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General decay for viscoelastic plate equation with p-Laplacian and time-varying delay

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Abstract

In this paper we study the viscoelastic plate equation with p-Laplacian and time-varying delay. We establish a general decay rate result under some restrictions on the coefficients of strong damping and strong time-varying delay and weakening the usual assumptions on the relaxation function, using the energy perturbation method.

MSC: 35B35; 37B25; 74Dxx; 93D20

Keywords: Memory; Plate equation; General decay rate; p-Laplacian; Time-varying delay

1 Introduction

In this paper, we are concerned with the following problem:

$$u_{tt} + \alpha \Delta^2 u - \Delta_p u - \int_{-\infty}^{t} g(t-s)\Delta^2 u(s)\,ds - \mu_1 \Delta u_t$$

$$- \mu_2 \Delta u_t(t - \tau(t)) + f(u) = h, \quad \Omega \times \mathbb{R}^+, \tag{1.1}$$

$$u = \Delta u = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.3}$$

$$u_t(x,t) = f_0(x,t), \quad (x,t) \in \Omega \times \big[-\tau(0), 0\big), \tag{1.4}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a sufficiently smooth boundary $\partial\Omega$, and

$$\Delta_p u = \operatorname{div}\big(\|\nabla u\|^{p-2}\nabla u\big)$$

is the p-Laplacian operator. The unknown function $u(x,t)$ denotes the transverse displacement of a plate filament with prescribed history $u_0(x,t), t \leq 0$. The constants α and μ_1 are positive and μ_2 is a real number. The function $\tau(t) > 0$ represents the time-varying delay, $g > 0$ is the memory kernel and f is forcing term.

The plate equation with lower order perturbation of p-Laplacian type,

$$u_{tt} + \Delta^2 u - \Delta_p u - \Delta u_t = h(x),$$

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has been extensively studied (see [1–3]) and results concerning existence, nonexistence and long-time behavior of solutions have been considered. This model can be regarded as describing elastoplastic microstructure flows. On the other hand, the elliptic problem for p-Laplace operator can be found in [4]. Recently, Torres [5] showed the existence of a solution for the fractional p-Laplacian Dirichlet problem with mixed derivatives.

When $\mu_2 = 0$ in Eq. (1.1), that is, in the absence of delay, problem (1.1) with strong damping was investigated by Jorge Silva and Ma [6]. They established exponential stability of solutions under the condition

$$g'(t) \leq -cg(t), \quad \forall t \geq 0, \tag{1.5}$$

for some $c > 0$. Andrade et al. [7] proved exponential stability of solutions for the plate equation with finite memory and p-Laplacian. The viscosity term $-\Delta u_t$ is often called a Kelvin–Voigt type dissipation or strong dissipation; it appears in phenomena of wave propagation in a viscoelastic material. Nakao [8] obtained the existence and uniqueness of a global decaying solution for the quasilinear wave equation with Kelvin–Voigt dissipation and a derivative nonlinearity. Pukach et al. [9] studied sufficient conditions of nonexistence of global in time solution for a nonlinear evolution equation with memory generalizing the Voigt–Kelvin model. Recently, Cavalcanti et al. [10] considered intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density.

Time delays so often arise in many physical, chemical, biological, thermal, and economical phenomena because these phenomena depend not only on the present state but also on the past history of the system in a more complicated way. In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay effects. Nicaise and Pignotti [11] investigated some stability results for the following wave equation with a linear damping and delay term in the domain:

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(t - \tau) = 0 \tag{1.6}$$

in the case $0 < \mu_2 < \mu_1$. Moreover, the same results were obtained when both the damping and the delay act on the boundary. Nicaise and Pignotti [12] studied exponential stability results for the following wave equation with time-dependent delay:

