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Inverse spectral problems for first order integro-differential operators

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Abstract
Inverse spectral problems are studied for the first order integro-differential operators on a finite interval. Properties of spectral characteristic are established, and the uniqueness theorem is proved for this class of inverse problems.

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1 Introduction

The paper is devoted to studying inverse spectral problems for integro-differential operators of the form

$$\ell y := iy'(x) + R(x) \int_0^x V(t)y(t)\,dt.$$

Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics, meteorology and other branches of natural sciences. Inverse problems also play an important role in solving nonlinear evolution equations in mathematical physics. Interest in this subject has been increasing permanently because of the appearance of new important applications, and nowadays the inverse problem theory is being developed intensively all over the world. The greatest success in the inverse problem theory has been achieved for the Sturm-Liouville operator (see, e.g., [1–3]) and afterwards for higher order differential operators [4–6] and other classes of differential operators. For integro-differential and other classes of nonlocal operators, inverse problems are more difficult to investigate, and the main classical methods (transformation operator method and the method of spectral mappings) either are not applicable to them or require essential modifications; and for such operators, the general inverse problem theory does not exist. At the same time, nonlocal and, in particular, integro-differential operators are of great interest, because they have many applications (see, e.g., [7]). We note that some aspects of inverse problems for integro-differential and integral operators were studied in [8–12] and other works. In the present paper, we study the inverse spectral problem for the first order integro-differential operator ℓ on a finite interval. Properties of spectral characteristics are established, and the uniqueness theorem is proved for the inverse problem of recovering the function $R(x)$ and $V(x)$ from the given spectral data.

2 Preliminary information

Consider the integro-differential equation

$$\ell y := iy'(x) + R(x) \int_0^x V(t)y(t)\,dt = \lambda y(x), \quad x \in [0,\pi], \tag{1}$$

where $R(x)$, $V(t)$ are continuous complex-valued functions, and

$$R(\pi - x) \sim C_\alpha x^\alpha, \qquad V(x) \sim D_\beta x^\beta, \quad x \to +0, \qquad C_\alpha D_\beta \neq 0.$$

Let $\varphi(x,\lambda)$ be the solution of Eq. (1) with the condition $\varphi(0,\lambda) = 1$. Then the following representation holds (see [3]):

$$\varphi(x,\lambda) = \exp(-i\lambda x) + \int_0^x K(x,t)\exp(-i\lambda t)\,dt, \tag{2}$$

where $K(x,t)$ is a continuous function, and $K(x,0) = 0$. Denote

$$\Pi_+ := \{\lambda : \operatorname{Im}\lambda \geq 0\}, \qquad \Pi_-^\delta := \bigl\{\lambda : \arg\lambda \in [\pi+\delta, 2\pi-\delta]\bigr\}.$$

It follows from (2) that for $|\lambda| \to \infty$ uniformly in $x \in [0,\pi]$:

$$\left.\begin{array}{l} \varphi^{(\nu)}(x,\lambda) = (-i\lambda)^\nu \exp(-i\lambda x)(1+o(1)), \quad \lambda \in \Pi_+, \nu = 0,1, \\ \varphi^{(\nu)}(x,\lambda) = o(\lambda^\nu), \quad \lambda \in \Pi_-^\delta, \nu = 0,1. \end{array}\right\} \tag{3}$$

Denote

$$\varphi_\nu(x,\lambda) := \frac{1}{\nu!}\frac{\partial^\nu \varphi(x,\lambda)}{\partial\lambda^\nu}, \quad \nu \geq 0, \qquad \Delta(\lambda) := \varphi(\pi,\lambda).$$

The function $\Delta(\lambda)$ is entire in λ of exponential type, and its zeros $\Lambda := \{\lambda_n\}_{n\geq 1}$ (counting with multiplicities) coincide with the eigenvalues of the boundary value problem $L = L(R,V)$ for Eq. (1) with the condition $y(\pi) = 0$. Let m_n be the multiplicity of λ_n ($\lambda_n = \lambda_{n+1} = \cdots = \lambda_{n+m_n-1}$). Denote

$$S := \{n : n-1 \in \mathbf{N}, \lambda_{n-1} \neq \lambda_n\} \cup \{1\}, \qquad s_{n+\nu}(x) := \varphi_\nu(x,\lambda_n), \quad n \in S, \nu = \overline{m_n - 1}.$$

The functions $\{s_n(x)\}_{n\geq 1}$ are eigen and associated functions for L.

Example 1 Let $\lambda_1 = \lambda_2 < \lambda_3 < \lambda_4 = \lambda_5 = \lambda_6 < \lambda_7 < \lambda_8 < \cdots$. Then $S = \{1,3,4,7,8,\ldots\}$, $s_1(x) = \varphi_0(x,\lambda_1)$, $s_2(x) = \varphi_1(x,\lambda_1)$, $s_3(x) = \varphi_0(x,\lambda_3)$, $s_4(x) = \varphi_0(x,\lambda_4)$, $s_5(x) = \varphi_1(x,\lambda_4)$, $s_6(x) = \varphi_2(x,\lambda_4)$, $s_7(x) = \varphi_0(x,\lambda_7), \ldots$.

