Du et al. *Boundary Value Problems* (2018) 2018:50
https://doi.org/10.1186/s13661-018-0964-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

Partial differential equation modeling with Dirichlet boundary conditions on social networks

Bo Du*, Xiuguo Lian and Xiwang Cheng

*Correspondence:
dubo7307@163.com
Department of Mathematics, Huaiyin Normal University, Huaian, P.R. China

Abstract

The being a wide range of applications of the Internet, social networks have become an effective and convenient platform for information communication, propagation and diffusion. Most of information exchange and spreading exist in social networks. The issue of information diffusion in social networks is getting more and more attention by government and individuals. The researchers investigated either empirical studies or focused on ordinary differential equation (ODE) models with only consideration of temporal dimension in most prior work. As is well known, partial differential equations (PDEs) can describe temporal and spatial patterns of information diffusion over online social networks; however, until now, results for understanding information propagation of social networks over both temporal and spatial dimensions are few. This paper is devoted to investigating a non-autonomous diffusive logistic model with Dirichlet boundary conditions to describe the process of information propagation in social networks. By constructing upper and lower solutions we obtain the dynamic behavior of the solution to the non-autonomous diffusive logistic model. Our results show that information diffusion is greatly affected by the diffusion coefficient $d(t)$ and the intrinsic growth rate $r(t)$.

Keywords: Social networks; Spatio-temporal patterns; Logistic model; Stability

1 Introduction

With the rapid development of Internet technology, a new platform for information communication and diffusion has been constructed by online social networking which has established a wider range of social relations [1–5]. Online social networks have better features, including the speed and range of information transmission, rather than the traditional ways of information exchange. Moreover, social networks also play a significant role in business negotiation and information sharing. A lot of people used some new social media sites such as Twitter, Facebook and Wechat as they first appeared. We believe that these new social media sites will have in-depth development and play a key role in contemporary society. Facing large amounts of data, the laws of communication for big data on online social networks should be studied by the researchers. For reducing unwanted information over social media, studying information diffusion process is necessary. However, due to the complexity of the network structure and the rapid change of social network platforms, for studying information spreading on online social networks there exist many difficulties.

The mechanism of information diffusion on online social networks including characterizing user behavior, characterizing social cascades in Flickr, network level footprints of Facebook, applications etc. [6–15] has been studied by many authors. Recently, some results have been obtained by using mathematical models to predict information diffusion over a time period in online social networks; see e.g. [16–20]. Inspired by empirical studies of networked systems, Newman [21] studied the structure and function of complex networks. Banavar, Maritan and Rinaldo [22] derived a general relationship between size and flow rates in general networks with local connectivity which can predict scaling relations applicable to all efficient transportation networks.

A number of concerns in modeling information diffusion in online social networks have been put forward on the basis of PDE-based models. Specially, PDEs can establish many complex models. However, the results are uncommonly poor for the diffusion models by PDEs in online social networks. Recently, both temporal and spatial patterns of information diffusion process on social networks were studied by Wang et al. [23–25] through constructing an intuitive cyber-distance among online users. Using real data coming from Digg (an online social network), Wang verified the reliability of the PDE model. Zhu, Zhao and Wang [26–28] studied several reaction–diffusion malware propagation models and obtained some results for stability and bifurcation of positive equilibrium points. Dai et al. [29] studied a partial differential equation with a Robin boundary condition in online social networks and discussed temporal and spatial properties of social networks. Since PDEs have complex natures and no standard characteristic equations, it is difficult to study PDE models by using matrix theory. Hence, the authors only have begun to study propagation models with PDEs in online social networks, and there are still many problems to be solved.

As is well known, studying information diffusion in online social networks by PDE-based models is very difficult, and this presents a new opportunity and challenge for mathematicians. In the ecological and physics models, the researchers focused on different boundary conditions, eigenvalue analysis and dynamic properties of diffusion. Cantrell and Cosner [30] studied a diffusive logistic equation with spatially varying growth rate. The authors obtained the theorem for the principal eigenvalue of the corresponding linearized equation, which is significant for the research of the dynamics of a population inhabiting a heterogeneous environment. Afrozi and Brown [31] discussed the existence of principal eigenvalues for Robin boundary conditions with an indefinite weight function.