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t\big(t - \tau(t)\big) = 0 \tag{1.7}$$

under the condition $|\mu_2| < \sqrt{1 - d}\,\mu_1$. Kirane and Said-Houari [13] considered the following viscoelastic wave equation with a linear damping and a delay term:

$$u_{tt} - \Delta u + \int_0^t g(t - s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t - \tau) = 0, \tag{1.8}$$

where μ_1 and μ_2 are positive constants. When $\mu_2 \leq \mu_1$, they proved general decay of the energy under the condition

$$g'(t) \leq -\xi(t)g(t), \quad \forall t \geq 0, \tag{1.9}$$

where $\xi : R^+ \rightarrow R^+$ is a nonincreasing differentiable function. Dai and Yang [14] improved the results of [13]. They also obtained an exponentially decay results for the energy of the problem (1.8) in the case $\mu_1 = 0$. Liu [15] studied a general decay result for the following viscoelastic wave equation with time-dependent delay:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t\big(t - \tau(t)\big) = 0 \qquad (1.10)$$

under the conditions (1.9) and $|\mu_2| < \sqrt{1-d}\,\mu_1$. For the plate equation with time delay term, Yang [16] considered the stability for an Euler–Bernoulli viscoelastic equation with constant delay

$$u_{tt} + \Delta^2 u + \int_0^t g(t-s)\Delta^2 u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t - \tau) = 0 \qquad (1.11)$$

under the conditions (1.5) and $0 < |\mu_2| < \mu_1$. Moreover, he proved the exponential decay results of the energy in the case $\mu_1 = 0$. Recently, Feng [17] investigated an exponential stability results for the following plate equation with time-varying delay and past history:

$$u_{tt} + \alpha\Delta^2 u - \int_{-\infty}^t g(t-s)\Delta^2 u(s)\,ds + \mu_1 u_t + \mu_2 u_t\big(t - \tau(t)\big) + f(u) = 0 \qquad (1.12)$$

under the conditions (1.5) and $0 < |\mu_2| < \sqrt{1-d}\,\mu_1$. Mustafa and Kafini [18] showed the decay rates for memory type plate system (1.12) with $\tau(t) = \tau$ and $f(u) = -u|u|^\gamma$. Park [19] obtained the general decay estimates for a viscoelastic plate equation with time-varying delay under the condition (1.9). The stability of the solutions to a viscoelastic system under the condition (1.9) was studied in [20–23] and the references therein. With respect to wave equation with strong time delay, there is just little published work. Messaoudi et al. [24] considered the following wave equation with strong time delay:

$$u_{tt} - \Delta u - \mu_1\Delta u_t - \mu_2\Delta u_t(t - \tau) = 0 \qquad (1.13)$$

and proved the well-posedness under the condition $|\mu_2| \leq \mu_1$ and obtained exponential decay of energy under the condition $|\mu_2| < \mu_1$. Recently, Feng [25] established the general decay result for the following viscoelastic wave equation with strong time-dependent delay:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds - \mu_1\Delta u_t - \mu_2\Delta u_t\big(t - \tau(t)\big) = 0 \qquad (1.14)$$

under the conditions (1.9) and $|\mu_2| < \sqrt{1-d}\,\mu_1$.

However, to the best of my knowledge, there is no stability result for the viscoelastic plate equation with strong time-varying delay. Motivated by [24, 25], we study a general decay result for viscoelastic plate equation with p-Laplacian and time-varying delay (1.1)–(1.4) for relaxation function g satisfying the condition (1.9). This result improves on earlier ones in the literature because it allows for certain relaxation functions which are not necessarily of exponential or polynomial decay.

