^2,
\tag{4.20}
$$

where $C_{1r} = C_8(C_5C_8 + C_{11})$.

Recall that $[\tilde{u},\tilde{v}]$ is the even extension of $[u,v]$. Plugging (4.12), (4.13), (4.15), (4.16), (4.17), (4.18), (4.19), (4.20) into (4.4),

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \sum_{|\sigma|=2} \int_{2\mathbb{T}^d} \bigl(|D^\sigma\tilde{u}|^2 + K|D^\sigma\tilde{v}|^2 \bigr)\, d\mathbf{x} \\
&\quad + \sum_{|\sigma|=2} \int_{2\mathbb{T}^d} \left(\frac{d_1k^n}{4} |\nabla D^\sigma\tilde{u}|^2 + \frac{d_2K}{2} \bigl|\nabla\bigl(D^\sigma\tilde{v}\bigr)\bigr|^2 \right) d\mathbf{x} \\
&\quad + r\sum_{|\sigma|=2} \int_{2\mathbb{T}^d} |D^\sigma\tilde{u}|^2\, d\mathbf{x} + \frac{K}{2} \sum_{|\sigma|=2} \int_{2\mathbb{T}^d} |D^\sigma\tilde{v}|^2\, d\mathbf{x} \\
&\leq \hat{C}_2 \left(\chi + d_1 \sum_{i=1}^n C_n^i k^{n-i} + \frac{2r}{k} \right) \left(\sum_{i=1}^n \|\tilde{\mathbf{w}}\|_{H^2}^i \right) \bigl\|\nabla^3\tilde{\mathbf{w}}\bigr\|^2 + \hat{C}_3\|\tilde{u}\|^2,
\end{aligned}
$$

where $\hat{C}_2 = C_{12} + \sum_{i=2}^{n+1} C_{1i} + C_{1r}$. This completes the proof. □

Lemma 4.3 *Let $\mathbf{w}(\mathbf{x},t)$ be a solution of (3.1) such that, for $0 \leq t \leq T < T_0$,*

$$
\sum_{i=1}^n \bigl\|\mathbf{w}(\cdot,t)\bigr\|_{H^2}^i \leq \frac{1}{\hat{C}_2} \min\left\{ \frac{d_1k^n}{4}, \frac{d_2K}{2} \right\},
\tag{4.21}
$$

and

$$
\bigl\|\mathbf{w}(\cdot,t)\bigr\| \leq \hat{C}_1 e^{\lambda_{\max}t} \bigl\|\mathbf{w}(\cdot,0)\bigr\|.
\tag{4.22}
$$

Then

$$
\bigl\|\mathbf{w}(\cdot,t)\bigr\|_{H^2}^n \leq \hat{C}_4 \bigl\{ \bigl\|\mathbf{w}(\cdot,0)\bigr\|_{H^2}^2 + e^{2\lambda_{\max}t} \bigl\|\mathbf{w}(\cdot,0)\bigr\|^2 \bigr\}^{\frac{n}{2}}, \quad 0 \leq t \leq T,
$$

where $\hat{C}_4 = \max\{((1+C_9^2)K)^{\frac{n}{2}}, [\hat{C}_1^2(1+\frac{(1+C_9^2)\hat{C}_3^2}{\lambda_{\max}})]^{\frac{n}{2}}\}$, *if* $K \geq 1$ *and* $\hat{C}_4 = \max\{(\frac{1+C_9^2}{K})^{\frac{n}{2}}, [\hat{C}_1^2(1+\frac{(1+C_9^2)\hat{C}_3^2}{K\lambda_{\max}})]^{\frac{n}{2}}\}$, *if* $K < 1$.