This paper is devoted to investigating a non-autonomous diffusive logistic model with Dirichlet boundary conditions. In view of Digg.com and the simulation, there exists a real data set from an online social network for the above model. In [23], the overall accuracy for the logistic model with the Robin boundary conditions is 96.61% and an overall accuracy for the Neumann boundary condition of 92.85% was obtained. Moreover, the dynamic properties of a non-autonomous logistic model with Dirichlet boundary conditions are obtained by constructing upper and lower solutions. In this paper, our main contributions are summarized as follows.

(1) We develop a non-autonomous model with Dirichlet boundary conditions by using PDEs on social networks. To the best of our knowledge, few results have been obtained for our model. Our new model is more accurate as regards the actual situation; hence, it will be more important for the applications.

(2) We study the sensitivity of some parameters in the present model, guaranteeing the controllability of the model. Thus, we can control the stability interval by sensitive parameters which have important implications for practical applications.
(3) It makes a lot of sense to establish a unified framework to handle the reaction–diffusion terms and the influence of the variable coefficients. We develop some mathematical techniques (including upper and lower solutions method, comparison principle, and the like) for overcoming these difficulties.
(4) Our model is based on non-autonomous partial differential equations which are proposed to characterize temporal and spatial patterns of information diffusion over online social networks.

The remaining structure of this paper is arranged as follows. In Sect. 2, a non-autonomous PDE model with Dirichlet boundary conditions is developed in n-dimensional space $\mathbb{R}^n$. In Sect. 3, a number of dynamic properties for the present model are presented. Finally, conclusions are drawn in Sect. 4.

2 Non-autonomous PDE model in social networks

In view of the PDE model, by constructing an intuitive cyber-distance among online users, Wang et al. [23] studied both temporal and spatial patterns of information diffusion process in social media. In generally, we divide the information diffusion into two sections: content-based and structure-based processes in an online social network. Myers, Zhu and Leskovec [20] pointed out that the content-based and structure-based processes are analogous to the external and internal influences, respectively. It is challenging to reflect the above two process of information propagation in a mathematical model. PDEs can combine with time scale and space scale effectively and characterize the space-time developing properties of the complex models.

In generally, the spatial distance and abstractly translated information propagation process can be described by friendship hops which play dominant roles in social networks. In the social network graph, we describe the number of friendship links by using the shortest path from one user to another for defining the distance between two users. Particularly, the social distance is defined by x-axis and the density at the location x is defined by U_x. Let $I(t,x)$ be the density of influenced users at distance x during time t. The flux of the influenced users at distance x during time t can be denoted by $J = -c\frac{\partial I}{\partial x}$, where c represents the popularity of information. In Twitter, marking keywords or topics can be described by a hashtag. Since online users in social media show heterogeneity, c can be controlled by the distance x from the source in the content-based process. The distance metric can characterize the shortest friendship hops, and the average distance on Twitter is 4.67 [32]. Wang et al. [23] studied the following diffusion model:

$$\begin{cases} I_t = d\Delta I - rI(1 - \frac{I}{K}), & t > 0, \\ I(x,1) = \phi(x), & l < x < L, \\ I_x(l,t) = I_x(L,t) = 0, & t \geq 0. \end{cases}$$

Specifically, the majority of online social network users in Digg have a distance of 2 to 5 from the initiators and the predicted results for the most popular news with 24,099 votes in Digg.com can be found in [23].

In [33], Tang and Lin studied a nonlinear reaction–diffusion equation as follows:

$$\begin{cases} u_t = d\Delta u + u(a - bu^q), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where d, a and b are positive constants, Ω is an open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Then they changed (2.1) into a non-autonomous reaction–diffusion problem:

$$\begin{cases} v_t = \frac{d}{\rho^2(t)}\Delta u - \frac{n\dot{\rho}(t)}{\rho(t)} + v(a - bv^q), & y \in \Omega(0), t > 0, \\ v(y,t) = 0, & y \in \partial\Omega(0), t > 0, \\ v(y,0) = v_0(y), & y \in \Omega(0), \end{cases} \tag{2.2}$$

where

$$\begin{aligned} &\Omega(t) \subset \mathbb{R}^n \text{is a simply connected bounded growing domain at time } t \geq 0 \\ &\quad \text{with its growing boundary } \partial\Omega(t). \end{aligned} \tag{2.3}$$

For more details as regards $\Omega(t)$ in (2.3), see [34]. The model (2.1) is an insect dispersal model on a growing domain. By constructing upper and lower solutions of (2.2), the authors obtained the asymptotic behavior of the solution to (2.1). From the research of [34], we find that the properties of solutions for non-autonomous reaction–diffusion model (2.2) are key to the asymptotic behavior of the solution to (2.1).