We end this section by establishing the usual history setting of problem (1.1)–(1.4). Following a method devised in [26–29], we shall use a new variable η^t to the system with past history. Let us define

$$\eta = \eta^t(x,s) = u(x,t) - u(x,t-s), \quad (x,s) \in \Omega \times \mathbb{R}^+, t \geq 0. \tag{1.15}$$

Differentiating in (1.15) we have

$$\eta_t^t(x,s) + \eta_s^t(x,s) = u_t(x,t), \quad (x,s) \in \Omega \times \mathbb{R}^+, t \geq 0.$$

Taking $\alpha = 1 + \int_0^\infty g(s)\,ds$, the original problem (1.1)–(1.4) can be transformed into the new system

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta_p u + \int_0^\infty g(s)\Delta^2 \eta^t(s)\,ds \\ \quad - \mu_1 \Delta u_t - \mu_2 \Delta u_t(t-\tau(t)) + f(u) = h, \quad \text{in } \Omega \times \mathbb{R}^+, \\ \eta_t^t = -\eta_s^t + u_t, \quad (x,t,s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \\ u_t(x,t) = f_0(x,t), \quad (x,t) \in \Omega \times [-\tau(0),0), \end{cases} \tag{1.16}$$

with boundary conditions

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \qquad \eta = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \tag{1.17}$$

and initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \eta^t(x,0) = 0, \qquad \eta^0(x,s) = \eta_0(x,s), \tag{1.18}$$

where

$$\begin{cases} u_0(x) = u_0(x,0), \quad x \in \Omega, \\ u_1(x) = \partial_t u_0(x,t)|_{t=0}, \quad x \in \Omega, \\ \eta_0(x,s) = u_0(x,0) - u_0(x,-s), \quad (x,s) \in \Omega \times \mathbb{R}^+. \end{cases}$$

The paper is organized as follows. In Sect. 2, we state the notation and main result. In Sect. 3, we prove the general decay of the solutions to the viscoelastic plate equation with p-Laplacian and time-varying delay by using the energy perturbation method.

2 Preliminaries

In this section, we present some material needed in the proof of our result and state the main result.

For a Banach space X, $\|\cdot\|_X$ denotes the norm of X. For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}$ by $\|\cdot\|$. We use the standard Lebesgue and Sobolev spaces, with their usual scalar products and norms, and the Sobolev–Poincaré inequality

$$\|u\|_q^2 \leq C_* \|\Delta u\|^2, \quad u \in H_0^2(\Omega), \tag{2.1}$$

for $q \geq 2$ if $1 \leq n \leq 4$ or $2 \leq q \leq \frac{2n}{n-4}$ if $n \geq 5$.

In the following, we fix some notations on the function spaces that will be used. Let

$$V_0 = L^2(\Omega), \qquad V_1 = H_0^1(\Omega), \qquad V_2 = H^2(\Omega) \cap H_0^1(\Omega)$$

and

$$V_3 = \left\{ u \in H^3(\Omega) | u = \Delta u = 0 \text{ on } \partial\Omega \right\}.$$

In order to consider the new variable η, we introduce the weighted L^2-spaces

$$\mathcal{M}_i = L_g^2(\mathbb{R}^+; V_i) = \left\{ \eta : \mathbb{R}^+ \to V_i \middle| \int_0^\infty g(s) \|\eta(s)\|_{V_i}^2 \, ds < \infty \right\}, \quad i = 0, 1, 2, 3,$$

which are Hilbert spaces endowed with the inner products and norms

$$(\eta, \xi)_{\mathcal{M}_i} = \int_0^\infty g(s) \big(\eta(s), \xi(s)\big)_{V_i} \, ds \quad \text{and} \quad \|\eta\|_{\mathcal{M}_i}^2 = \int_0^\infty g(s) \|\eta(s)\|_{V_i}^2 \, ds, \quad i = 0, 1, 2, 3,$$

respectively. To simplify the notation, we define the Hilbert spaces

$$\mathcal{H} = V_2 \times V_0 \times \mathcal{M}_2 \quad \text{and} \quad \mathcal{H}_1 = V_3 \times V_1 \times \mathcal{M}_3.$$

Let us begin with the precise hypotheses on the constant p and the functions f and g. For $n \in \mathbb{N}$, we assume that

$$2 \le p \le \frac{2n-2}{n-2} \quad \text{if } n \ge 3 \quad \text{and} \quad p \ge 2 \quad \text{if } n = 1, 2. \tag{2.2}$$