Proof From (4.10), one knows that

$$\|\nabla \mathbf{w}(\cdot,t)\|^2 \leq C_9^2 \sum_{|\sigma|=2} \|D^\sigma \mathbf{w}(\cdot,t)\|^2, \tag{4.23}$$

which leads to

$$\|\mathbf{w}(\cdot,t)\|_{H^2}^n \leq \left(\|\mathbf{w}(\cdot,t)\|^2 + (C_9^2+1)\sum_{|\sigma|=2}\|D^\sigma \mathbf{w}(\cdot,t)\|^2\right)^{\frac{n}{2}}. \tag{4.24}$$

By Lemma 4.2 and (4.21), we can see that

$$\frac{1}{2}\frac{d}{dt}\sum_{|\sigma|=2}\int_{\mathbb{T}^d}\left(|D^\sigma u|^2 + K|D^\sigma v|^2\right)d\mathbf{x} \leq \hat{C}_3\|u\|^2 \leq \hat{C}_3\|\mathbf{w}(\cdot,t)\|^2. \tag{4.25}$$

From this and (4.22), it follows that

$$\begin{aligned}&\sum_{|\sigma|=2}\int_{\mathbb{T}^d}\left(|D^\sigma u(\cdot,t)|^2 + K|D^\sigma v(\cdot,t)|^2\right)d\mathbf{x}\\ &\quad\leq \sum_{|\sigma|=2}\int_{\mathbb{T}^d}\left(|D^\sigma u(\cdot,0)|^2 + K|D^\sigma v(\cdot,0)|^2\right)d\mathbf{x} + \frac{\hat{C}_1^2\hat{C}_3}{\lambda_{\max}}e^{2\lambda_{\max}t}\|\mathbf{w}(\cdot,0)\|^2.\end{aligned} \tag{4.26}$$

We will proceed in the following two cases: $K \geq 1$, $K < 1$.

(1) If $K \geq 1$, using (4.26) yields

$$\sum_{|\sigma|=2}\|D^\sigma \mathbf{w}(\cdot,t)\|^2 \leq K\sum_{|\sigma|=2}\|D^\sigma \mathbf{w}(\cdot,0)\|^2 + \frac{\hat{C}_1^2\hat{C}_3}{\lambda_{\max}}e^{2\lambda_{\max}t}\|\mathbf{w}(\cdot,0)\|^2.$$

Then it is not hard to verify from (4.22), (4.24) and (4.26) that

$$\|\mathbf{w}(\cdot,t)\|_{H^2}^n \leq \hat{C}_4\left(\|\mathbf{w}(\cdot,0)\|_{H^2}^2 + \|\mathbf{w}(\cdot,0)\|^2 e^{2\lambda_{\max}t}\right)^{\frac{n}{2}},$$

where $\hat{C}_4 = \max\{((1+C_9^2)K)^{\frac{n}{2}}, [\hat{C}_1^2(1+\frac{(1+C_9^2)\hat{C}_3^2}{\lambda_{\max}})]^{\frac{n}{2}}\}$.

(2) If $K < 1$, notice that

$$K\sum_{|\sigma|=2}\|D^\sigma \mathbf{w}(\cdot,t)\|^2 \leq \sum_{|\sigma|=2}\|D^\sigma \mathbf{w}(\cdot,0)\|^2 + \frac{\hat{C}_1^2\hat{C}_3}{\lambda_{\max}}e^{2\lambda_{\max}t}\|\mathbf{w}(\cdot,0)\|^2.$$

Moreover, applying (4.22) and (4.24), we can see that

$$\|\mathbf{w}(\cdot,t)\|_{H^2}^n \leq \hat{C}_4\left(\|\mathbf{w}(\cdot,0)\|_{H^2}^2 + \|\mathbf{w}(\cdot,0)\|^2 e^{2\lambda_{\max}t}\right)^{\frac{n}{2}},$$

where $\hat{C}_4 = \max\{(\frac{1+C_9^2}{K})^{\frac{n}{2}}, [\hat{C}_1^2(1+\frac{(1+C_9^2)\hat{C}_3^2}{\lambda_{\max}K})]^{\frac{n}{2}}\}$ and thereby we complete the proof. □

5 Main result

Let θ be a small fixed constant, and $\lambda_{\max}$ be the dominant eigenvalue which is the maximal growth rate. We also denote the gap between the largest growth rate $\lambda_{\max}$ and the rest by $\rho > 0$ i.e., $\rho = \min_{\mathbf{q}\in\Omega\setminus\Omega_{\max}} |\lambda_{\max} - \lambda_{\mathbf{q}}|$. Then for $\delta > 0$ arbitrary small, we define the escape time T^δ by

$$\theta = \delta e^{\lambda_{\max} T^\delta}, \tag{5.1}$$

or equivalently

$$T^\delta = \frac{1}{\lambda_{\max}} \ln \frac{\theta}{\delta}. \tag{5.2}$$

Our main theorem in this paper is as follows.