Motivated by the above work, this paper is devoted to investigating the following non-autonomous PDE model with Dirichlet boundary conditions:

$$\begin{cases} u_t - d(t)\Delta u = r(t)u(a - bu^q), & t > 0, x \in \Omega(0), \\ u(t,x) = 0, & t > 0, x \in \partial\Omega(0), \\ u(0,x) = u_0(x), & x \in \Omega(0), \end{cases} \tag{2.4}$$

where $\Omega(t)$ is defined by (2.3), q is positive constant, $u(t,x)$ represents the density of influenced users with a distance of x at time t. For the other parameters' meanings, see Table 1.

Remark 2.1 When $q = 1$, $d(t)$ and $r(t)$ are constants, system (2.4) is a classic diffusive logistic equation which has been widely studied in [35–37]. When $d(t)$ and $r(t)$ are not constants, system (2.4) is a non-autonomous reaction–diffusion system. For a general non-autonomous reaction–diffusion system, Rodriguez-Bernal and Vidal-Lopez [38] studied

Table 1 Symbols and their meanings of system (2.4)

Parameters	Meaning
$d(t)$	The popularity of information which promotes the spread of the information
$r(t)$	The intrinsic growth rate of influenced users with the same distance
b	The carrying capacity which is the maximum possible density of users
a	The intra-specific competition rate with influenced users at a given distance

the following general non-autonomous reaction–diffusion problem:

$$\begin{cases} u_t = d\Delta u + a(t,x)u - b(t,x)u^{q+1}, & t > 0, x \in \Omega, \\ u(t,x) = 0, & t > 0, x \in \partial\Omega, \\ u(0,x) = u_0(x) \geq 0, & x \in \Omega, \end{cases}$$

and they obtained some dynamic properties of positive complete trajectories for the above problem.

Firstly, we consider the following eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.5}$$

For problem (2.1), the following result is well known.

Theorem 2.1 ([39]) *Let λ_1 be the principal positive eigenvalue of* (2.5).

(1) *If $a \leq d\lambda_1$, then* (2.1) *admits only one nonnegative steady state solution $u = 0$, which is globally asymptotically stable.*
(2) *If $a > d\lambda_1$, then* (2.1) *has only one positive steady state solution $u = u^*(x)$, which is globally asymptotically stable.*

3 Dynamic analysis for a diffusion logistic model

In this section we will study the dynamic behavior of the solution of (2.4). First we give the following assumptions:

(H_1) $d(t)$ is continuous differentiable decreasing positive function on $[0,+\infty)$ and satisfies

$$\lim_{t\to+\infty} d(t) = d_\infty.$$

(H_2) $r(t)$ is continuous differentiable function on $[0,+\infty)$ and satisfies

$$r_1 \leq \lim_{t\to+\infty} r(t) \leq r_2,$$

where r_1, r_2 are positive constants.

Definition 3.1 A function $\hat{u} \in C^{2,1}(\Omega(0) \times (0,\infty)) \cap C(\bar{\Omega}(0) \times [0,\infty))$ is called an upper solution of (2.4) if it satisfies

$$\begin{cases} \hat{u}_t - d(t)\Delta\hat{u} \geq f(t,\hat{u}), & t > 0, x \in \Omega(0), \\ \hat{u}(t,x) \geq 0, & t > 0, x \in \partial\Omega(0), \\ \hat{u}(0,x) \geq u_0(x), & x \in \Omega(0), \end{cases} \tag{3.1}$$

where $f(t,u) = r(t)u(a - bu^q)$. Similarly, $\check{u}$ is called a lower solution of (2.4) if it satisfies all the reversed inequalities in (3.1).

Lemma 3.1 (Comparison principle) *Let $v(t,x)$ be a solution of* (2.4), *$\hat{v}(t,x)$ and $\check{v}(t,x)$ are upper and lower solutions of* (2.4) *respectively, then $\check{v}(t,x) \le v(t,x) \le \hat{v}(t,x)$ in $\bar{\Omega} \times [0,+\infty)$.*

Lemma 3.2 *Let $u(t,x)$ be a nonnegative nontrivial solution of the following problem*:

$$\begin{cases} u_t = d(t)\Delta u + r(t)u(a - bu^q), & t > 0, x \in \Omega(0), \\ u(t,x) = 0, & t > 0, x \in \partial\Omega(0), \\ u(0,x) = u_0(x) \ge 0, & x \in \Omega(0). \end{cases} \tag{3.2}$$