Then

$$H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow W_0^{1,2(p-1)}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega).$$

The nonlinear function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 0$ and the growth condition,

$$\left| f(w) - f(z) \right| \le k_0 \big(1 + |w|^\rho + |z|^\rho\big) |w - z|, \quad \forall w, z \in \mathbb{R}, \tag{2.3}$$

where $k_0 > 0$ and

$$0 < \rho \le \frac{4}{n-4} \quad \text{if } n \ge 5 \quad \text{and} \quad \rho > 0 \quad \text{if } 1 \le n \le 4, \tag{2.4}$$

which implies that $H^2(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$. In addition, we assume that

$$0 \le F(u) \le f(u)u, \quad \forall u \in \mathbb{R}, \tag{2.5}$$

where $F(z) = \int_0^z f(s) \, ds$.

For the relaxation function g, we assume that $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing C^1 function satisfying

$$g(0) > 0, \quad l := 1 - \int_0^\infty g(s) \, ds > 0, \tag{2.6}$$

and there exists a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\xi(t) > 0, g'(t) \le -\xi(t)g(t), \quad \forall t \ge 0 \tag{2.7}$$

and

$$\int_0^{+\infty} \xi(t)\,dt = \infty.$$

As in [12], for the time-varying delay, we assume that $\tau \in W^{2,\infty}([0,T])$ for $T > 0$, and there exist positive constants τ_0, τ_1 and d satisfying

$$0 < \tau_0 \le \tau(t) \le \tau_1 \quad \text{and} \quad \tau'(t) \le d < 1 \quad \forall t > 0, \tag{2.8}$$

and that μ_1 and μ_2 satisfy

$$|\mu_2| < \sqrt{1-d}\,\mu_1. \tag{2.9}$$

We can prove the existence of weak solution by making use of the classical Faedo–Galerkin method. Then using elliptic regularity and second order estimates we can show the regularity of the solution. We state a well-posedness result without a proof here (see [6, 7, 13, 17]).

Theorem 2.1 *Suppose that hypotheses* (2.2)–(2.9) *hold.*

(i) *If the initial data* $(u_0, u_1, \eta_0) \in \mathcal{H}, f_0(x,t) \in H^1(\Omega \times (-\tau(0),0))$ *and* $h(x) \in L^2(\Omega)$, *then problem* (1.16)–(1.18) *has a unique weak solution*

$$\left(u, u_t, \eta^t\right) \in C(0, T; \mathcal{H}), \quad \forall T > 0,$$

satisfying

$$u \in L^\infty(0, T; V_2), \qquad u_t \in L^\infty(0, T; V_0) \cap L^2(0, T; V_1), \qquad \eta^t \in L^\infty(0, T; \mathcal{M}_2).$$

(ii) *If the initial data* $(u_0, u_1, \eta_0) \in \mathcal{H}_1, f_0(x,t) \in H^2(\Omega \times (-\tau(0),0))$ *and* $h(x) \in H^1(\Omega)$, *then the above weak solution has higher regularity*

$$u \in L^\infty(0, T; V_3), \qquad u_t \in L^\infty(0, T; V_1) \cap L^2(0, T; V_2),$$

$$\eta^t \in L^\infty(0, T; \mathcal{M}_3).$$

In order to state our main result, we define the energy of problem (1.16)–(1.18) by

$$E(t) = \frac{1}{2}\left\|u_t(t)\right\|^2 + \frac{1}{2}\left\|\Delta u(t)\right\|^2 + \frac{1}{p}\left\|\nabla u(t)\right\|_p^p + \int_\Omega F(u)\,dx$$

$$+ \frac{1}{2}\left\|\eta^t\right\|_{\mathcal{M}_2}^2 + \frac{\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left\|\nabla u_t(s)\right\|^2 ds,$$

where ξ and λ are positive constants satisfying

$$\frac{|\mu_2|}{\sqrt{1-d}} < \xi < 2\mu_1 - \frac{|\mu_2|}{\sqrt{1-d}} \quad \text{and} \quad \lambda < \frac{1}{\tau_1}\log\left|\frac{\xi\sqrt{1-d}}{|\mu_2|}\right|. \tag{2.10}$$

Note that this choice of ξ is possible from assumption (2.9).