Theorem 5.1 *Assume that the set of* $q^2 = \sum_{i=1}^d q_i^2$ *satisfying instability criterion* (3.5) *is not empty for given parameters* $d_1, d_2, k, \chi, \alpha, \beta, r$. *Let*

$$\mathbf{w}_0(\mathbf{x}) = \sum_{\mathbf{q}\in\Omega} \bigl\{ w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) \bigr\} e_{\mathbf{q}}(\mathbf{x}) \in H^2, \tag{5.3}$$

such that $\|\mathbf{w}_0\| = 1$. *Then there exist constants* $\delta_0 > 0$, $\hat{C} > 0$ *and* $\theta > 0$ *depending on* $d_1, d_2, k, \chi, \alpha, \beta, r$, *such that, for all* $0 < \delta \le \delta_0$, *if the initial perturbation of the steady state* $[\bar{U}, \bar{V}]$ *is* $\mathbf{w}^\delta(\cdot, 0) = \delta \mathbf{w}_0$, *then its nonlinear evolution* $\mathbf{w}^\delta(\cdot, t)$ *satisfies*

$$\begin{aligned} &\Bigl\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q}\in\Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \Bigr\| \\ &\quad \le \hat{C} \left\{ e^{-\rho t} + \sum_{i=1}^n \bigl(\delta^i \|\mathbf{w}_0\|_{H^2}^{i+1} + \delta^i e^{i\lambda_{\max} t} \bigr) \right\} \delta e^{\lambda_{\max} t} \end{aligned} \tag{5.4}$$

for $0 \le t \le T^\delta$, *and* $\rho > 0$ *is the gap between* $\lambda_{\max}$ *and the rest of* $\lambda_{\mathbf{q}}$ *in* (5.4).

Proof Let $\mathbf{w}^\delta(\mathbf{x}, t)$ be the solutions to (3.1) with initial data $\mathbf{w}^\delta(\cdot, 0) = \delta \mathbf{w}_0$. We define

$$T^* = \sup \left\{ t \Bigm| \bigl\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\mathfrak{L} t} \mathbf{w}_0 \bigr\| \le \frac{\hat{C}_1}{2} \delta e^{\lambda_{\max} t} \right\} \tag{5.5}$$

and

$$T^{**} = \sup \left\{ t \Bigm| \sum_{i=1}^n \bigl\| \mathbf{w}^\delta(\cdot, t) \bigr\|_{H^2}^i \le \frac{1}{\hat{C}_2^*} \min \left\{ \frac{d_1 k^n}{4}, \frac{d_2 K}{2} \right\} \right\}. \tag{5.6}$$

First, one can estimate the H^2 norm of $\mathbf{w}^\delta(\mathbf{x}, t)$ for $0 \le t \le \min\{T^\delta, T^*, T^{**}\}$. From the definition of T^* and Lemma 3.1, for $0 < t \le T^*$, it derives that

$$\bigl\| \mathbf{w}^\delta(\cdot, t) \bigr\| \le \frac{3}{2} \hat{C}_1 \delta e^{\lambda_{\max} t}. \tag{5.7}$$

In the light of Lemma 4.3, it is easy to see that

$$\left\|\mathbf{w}^\delta(\cdot,t)\right\|_{H^2} \le \hat{C}_4^{\frac{1}{n}}\left\{\delta\|\mathbf{w}_0\|_{H^2} + \delta e^{\lambda_{\max}t}\right\} \tag{5.8}$$

and

$$\left\|\mathbf{w}^\delta(\cdot,t)\right\|_{H^2}^{i+1} \le 2^i\hat{C}_4^{\frac{i+1}{n}}\left\{\delta^{i+1}\|\mathbf{w}_0\|_{H^2}^{i+1} + \delta^{i+1}e^{(i+1)\lambda_{\max}t}\right\}, \quad 1\le i\le n. \tag{5.9}$$