If $u_0(x) \in C^2(\bar{\Omega}(0))$, $u_0(x) = \Delta u(0,x) = 0$ for $x \in \partial\Omega(0)$ and $\Delta u(0,x) \le 0$ for $x \in \bar{\Omega}(0)$, then $u(t,x) \in C^{2,1}(\Omega(0) \times (0,\infty))$ and

$$\Delta u(t,x) \le 0 \quad \textit{for } x \in \Omega(0), t > 0.$$

Proof In view of u_0 being smooth and for $x \in \partial\Omega(0)$, $d(t)u_0(x) + r(t)u_0(a - bu_0^q) = 0$, by the standard parabolic regularity theory it follows that the solution $u(t,x) \in C^{2,1}(\Omega(0) \times (0,\infty))$. Denote $\omega = \Delta u$. For $t > 0$, $x \in \Omega(0)$, we have

$$\omega_t \le d(t)\Delta\omega + \bigl(r(t)a - b(q+1)u^q\bigr)\omega.$$

From the condition $\Delta u(x,0) \le 0$ for $x \in \Omega(0)$, we have

$$\omega(0,x) \le 0 \quad \text{for } x \in \Omega(0).$$

By $u(t,x) = 0$ for $x \in \partial\Omega(0)$, $t > 0$, we have

$$\omega(t,x) = \frac{1}{d(t)}\bigl[u_t - r(t)u\bigl(a - bu^q\bigr)\bigr] = 0, \quad x \in \partial\Omega(0), t > 0.$$

By Lemma 3.1, we can obtain that

$$\omega(t,x) \le 0 \quad \text{for } x \in \partial\Omega(0), \quad t > 0,$$

which implies that $\Delta u(t,x) \le 0$ for $x \in \partial\Omega(0)$, $t > 0$. The proof is completed. □

Let λ_1 be the principal positive eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u, & y \in \Omega(0), \\ u(t,y) = 0, & y \in \partial\Omega(0). \end{cases}$$

Theorem 3.1 *If the assumptions* (H_1) *and* (H_2) *hold and $a \le \frac{d_\infty \lambda_1}{r_2}$, then the solution of problem* (2.4) *satisfies*

$$\lim_{t\to\infty} u(t,x) = 0 \quad \textit{for } x \in \bar{\Omega}(0).$$

Proof Obviously, $\check{u} = 0$ is a lower solution of (2.4). Next we construct a upper solution of (2.4). Let $\hat{u}(t,x)$ be the unique solution of the problem:

$$\begin{cases} \hat{u}_t - d(t)\Delta\hat{u} = r(t)\hat{u}(a - b\hat{u}^q), & t > 0, x \in \Omega(0), \\ \hat{u}(t,x) = 0, & t > 0, x \in \partial\Omega(0), \\ \hat{u}(0,x) = M\phi(x), & x \in \Omega(0), \end{cases} \tag{3.3}$$

where $\phi(x)$ is the corresponding eigenfunction of λ_1, M is a positive constant. We choose M so large that $M\phi(x) \geq u_0(x)$, For any solution $\hat{u}(t,x)$ of (3.3), we can see that $\hat{u}(t,x)$ is an upper solution of (2.4). Let u be any solution of (2.4). It follows from Lemma 3.1 that

$$0 \leq u(t,x) \leq \hat{u}(t,x), \quad t > 0, x \in \Omega(0).$$

Since $\Delta\hat{u}(0,x) = M\Delta\phi(x) = -\lambda_1 M\phi(x) \leq 0$, it follows from Lemma 3.2 that

$$\Delta\hat{u}(t,x) \leq 0 \quad \text{for } x \in \Omega(0), t > 0.$$

On the other hand, by assumption (H_1), $d(t)$ tends decreasingly to d_∞ as $t \to \infty$, then $d(t) \geq d_\infty$ for $t \geq 0$. By assumption (H_2), $r_1 < r(t) \leq r_2$ for $t \geq 0$. Thus, $\hat{u}(t,x)$ satisfies

$$\hat{u}_t \leq d_\infty\Delta\hat{u} + r_2 a\hat{u} - r_1 b\hat{u}^{q+1}, \quad t > 0, x \in \Omega(0).$$