Theorem 2.2 *Suppose that assumptions* (2.2)–(2.9) *hold. Let $h = 0$ and the initial data $(u_0, u_1, \eta_0) \in \mathcal{H}, f_0(x,t) \in H^1(\Omega \times (-\tau(0), 0))$. Then there exist two positive constants k_1 and k_2 such that the energy $E(t)$ satisfies*

$$E(t) \leq k_1 e^{-k_2 \int_0^t \xi(s)\,ds}, \quad \forall t \geq 0. \tag{2.11}$$

3 General decay of the energy

In this section we shall establish the decay rates in Theorem 2.2. To demonstrate the stability of the system (1.16)–(1.18) the lemmas below are essential.

Lemma 3.1 *Under the assumptions of Theorem* 2.2, *the energy functional $E(t)$ satisfies, for any $t \geq 0$,*

$$E'(t) \leq \left(\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2}\right)\|\nabla u_t\|^2 + \left(\frac{|\mu_2|\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2e^{\lambda\tau_1}}\right)\left\|\nabla u_t(t - \tau(t))\right\|^2$$

$$+ \frac{1}{2}\int_0^\infty g'(s)\left\|\Delta\eta^t(s)\right\|^2 ds - \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left\|\nabla u_t(s)\right\|^2 ds. \tag{3.1}$$

Proof Multiplying the first equation of (1.16) by $u_t(t)$, we get the identity

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{p}\|\nabla u\|_p^p + \int_\Omega F(u)\,dx\right)$$

$$= -\mu_1\|\nabla u_t\|^2 - \mu_2\int_\Omega \nabla u_t(t - \tau(t))\nabla u_t\,dx - \int_\Omega\int_0^\infty g(s)\Delta\eta^t(s)\Delta u_t(t)\,ds\,dx. \tag{3.2}$$

Applying the second equation of (1.16) to (3.2), we have

$$E'(t) = -\mu_1\|\nabla u_t\|^2 - \mu_2\int_\Omega \nabla u_t(t - \tau(t))\nabla u_t\,dx - (\eta_s^t, \eta^t)_{\mathcal{M}_2} + \frac{\xi}{2}\|\nabla u_t\|^2$$

$$- \frac{\xi}{2}e^{-\lambda\tau(t)}\left(1 - \tau'(t)\right)\left\|\nabla u_t(t - \tau(t))\right\|^2 - \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left\|\nabla u_t(s)\right\|^2 ds.$$

Using Young's inequality, we obtain

$$-\mu_2\int_\Omega \nabla u_t(t - \tau(t))\nabla u_t\,dx \leq \frac{|\mu_2|}{2\sqrt{1-d}}\|\nabla u_t\|^2 + \frac{|\mu_2|\sqrt{1-d}}{2}\left\|\nabla u_t(t - \tau(t))\right\|^2.$$

By (2.8) we get

$$-\frac{\xi}{2}e^{-\lambda\tau(t)}\left(1 - \tau'(t)\right)\left\|\nabla u_t(t - \tau(t))\right\|^2 \leq -\frac{\xi}{2}e^{-\lambda\tau_1}(1 - d)\left\|\nabla u_t(t - \tau(t))\right\|^2.$$

Since $\eta^t(0) = 0$, we have

$$-\left(\eta_s^t, \eta^t\right)_{\mathcal{M}_2} = \frac{1}{2} \int_0^\infty g'(s) \left\| \Delta \eta^t(s) \right\|^2 ds.$$

Combining with the above estimates, we obtain the desired inequality (3.1). The proof is now complete. □

Now, let us define the perturbed modified energy by

$$L(t) = E(t) + \epsilon \Phi(t), \tag{3.3}$$

where

$$\Phi(t) = \int_\Omega u_t u \, dx.$$

Then it is easily shown that there exists $C_1 > 0$ such that

$$\left| L(t) - E(t) \right| \le \epsilon C_1 E(t), \quad \forall t \ge 0, \forall \epsilon > 0,$$

where $C_1 = \max\{1, C_*\}$.