Second, we establish a sharper L^2 estimate for $\mathbf{w}^\delta(\mathbf{x},t)$ for $0\le t\le\min\{T^\delta,T^*,T^{**}\}$. By Duhamel's principle, the solution of (3.1) is written as

$$\begin{aligned}\mathbf{w}^\delta(\cdot,t) = \delta e^{\mathfrak{L}t}\mathbf{w}_0 - \int_0^t e^{\mathfrak{L}(t-\tau)}\Bigg[&-d_1\nabla\left(\left(\sum_{i=1}^n C_n^i k^{n-i}\big(u^\delta(\tau)\big)^i\right)\nabla\big(u^\delta(\tau)\big)\right)\\ &+\chi\nabla\big(u^\delta(\tau)\nabla u^\delta(\tau)\big) + \frac{r}{k}\big(u^\delta(\tau)\big)^2, 0\Bigg]\,d\tau.\end{aligned} \tag{5.10}$$

By Lemma 3.1, (4.7), (4.9) and (5.9), notice that $t\le\min\{T^\delta,T^*,T^{**}\}$; it is bounded by

$$\begin{aligned}&\left\|\mathbf{w}^\delta(t) - \delta e^{\mathfrak{L}t}\mathbf{w}_0\right\|\\ &\quad\le \hat{C}_1\int_0^t e^{\lambda_{\max}(t-\tau)}\Bigg[\chi\left\|\nabla u^\delta(\tau)\right\|_{L^4}\left\|\nabla v^\delta(\tau)\right\|_{L^4} + \chi\left\|u^\delta(\tau)\right\|_{L^\infty}\left\|\nabla^2\big(v^\delta(\tau)\big)\right\|\\ &\qquad + d_1\left(\sum_{i=1}^n C_n^i k^{n-i} i\left\|u^\delta(\tau)\right\|_{L^\infty}^{i-1}\right)\left\|\nabla u^\delta(\tau)\right\|_{L^4}^2 + \frac{r}{k}\left\|u^\delta(\tau)\right\|_{L^4}^2\Bigg]\,d\tau\\ &\quad\le \hat{C}_1\int_0^t e^{\lambda_{\max}(t-\tau)}\left\{\left[\chi\left(C_5 + C_6^2\right) + rC_6^2/k\right]\left\|\mathbf{w}^\delta(\tau)\right\|_{H^2}^2 + \bar{C}\sum_{i=1}^n\left\|\mathbf{w}^\delta(\tau)\right\|_{H^2}^{i+1}\right\}d\tau\\ &\quad\le \hat{C}_1\hat{C}_5\int_0^t e^{\lambda_{\max}(t-\tau)}\sum_{i=1}^n\left\|\mathbf{w}^\delta(\tau)\right\|_{H^2}^{i+1}\,d\tau\\ &\quad\le \hat{C}_1\hat{C}_5\left\{\frac{1}{\lambda_{\max}}\sum_{i=1}^n\hat{C}_4^{\frac{i+1}{n}}2^i\delta^i\|\mathbf{w}_0\|_{H^2}^{i+1} + \frac{1}{\lambda_{\max}}\sum_{i=1}^n\hat{C}_4^{\frac{i+1}{n}}\frac{2^i}{i}\delta^i e^{i\lambda_{\max}t}\right\}\delta e^{\lambda_{\max}t}.\end{aligned} \tag{5.11}$$