Now consider the following problem:

$$\begin{cases} \bar{u}_t = d_\infty\Delta\bar{u} + r_2 a\bar{u} - r_1 b\bar{u}^{q+1}, & t > 0, x \in \Omega(0), \\ \bar{u}(t,x) = 0, & t > 0, x \in \partial\Omega(0), \\ \bar{u}(0,x) = M\phi(x), & x \in \Omega(0). \end{cases} \tag{3.4}$$

Let $\bar{u}(t,x)$ be a solution of (3.4). By the comparison principle, for $t > 0$, $x \in \Omega(0)$, we have

$$\hat{u}(t,x) \leq \bar{u}(t,x)$$

and

$$0 \leq u(t,x) \leq \hat{u}(t,x) \leq \bar{u}(t,x).$$

Since $ar_2 \leq d_\infty\lambda_1$, by Theorem 2.1, $\lim_{t\to\infty} \bar{u}(t,x) = 0$ for $x \in \bar{\Omega}(0)$. Hence,

$$\lim_{t\to\infty} u(t,x) = 0 \quad \text{for } x \in \bar{\Omega}(0).$$ □

Theorem 3.2 *If the assumptions* (H_1) *and* (H_2) *hold and* $a > \frac{d_\infty\lambda_1}{r_1}$, *then the solution of problem* (3.14) *satisfies*

$$\check{u}^*(x) \leq u(x) \leq \bar{u}^*(x) \quad \text{for } x \in \bar{\Omega}(0),$$

where $u^*(x)$ *is a unique positive solution of* (3.8), $\check{u}^*(x)$ *is a unique positive solution of* (3.15).

Proof In view of assumption (H_1), we have $\lim_{t\to\infty} d(t) = d_\infty$ and $d(t)$ is decreasing for $t > 0$, there exists a $T_1 > 0$ such that $d_\infty \le d(t) \le d_\infty + \varepsilon$ for $t > T_1$. Let $\hat{u}(t,x)$ be the unique solution of the problem:

$$\begin{cases} \hat{u}_t - d(t)\Delta\hat{u} = r(t)\hat{u}(a - b\hat{u}^q), & t > T_1, x \in \Omega(0), \\ \hat{u}(t,x) = 0, & t > T_1, x \in \partial\Omega(0), \\ \hat{u}(T_1,x) = M\phi(x), & x \in \Omega(0), \end{cases} \tag{3.5}$$

where $\phi(x)$ is the corresponding eigenfunction of λ_1, M is a positive constant. We choose M so large that $M\phi(x) \ge u_0(x)$, For any solution $\hat{u}(t,x)$ of (3.5), we can see that $\hat{u}(t,x)$ is an upper solution of (2.4). Let u be any solution of (2.4). It follows from Lemma 3.1 that

$$0 \le u(t,x) \le \hat{u}(t,x), \quad t > T_1, x \in \Omega(0).$$

From $\Delta\hat{u}(T_1,x) \le 0$ and Lemma 3.2, we have $\Delta\hat{u}(t,x) \le 0$ in $[T_1,\infty) \times \Omega(0)$. Thus,

$$\hat{u}_t \le d_\infty\Delta\hat{u} + r_2 a\hat{u} - r_1 b\hat{u}^{q+1}, \quad t > T_1, x \in \Omega(0). \tag{3.6}$$

Now consider the following problem:

$$\begin{cases} u_t = d_\infty\Delta u + r_2 au - r_1 bu^{q+1}, & t > T_1, x \in \Omega(0), \\ u(t,x) = 0, & t > T_1, x \in \partial\Omega(0), \\ u(T_1,x) = M\phi(x), & x \in \Omega(0). \end{cases} \tag{3.7}$$

The problem (3.7) admits a unique solution $\bar{u}(t,x)$. In view of $a > \frac{d_\infty\lambda_1}{r_1}$ and $r_1 \le r_2$, we have $a > \frac{d_\infty\lambda_1}{r_2}$. Thus, the result of Theorem 2.1 shows that

$$\bar{u}(t,x) \to \bar{u}^*(x) \quad \text{as } t \to \infty,$$

where $\bar{u}^*(x)$ is the unique positive solution of the following problem:

$$\begin{cases} u_t = d_\infty\Delta u + r_2 au - r_1 bu^{q+1}, & t > T_1, x \in \Omega(0), \\ u(x) = 0, & x \in \partial\Omega(0). \end{cases} \tag{3.8}$$