Lemma 3.2 *There exist positive constants C_2 and C_3 such that*

$$L'(t) \le -C_2 E(t) + C_3 \int_0^\infty g(s) \left\| \Delta \eta^t(s) \right\|^2 ds, \quad \forall t \ge 0. \tag{3.4}$$

Proof By using (1.16), we get

$$\Phi'(t) = \|u_t\|^2 - \|\Delta u\|^2 - \|\nabla u\|_p^p - \int_\Omega \int_0^\infty g(s) \Delta \eta^t(s) \Delta u(t) \, ds \, dx$$

$$- \mu_1 \int_\Omega \nabla u_t \nabla u \, dx - \mu_2 \int_\Omega \nabla u_t \left(t - \tau(t) \right) \nabla u \, dx - \int_\Omega f(u) u \, dx.$$

Adding and subtracting $E(t)$, we see that

$$\Phi'(t) = -E(t) + \frac{3}{2} \|u_t\|^2 - \frac{1}{2} \|\Delta u\|^2 - \left(1 - \frac{1}{p} \right) \|\nabla u\|_p^p + \frac{1}{2} \left\| \eta^t \right\|_{\mathcal{M}_2}^2$$

$$+ \int_\Omega \left(F(u) - f(u) u \right) dx$$

$$- \int_\Omega \int_0^\infty g(s) \Delta \eta^t(s) \Delta u(t) \, ds \, dx - \mu_1 \int_\Omega \nabla u_t \nabla u \, dx$$

$$- \mu_2 \int_\Omega \nabla u_t \left(t - \tau(t) \right) \nabla u \, dx$$

$$+ \frac{\xi}{2} \int_{t-\tau(t)}^t e^{\lambda(s-t)} \left\| \nabla u_t(s) \right\|^2 ds.$$

Applying Young's inequality and (2.6), we have

$$\int_\Omega \int_0^\infty g(s)\Delta\eta^t(s)\Delta u(t)\,ds\,dx \le \frac{1}{8}\|\Delta u\|^2 + 2(1-l)\|\eta^t\|^2_{\mathcal{M}_2},$$

$$-\mu_1 \int_\Omega \nabla u_t \nabla u\,dx \le \frac{1}{16}\|\Delta u\|^2 + 4d_1\mu_1^2\|\nabla u_t\|^2$$

and

$$-\mu_2 \int_\Omega \nabla u_t\big(t-\tau(t)\big)\nabla u\,dx \le \frac{1}{16}\|\Delta u\|^2 + 4d_1\mu_2^2\big\|\nabla u_t\big(t-\tau(t)\big)\big\|^2,$$

where $d_1 > 0$ is an embedding constant for $H^2(\Omega)\cap H_0^1(\Omega) \hookrightarrow H_0^1(\Omega)$. Combining all above estimates and using (2.5), we obtain

$$\Phi'(t) \le -E(t) + \left(4d_1\mu_1^2 + \frac{3}{2}d_2\right)\|\nabla u_t\|^2 - \frac{1}{4}\|\Delta u\|^2$$

$$- \left(1 - \frac{1}{p}\right)\|\nabla u\|_p^p + \left(2(1-l) + \frac{1}{2}\right)\|\eta^t\|^2_{\mathcal{M}_2}$$

$$+ 4d_1\mu_2^2\big\|\nabla u_t\big(t-\tau(t)\big)\big\|^2 + \frac{\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\big\|\nabla u_t(s)\big\|^2\,ds,$$