Next, there exists sufficiently small δ_0, for $0<\delta\le\delta_0$, so we can prove that

$$T^\delta = \min\left\{T^\delta, T^*, T^{**}\right\}.$$

If T^{**} is the smallest, we can let $t = T^{**}\le T^\delta$ in (5.8) and (5.9) to obtain

$$\begin{aligned}\sum_{i=1}^n\left\|\mathbf{w}^\delta\left(\cdot,T^{**}\right)\right\|_{H^2}^i &\le \sum_{i=1}^n\hat{C}_4^{\frac{i}{n}}2^{i-1}\left(\delta^i\left\|\mathbf{w}(\cdot,0)\right\|_{H^2}^i + \delta^i e^{i\lambda_{\max}T^{**}}\right)\\ &\le \sum_{i=1}^n\hat{C}_4^{\frac{i}{n}}2^{i-1}\left\|\mathbf{w}^\delta(\cdot,0)\right\|_{H^2}^i + \sum_{i=1}^n\hat{C}_4^{\frac{i}{n}}2^i\theta^i.\end{aligned}$$

Choosing θ satisfies

$$\hat{C}_2^* \sum_{i=1}^{n} \hat{C}_4^{\frac{i}{n}} 2^i \theta^i \leq \frac{1}{2} \min\left\{ \frac{d_1 k^n}{4}, \frac{d_2 K}{2}, \frac{\lambda_{\max}}{2} \right\} \tag{5.12}$$

and for δ sufficiently small, such that

$$\sum_{i=1}^{n} \hat{C}_4^{\frac{i}{n}} 2^{i-1} \left\| \mathbf{w}^\delta(\cdot, 0) \right\|_{H^2}^{i} \leq \frac{1}{2\hat{C}^*} \min\left\{ \frac{d_1 k^n}{4}, \frac{d_2 K}{2} \right\}.$$

Thus, one can get

$$\sum_{i=1}^{n} \left\| \mathbf{w}^\delta\left(\cdot, T^{**}\right) \right\|_{H^2}^{i} \leq \frac{1}{\hat{C}_2^*} \min\left\{ \frac{d_1 k^n}{4}, \frac{d_2 K}{2} \right\}.$$

This is a contradiction to the definition of T^{**}. On the other hand, if T^* is the minimum, we let $t = T^* \leq T^\delta$ in (5.11). By (5.12) and for $\delta \leq \delta_0$,

$$\hat{C}_4 \hat{C}_5 \frac{1}{\lambda_{\max}} \sum_{i=1}^{n} \hat{C}_4^{\frac{i}{n}} 2^i \delta_0^i \left\| \mathbf{w}(\cdot, 0) \right\|_{H^2}^{i+1} < \frac{1}{4},$$

then

$$\begin{aligned}
&\left\| \mathbf{w}^\delta\left(\cdot, T^*\right) - \delta e^{\mathfrak{L} T^*} \mathbf{w}_0 \right\| \\
&\quad \leq \hat{C}_1 \hat{C}_4 \hat{C}_5 \left\{ \frac{1}{\lambda_{\max}} \sum_{i=1}^{n} \hat{C}_4^{\frac{i}{n}} 2^i \delta^i \left\| \mathbf{w}(\cdot, 0) \right\|_{H^2}^{i+1} + \frac{1}{\lambda_{\max}} \sum_{i=1}^{n} \hat{C}_4^{\frac{i}{n}} \frac{2^i}{i} \theta^i \right\} \delta e^{\lambda_{\max} T^*} \\
&\quad < \frac{1}{2} \hat{C} \delta e^{\lambda_{\max} T^*},
\end{aligned}$$

where let $\hat{C}_4 \hat{C}_5 / \hat{C}_2^* \leq 1$. This again contradicts the definition of T^*. Hence, T^δ is the smallest.

Now, it follows from (3.10) that

$$\begin{aligned}
&\left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^{+} \mathbf{r}_{+}(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \right\| \\
&\quad \leq \left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\mathfrak{L} t} \mathbf{w}_0 \right\| + \left\| \delta \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^{-} \mathbf{r}_{-}(\mathbf{q}) e^{\lambda_{\mathbf{q}}^{-} t} e_{\mathbf{q}}(\mathbf{x}) \right\| \\
&\qquad + \left\| \delta \sum_{\mathbf{q} \in \Omega \setminus \Omega_{\max}} \left\{ w_{\mathbf{q}}^{-} \mathbf{r}_{-}(\mathbf{q}) e^{\lambda_{\mathbf{q}}^{-} t} + w_{\mathbf{q}}^{+} \mathbf{r}_{+}(\mathbf{q}) e^{\lambda_{\mathbf{q}}^{+} t} \right\} e_{\mathbf{q}}(\mathbf{x}) \right\| \\
&\quad = \left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\mathfrak{L} t} \mathbf{w}_0 \right\| + I_1 + I_2.
\end{aligned} \tag{5.13}$$