Using (3.6), (3.7) and the comparison principle yields

$$\hat{u}(t,x) \le \bar{u}(t,x) \quad \text{for } t > T_1, x \in \Omega(0).$$

This implies that

$$\lim_{t\to\infty} \sup u(t,x) \le u^*(x), \quad x \in \Omega(0). \tag{3.9}$$

On the other hand, let $\check{u}(t,x)$ be the unique solution of the problem:

$$\begin{cases} \check{u}_t - d(t)\Delta\check{u} = r(t)\check{u}(a - b\check{u}^q), & t > T_1, x \in \Omega(0), \\ \check{u}(t,x) = 0, & t > T_1, x \in \partial\Omega(0), \\ \check{u}(T_1,x) = \delta\phi(x), & x \in \Omega(0), \end{cases} \tag{3.10}$$

where δ is a sufficiently small constant such that $\delta\phi(x) \le u_0$. It is easy to see that $\check{u}(t,x)$ is a lower solution of (2.4) in $[T_1,\infty) \times \bar{\Omega}(0)$. From $\Delta\check{u}(T_1,x) = -\delta\lambda_1\phi(x) \le 0$ and Lemma 3.1, we have $\Delta\check{u}(t,x) \le 0$ in $[T_1,\infty) \times \Omega(0)$. Thus,

$$\check{u}_t \ge (d_\infty + \varepsilon)\Delta\check{u} + r_1 a\hat{u} - r_2 b\hat{u}^{q+1}, \quad t > T_1, x \in \Omega(0). \tag{3.11}$$

Now consider the following problem:

$$\begin{cases} \check{u}_t = (d_\infty + \varepsilon)\Delta\check{u} + r_1 a\check{u} - r_2 b\hat{u}^{q+1}, & t > T_1, x \in \Omega(0), \\ \check{u}(t,x) = 0, & t > T_1, x \in \partial\Omega(0), \\ \check{u}(T_1,x) = \delta\phi(x), & x \in \Omega(0). \end{cases} \tag{3.12}$$

Clearly, (3.12) admits a unique positive solution, denoted by $\check{u}_\varepsilon(t,x)$. Using the comparison principle yields that $\check{u}_\varepsilon(t,x) \le \check{u}(t,x)$. Since $ar_1 > d_\infty\lambda_1$, we can choose $\varepsilon > 0$ sufficiently small such that $ar_1 > (d_\infty + \varepsilon)\lambda_1$. Thus, by Theorem 2.1 we have

$$\lim_{t\to\infty} \check{u}_\varepsilon(t,x) = \check{u}_\varepsilon^*(x), \quad x \in \bar{\Omega}(0),$$

where $\check{u}_\varepsilon^*(x)$ is the unique positive solution of the problem

$$\begin{cases} -\check{u}_t = (d_\infty + \varepsilon)\Delta\check{u} + r_1 a\check{u} - r_2 b\hat{u}^{q+1}, & x \in \Omega(0), \\ \check{u}(x) = 0, & x \in \partial\Omega(0). \end{cases} \tag{3.13}$$

From (3.11) and (3.13), it follows that

$$\lim_{t\to\infty} \inf u(t,x) \ge \check{u}_\varepsilon^*(x), \quad x \in \Omega(0). \tag{3.14}$$

By the continuous dependence of $\check{u}_\varepsilon^*(x)$ on ε, obviously,

$$\check{u}_\varepsilon^*(x) \to \check{u}^*(x) \quad \text{as } \varepsilon \to 0^+,$$

where $\check{u}^*(x)$ is a solution of the problem

$$\begin{cases} -d_\infty\Delta\check{u} = r_1 a\check{u} - r_2 b\hat{u}^{q+1}, & x \in \Omega(0), \\ \check{u}(x) = 0, & x \in \partial\Omega(0). \end{cases} \tag{3.15}$$

By (3.14), we have

$$\lim_{t\to\infty} \inf u(t,x) \ge \check{u}^*(x), \quad x \in \Omega(0). \tag{3.16}$$

It follows from (3.9) and (3.16) that

$$\check{u}^*(x) \le u(x) \le \bar{u}^*(x) \quad \text{for } x \in \bar{\Omega}(0).$$

This completes the proof. □

Remark 3.1 If $r(t) = r$ in (2.4) is a positive constant, then $\check{u}^*(x) = \bar{u}^*(x) := u^*(x)$. Hence, for (2.4) there exists a unique positive steady state solution $u^*(x)$. It is obvious that the results of [33] are special results of the present paper in the case of $r(t) = r$ in (2.4) being a positive constant. On the other hand, by upper and lower solutions, we cannot find that for (2.4) there exists a unique positive steady state solution $u^*(x)$ and we only obtain the scope of the solution to (2.4). We hope that some new methods can be developed by the researchers for obtaining an unique positive solution of (2.4).