where $d_2 > 0$ is an embedding constant for $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. Thus, taking two positive constants

$$c_1 = \max\left\{4d_1\mu_1^2 + \frac{3}{2}d_2, 4d_1\mu_2^2\right\} \quad\text{and}\quad c_2 = 2(1-l) + \frac{1}{2},$$

we find that

$$\Phi'(t) \le -E(t) + c_1\|\nabla u_t\|^2 + c_1\big\|\nabla u_t\big(t-\tau(t)\big)\big\|^2$$

$$+ c_2\|\eta^t\|^2_{\mathcal{M}_2} + \frac{\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\big\|\nabla u_t(s)\big\|^2\,ds. \tag{3.5}$$

Using (3.1), (3.3) and (3.5), we deduce that

$$L'(t) \le -\epsilon E(t) + \left(\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} + \epsilon c_1\right)\|\nabla u_t\|^2$$

$$+ \left(\frac{|\mu_2|\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2e^{\lambda\tau_1}} + \epsilon c_1\right)\big\|\nabla u_t\big(t-\tau(t)\big)\big\|^2$$

$$+ \frac{1}{2}\int_0^\infty g'(s)\big\|\Delta\eta^t(s)\big\|^2\,ds + \epsilon c_2\|\eta^t\|^2_{\mathcal{M}_2} - \frac{\xi}{2}(\lambda-\epsilon)\int_{t-\tau(t)}^t e^{\lambda(s-t)}\big\|\nabla u_t(s)\big\|^2\,ds.$$

From (2.10), we see that

$$\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} < 0, \qquad \frac{|\mu_2|\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2e^{\lambda\tau_1}} < 0.$$

We choose $\epsilon > 0$ sufficiently small for

$$\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} + \epsilon c_1 < 0, \qquad \frac{|\mu_2|\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2e^{\lambda\tau_1}} + \epsilon c_1 < 0 \quad \text{and} \quad \lambda - \epsilon > 0.$$

Hence, we conclude that (3.4) holds for some constants $C_2, C_3 > 0$. □

Proof of Theorem 2.2 From the ideas presented in [19–23], we get the following results. Multiplying (3.4) by $\xi(t)$ and using (2.7) and (3.1), we get

$$\begin{aligned}
\xi(t)L'(t) &\leq -C_2\xi(t)E(t) + C_3 \lim_{t\to\infty} \int_0^t \xi(s)g(s)\|\Delta\eta^t(s)\|^2 \, ds \\
&\leq -C_2\xi(t)E(t) - C_3 \int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2 \, ds \\
&\leq -C_2\xi(t)E(t) - 2C_3 E'(t).
\end{aligned}$$

Using the fact that $\xi'(t) \leq 0$ and letting

$$\mathcal{L}(t) = \xi(t)L(t) + 2C_3 E(t) \sim E(t) \tag{3.6}$$

we have

$$\mathcal{L}'(t) \leq -C_2\xi(t)E(t) \leq -C_4\xi(t)\mathcal{L}(t), \quad \forall t \geq 0, \tag{3.7}$$

where C_4 is a positive constant. A simple integrating of (3.7) over $(0, t)$ leads to

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-C_4 \int_0^t \xi(s)\, ds}, \quad \forall t \geq 0. \tag{3.8}$$

Consequently, from (3.6) and (3.8), we obtain the desired inequality (2.11). □

4 Conclusions

In this paper, a viscoelastic plate equation with p-Laplacian and time-varying delay has been investigated. In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of my knowledge, there was no stability result for the viscoelastic plate equation with strong time delay. We have been proved that the general decay rate result under some restrictions on the coefficients of strong damping and strong time-varying delay. Furthermore, we have been obtained the general decay rate result under weakening the usual assumptions on the relaxation function, using the energy perturbation method. This result improves on earlier ones in the literature because it allows for certain relaxation functions which are not necessarily of exponential or polynomial decay.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

The work was realized by the author. The author read and approved the final manuscript.

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