By (3.7), we know that there is one (or two) q^2 satisfying $\lambda^+(q^2) = \lambda_{\max}$. If there is only one q^2 satisfying $\lambda^+(q^2) = \lambda_{\max}$, it is denoted by $q_{\max}^2$; if there are q_1^2 and q_2^2 satisfying $\lambda^+(q_i^2) = \lambda_{\max}$

$(i = 1,2)$, one can let $q_{\max}^2 = \max\{q_1^2, q_2^2\}$. Using (3.17) and(3.19) yields

$$\begin{aligned} I_1^2 &\le 4\delta^2 e^{2(\lambda_{\max}-\rho)t}\left(\frac{\pi}{2}\right)^d \sum_{\mathbf{q}\in\Omega_{\max}} \left|w_{\mathbf{q}}^-\right|^2 \left|\mathbf{r}_-(\mathbf{q})\right|^2 \\ &\le 4C_2^2 C_4^2 \delta^2 e^{2(\lambda_{\max}-\rho)t}\left(\frac{\pi}{2}\right)^d \sum_{\mathbf{q}\in\Omega_{\max}} q^6 |\mathbf{w}_q|^2 \\ &\le 4C_2^2 C_4^2 \delta^2 e^{2(\lambda_{\max}-\rho)t} q_{\max}^6, \end{aligned}$$

that is,

$$I_1 \le \hat{C}_6 \delta e^{(\lambda_{\max}-\rho)t}. \tag{5.14}$$

Similarly, one can verify that

$$I_2 \le \delta e^{(\lambda_{\max}-\rho)t}. \tag{5.15}$$

Combining (5.11) with (5.13)–(5.14) yields

$$\begin{aligned} &\left\| \mathbf{w}^\delta(\cdot,t) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q}\in\Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \right\| \\ &\quad \le \left\{ (\hat{C}_6+1)e^{-\rho t} + \frac{\hat{C}_1\hat{C}_4\hat{C}_5}{\lambda_{\max}} \left(\sum_{i=1}^n \hat{C}_4^{\frac{i}{n}} 2^i \delta^i \left\| \mathbf{w}(\cdot,0)\right\|_{H^2}^{i+1} + \frac{\sum_{i=1}^n \hat{C}_4^{\frac{i}{n}} \frac{2^i}{i} \delta^i e^{i\lambda_{\max} t}}{\lambda_{\max}} \right) \right\} \\ &\qquad \cdot \delta e^{\lambda_{\max} t} \\ &\quad \le \hat{C} \left\{ e^{-\rho t} + \sum_{i=1}^n \left(\delta^i \|\mathbf{w}_0\|_{H^2}^{i+1} + \delta^i e^{i\lambda_{\max} t} \right) \right\} \delta e^{\lambda_{\max} t}, \end{aligned}$$

where $\hat{C} = \max\{\hat{C}_6 + 1, \frac{\hat{C}_1\hat{C}_5}{\lambda_{\max}} \sum_{i=1}^n \hat{C}_4^{\frac{i+1}{n}} 2^i\}$ and thereby conclude the proof. □