4 Conclusions

In this article, we study a non-autonomous reaction–diffusion system on social networks. It is noted that the diffusion rate d and intrinsic growth rate r are not constants, which is different from the past results [25, 26, 33, 34].

How the non-autonomous case affects information propagation and the dynamic properties of solutions over social networks is obtained by upper and lower solutions methods and comparison principle. More importantly, we investigate the effects of the variable diffusion rate and the intrinsic growth rate on the scale of information spreading in networks. Our results show that a non-autonomous reaction–diffusion system has more complex asymptotic stability than an autonomous reaction–diffusion system.

Acknowledgements
The authors would like to thanks the editor and the referees for their valuable comments and suggestions, which improved the quality of our paper.

Funding
The work is supported by Natural Science Foundation of Jiangsu High Education Institutions of China (Grant No. 17KJB110001).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Publisher's Note

Received: 6 January 2018 Accepted: 20 March 2018 Published online: 04 April 2018

References

1. Xia, L., Jiang, G., Song, B., Song, Y.: Rumor spreading model considering hesitating mechanism in complex social networks. Physica A **437**, 295–303 (2015)
2. Zhao, J., Wu, J., Xu, K.: Weak ties: subtle role of information diffusion in online social networks. Phys. Rev. E **82**, 016105 (2010)
3. Zheng, M., Lu, L., Zhao, M.: Spreading in online social networks: the role of social reinforcement. Phys. Rev. E **88**, 012818 (2013)
4. Huberman, B.A.: The Laws of the Web. MIT Press, Cambridge (2001)
5. Jeong, H., Tombor, B., Albert, R., Oltvai, Z.N., Barabási, A.-L.: The large-scale organization of metabolic networks. Nature (London) **411**, 651–654 (2000). https://doi.org/10.1038/35036627
6. Benevenuto, F., Rodrigues, T., Cha, Y., Almeida, V.: Characterizing user behavior in online social networks. In: Proceedings of the 9th ACM SIGCOMM Conference on Internet Measurement (IMC'09), pp. 49–62. ACM, New York (2009). https://doi.org/10.1145/1644893.1644900
7. Cha, Y., Mislove, A., Adams, B., Gummadi, K.: Characterizing social cascades in Flickr. In: Proceedings of the First Workshop on Online Social Networks (WOSN'08), pp. 13–18 (2008)
8. Lerman, K., Ghosh, R.: Information contagion: an empirical study of spread of news on digg and Twitter social networks. In: Proceedings of 4th International Conference on Weblogs and Social, Media (2010)
9. Nazir, A., Raza, S., Gupta, D., Chuah, C., Krishnamurthy, B.: Network level footprints of Facebook applications. In: Proceedings of the 9th ACM SIGCOMM Conference on Internet Measurement (IMC'09), pp. 63–75. ACM, New York (2009). https://doi.org/10.1145/1644893.1644901
10. Du, B., Hu, M., Lian, X.: Dynamical behavior for a stochastic predator-prey model with HV type functional response. Bull. Malays. Math. Soc. **40**, 487–503 (2017)