Theorem 5.1 implies that the dynamics of a general perturbation is characterized by such linear dynamics over a long time period of $\varepsilon T^\delta \le t \le T^\delta$, for any $\varepsilon > 0$. In particular, we can choose a function $\mathbf{w}_0(x) \in H^2(\mathbb{T}^d)$ such that $w_{\mathbf{q}_0}^+ \neq 0$ for at least one $\mathbf{q}_0 = [q_{01}, \ldots, q_{0d}] \in \Omega_{\max}$. Let $\mathbf{w}_0(\mathbf{x}) = \kappa \frac{\mathbf{r}_+(\mathbf{q}_0)}{|\mathbf{r}_+(\mathbf{q}_0)|} e_{\mathbf{q}_0}(\mathbf{x})$, where $\kappa = 1/\|e_{\mathbf{q}_0}\| = \sqrt{(2/\pi)^d}$ so that $\|\mathbf{w}_0(\mathbf{x})\| = 1$. Then

$$\left\|\mathbf{w}_0(\mathbf{x})\right\|_{H^2} = \left(1 + |\mathbf{q}_0|^2 + |\mathbf{q}_0|^4\right)^{\frac{1}{2}} \equiv C(\mathbf{q}_0). \tag{5.16}$$

It follows from Theorem 5.1 that for $t = T^\delta$

$$\begin{aligned} &\left\|\mathbf{w}^\delta\left(\cdot, T^\delta\right) - \delta e^{\lambda_{\max} T^\delta} \mathbf{w}_0(\mathbf{x})\right\| \\ &\quad \le \hat{C} \left\{ e^{-\rho T^\delta} + \sum_{i=1}^n \delta^i \|\mathbf{w}_0\|_{H^2}^{i+1} + \sum_{i=1}^n \delta^i e^{i\lambda_{\max} T^\delta} \right\} \delta e^{\lambda_{\max} T^\delta} \\ &\quad \le \hat{C} \left\{ e^{-\rho T^\delta} + \sum_{i=1}^n \delta^i \|\mathbf{w}_0\|_{H^2}^{i+1} + \sum_{i=1}^n \theta^i \right\} \theta. \end{aligned}$$

Choose $\theta^i < \frac{1}{4n\hat{C}}$ and δ sufficiently small such that

$$e^{-\rho T^\delta} = \left(\frac{\delta}{\theta}\right)^{\frac{\rho}{\lambda_{\max}}} < \frac{1}{8\hat{C}}, \qquad \sum_{i=1}^{n} \delta^i \|\mathbf{w}_0\|_{H^2}^{i+1} = C(\mathbf{q}_0) \sum_{i=1}^{n} \bigl(C(\mathbf{q}_0)\delta\bigr)^i < \frac{1}{8\hat{C}}.$$

Thus

$$\bigl\|\mathbf{w}^\delta\bigl(\cdot, T^\delta\bigr) - \delta e^{\lambda_{\max} T^\delta} \mathbf{w}_0(\mathbf{x})\bigr\| < \frac{\theta}{2},$$

that is,

$$\bigl\|\mathbf{w}^\delta\bigl(\mathbf{x}, T^\delta\bigr)\bigr\| \geq \bigl\|\delta e^{\lambda_{\max} T^\delta} \mathbf{w}_0(\mathbf{x})\bigr\| - \frac{1}{2}\theta > \delta e^{\lambda_{\max} T^\delta} - \frac{1}{2}\theta = \frac{\theta}{2} > 0.$$

which shows that the solution can leave a positive distance before the escape time T^δ. This results also imply that the linearized unstable equilibrium point $\bar{\mathbf{W}} = [k, \frac{\alpha}{\beta}k]$ is nonlinearly unstable.

6 Conclusion and numerical simulations

In this paper we analyze the pattern formation of a chemotaxis model with nonlinear cell diffusion. We address verification that, given any general perturbation of magnitude δ, its nonlinear evolution is dominated by the corresponding linear dynamics along a finite

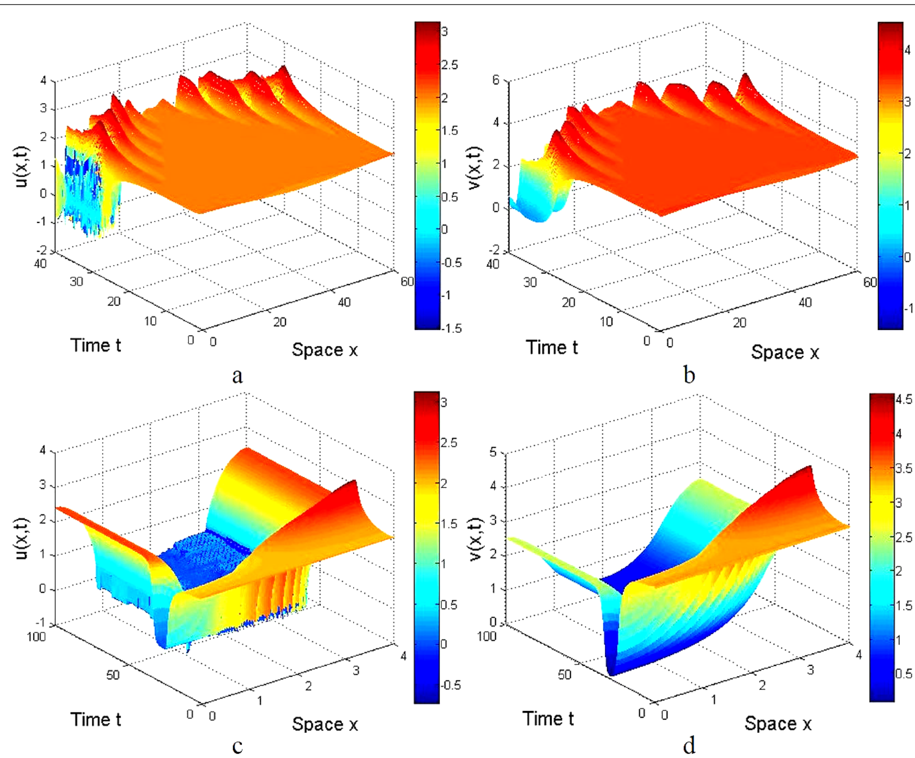

Figure 1 Pattern appears in system (1.2). In Fig. 1, the point $\bar{\mathbf{W}} = [2, 3.33]$ is nonlinearly unstable. Numerical solutions of the model (1.2) in one-dimensional domain $[0, \pi]$, $[0, 60]$, respectively, and a period of time $[0, 100]$, $[0, 40]$, respectively

number of fixed fastest growing modes, over a time period of the order $\ln(1/\delta)$. Therefore, our main results provide a quantitative characterization for nonlinear pattern formation in a nonlinear diffusion chemotaxis model with logistic source.

Finally, we give simple numerical simulations to illustrate the results got in Theorem 5.1. We apply a finite difference method on an equidistant grid and solve the resulting chemotaxis system by Newton's method. The Matlab PDE solver is implemented to solve system (1.2) subject to the Neumann boundary conditions, where the time step size $\Delta t = 0.1$ and spatial step size $\Delta x = 0.1$.

We consider the particular case of system (1.2) with fixed parameters $n = 2$, $d_1 = 1$, $d_2 = 0.5$, $\chi = 2.5$, $r = 1$, $k = 2$, $\alpha = 2$, $\beta = 1.2$, $\mathbf{q} = [0, 1]$, $q^2 = 1$, then the condition (3.5) is satisfied, and (3.4) has the two eigenvalues $\lambda_1(\mathbf{q}) = 0.215$, $\lambda_2(\mathbf{q}) = -6.915$. Initial data $u_0(x) = 2 + 0.1\cos(x/10)$, $v_0(x) = 3.33 + 0.1\sin(x/10)$. The positive equilibrium solution $\bar{\mathbf{W}} = [2, 3.33]$ becomes unstable under chemotaxis effects; see Fig. 1.

Funding
This work is supported by National Natural Science Foundation of China (No. 11661051), Sichuan province science and technology plan project (No. 2017JY0195), the Natural Science Foundation of Gansu Province (No. 1606RJZA038), the National Statistical Scientific Research Projects (No. 2017LZ41), the Scientific Study Project for Gansu Province Institutes of Higher Learning (No. 2017B-41), and the Doctor Start-up Funding of Neijiang Normal University (No. 16B01).

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Publisher's Note

Received: 6 December 2017 Accepted: 11 April 2018 Published online: 20 April 2018

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