11. Tang, S., Blenn, N., Doerr, C., Van Mieghem, P.: Digging in the digg social news website. IEEE Trans. Multimed. **13**(5), 1163–1175 (2011). https://doi.org/10.1109/TMM.2011.2159706
12. Du, B.: Stability analysis of periodic solution for a complex-valued neural networks with bounded and unbounded delays. Asian J. Control **20**, 1–12 (2018)
13. Yu, B., Fei, H.: Modeling social cascade in the Flickr social network. In: Sixth International Conference on Fuzzy Systems and Knowledge Discovery (FSKD'09). IEEE, New York (2009). https://doi.org/10.1109/FSKD.2009.719
14. Schneider, F., Feldmann, A., Krishnamurthy, B., Willinger, W.: Understanding online social network usage from a network perspective. In: Proceedings of the 9th ACM SIGCOMM Conference on Internet Measurement (IMC'09), pp. 35–48. ACM, New York (2009). https://doi.org/10.1145/1644893.1644899
15. Du, Y., Lin, Z.: Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary. SIAM J. Math. Anal. **42**, 377–405 (2010)
16. Barrat, A., Barthelemy, M., Vespignani, A.: Dynamical Processes on Complex Networks. Cambridge University Press, Cambridge (2008)
17. Ghosh, R., Lerman, K.: A framework for quantitative analysis of cascades on network. In: Proceedings of the Fourth ACM International Conference on Web Search and Data Mining, pp. 665–674. ACM, New York (2011). https://doi.org/10.1145/1935826.1935917
18. Jin, F., Dougherty, E., Saraf, P., Cao, Y., Ramakrishnan, N.: Epidemiological modeling of news and rumors on Twitter. In: Proceedings of the 7th Workshop on Social Network Mining and Analysis (SNAKDD'13), Article No. 8. ACM, New York (2013). https://doi.org/10.1145/2501025.2501027
19. Kumar, R., Novak, J., Tomkins, A.: Structure and evolution of online social networks. In: Proceedings of the 12th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, pp. 611–617 (2006)
20. Myers, S., Zhu, C., Leskovec, J.: Information diffusion and external influence in networks. In: Proceedings of the 18th ACM, SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD'12), pp. 33–41 (2012)
21. Newman, M.: The structure and function of complex networks. SIAM Rev. **45**, 167–256 (2003)
22. Banavar, J., Maritan, A., Rinaldo, A.: Size and form in efficient transportation networks. Nature **399**, 130–132 (1999)
23. Wang, F., Wang, H., Xu, K.: Diffusive logistic model towards predicting information diffusion in online social networks. In: Distributed Computing Systems Workshops (ICDCSW), 2012 32nd International Conference, pp. 133–139 (2012)
24. Wang, F., Wang, H., Xu, K.: Characterizing information diffusion in on-line social networks with linear diffusive model. In: 2013 IEEE 33rd International Conference on Distributed Computing Systems (ICDCS), pp. 307–316 (2013)
25. Lei, C., Lin, Z., Wang, H.: The free boundary problem describing information diffusion in online social networks. J. Differ. Equ. **254**, 1326–1341 (2013)
26. Zhu, L., Zhao, H., Wang, X.: Stability and bifurcation analysis in a delayed reaction-diffusion malware propagation model. Comput. Math. Appl. **69**, 852–887 (2015)
27. Zhu, L., Zhao, H., Wang, X.: Bifurcation analysis of a delay reaction–diffusion malware propagation model with feedback control. Commun. Nonlinear Sci. Numer. Simul. **22**, 747–768 (2015)
28. Zhu, L., Zhao, H., Wang, X.: Complex dynamic behavior of a rumor propagation model with spatial-temporal diffusion terms. Inf. Sci. **349–350**, 119–136 (2016)
29. Dai, G., Ma, R., Wang, H., Wang, F., Xu, K.: Partial differential equations with Robin boundary condition in online social networks. Discrete Contin. Dyn. Syst. **20**, 1609–1624 (2017)
30. Cantrell, R., Cosner, C.: The effects of spatial heterogeneity in population dynamics. J. Math. Biol. **29**, 315–338 (1991)
31. Afrozi, G., Brown, K.: On principal eigenvalues for boundary value problems with indefinite weight and Robin boundary conditions. Proc. Am. Math. Soc. **127**, 125–130 (1999)
32. http://en.wikipedia.org/wiki/Six-degrees-of-separation
33. Tang, Q., Lin, Z.: The asymptotic analysis of an insect dispersal model on a growing domain. J. Math. Anal. Appl. **378**, 649–656 (2011)
34. Crampin, E.J., Gaffney, E.A., Maini, P.K.: Reaction and diffusion on growing domains: scenarios for robust pattern formation. Bull. Math. Biol. **61**, 1093–1120 (1999)
35. Aronson, D.G., Weinberger, H.F.: Multidimensional nonlinear diffusions arising in population genetics. Adv. Math. **30**, 33–76 (1978)
36. Pao, C.V.: Nonlinear Parabolic and Elliptic Equations. Plenum Press, New York (1992)
37. Ludwig, D., Aronson, D.G., Weinberger, H.F.: Spatial patterning of the spruce budworm. J. Math. Biol. **8**, 217–258 (1979)
38. Rodriguez-Bernal, A., Vidal-Lopez, A.: Existence, uniqueness and attractivity properties of positive complete trajectories for non-autonomous reaction–diffusion problem. Discrete Contin. Dyn. Syst. **18**, 537–567 (2007)
39. Murray, J.D.: Mathematical Biology. Springer, Berlin (1993)