Let the function $\eta(x,\lambda)$ be the solution of the problem

$$i\eta'(x,\lambda) - R(x)\int_x^\pi V(t)\eta(t,\lambda)\,dt + R(x) = \lambda\eta(x,\lambda), \quad \eta(\pi,\lambda) = 0. \tag{4}$$

Denote $\theta(x,\lambda) := \eta(\pi - x,\lambda)$. Then

$$i\theta'(x,\lambda) + R_0(x)\int_0^x V_0(t)\theta(t,\lambda)\,dt - R_0(x) = -\lambda\theta(x,\lambda), \quad \theta(0,\lambda) = 0, \tag{5}$$

where $R_0(x) := R(\pi - x)$, $V_0(x) := V(\pi - x)$. It follows from (5) that

$$\theta(x,\lambda) = \int_0^x g(x,t,\lambda)R_0(t)\,dt, \tag{6}$$

where $g(x,t,\lambda)$ is Green's function of the Cauchy problem, and

$$ig_x(x+t,t,\lambda) - \lambda g(x+t,t,\lambda) + R(x+t)\int_0^x V(\tau+t)g(\tau+t,t,\lambda)\,d\tau = 0, \quad g(t,t,\lambda) = -i;$$

and consequently, $g(x+t,t,\lambda) = -i\varphi(x,\lambda;t)$, where $\varphi(x,\lambda;t)$ is the solution of the Cauchy problem

$$i\varphi'(x,\lambda;t) + R(x+t)\int_0^x V(\tau+t)\varphi(\tau,\lambda;t)\,d\tau = \lambda\varphi(x,\lambda;t), \quad \varphi(0,\lambda;t) = 1.$$

In view of (2) we get

$$\varphi(x,\lambda;t) = \exp(-i\lambda x) + \int_0^x K(x,\tau;t)\exp(-i\lambda\tau)\,d\tau,$$

where $K(x,\tau;t)$ is a continuous function. This yields

$$g(x,t,\lambda) = -i\exp\bigl(-i\lambda(x-t)\bigr) - i\int_0^{x-t} K(x-t,\tau;t)\exp(-i\lambda\tau)\,d\tau. \tag{7}$$

Substituting (7) into (6), we obtain

$$\theta(x,\lambda) = \int_0^x P(x,t)\exp(i\lambda t)\,dt, \tag{8}$$

where

$$P(x,t) = -iR_0(x-t) - i\int_0^{x-t} R_0(\tau)K(x-\tau,t;\tau)\,d\tau. \tag{9}$$

Clearly, $P(x,x) = -iR_0(0), P(x,0) = -iR_0(x)$. Using (8)-(9) and (4), we conclude that for $|\lambda| \to \infty$ uniformly in $x \in [0,\pi]$:

$$\left.\begin{aligned} &\eta^{(\nu)}(x,\lambda) = o(\lambda^\nu), \quad \lambda \in \Pi_+, \nu = 0,1, \\ &\eta^{(\nu)}(x,\lambda) = o(\lambda^\nu \exp(i\lambda(\pi - x))), \quad \lambda \in \Pi_-^\delta, \nu = 0,1. \end{aligned}\right\} \tag{10}$$

Denote

$$\Delta_0(\lambda) := 1 - \int_0^\pi V(t)\eta(t,\lambda)\,dt. \tag{11}$$

Using (4) and (11), we calculate

$$i\eta'(x,\lambda) + R(x)\left(\Delta_0(\lambda) + \int_0^x V(t)\eta(t,\lambda)\,dt\right) = \lambda\eta(x,\lambda), \quad \eta(\pi,\lambda) = 0. \tag{12}$$

In particular, it follows from (12) that the zeros of the entire function $\Delta_0(\lambda)$ coincide with the zeros of $\Delta(\lambda)$, and multiplicities of zeros of $\Delta_0(\lambda)$ are not more than multiplicities of zeros of $\Delta(\lambda)$. Therefore the function $\Delta(\lambda)/\Delta_0(\lambda)$ is entire in λ of exponential type. Denote $\Delta_1(\lambda) := \Delta_0(\lambda)\exp(-i\lambda\pi)$. Using (8), (9) and (11), by standard arguments (see, e.g., [3]), we obtain that for $|\lambda| \to \infty$, the following asymptotical formulae hold:

$$\left.\begin{aligned} &\Delta_1(\lambda) = \exp(-i\lambda\pi)(1 + o(1)), \quad \lambda \in \Pi_+, \\ &\Delta_1(\lambda) = B\lambda^{-\gamma-1}(1 + o(1)), \quad \lambda \in \Pi_-^\delta, \end{aligned}\right\} \tag{13}$$

where $B \neq 0, \gamma := \alpha + \beta + 1$. The function $F(\lambda) := \Delta(\lambda)/\Delta_1(\lambda)$ is entire in λ of exponential type. By virtue of (3),

$$\left.\begin{aligned} &\Delta(\lambda) = \exp(-i\lambda\pi)(1 + o(1)), \quad \lambda \in \Pi_+, \\ &\Delta(\lambda) = o(1), \quad \lambda \in \Pi_-^\delta. \end{aligned}\right\} \tag{14}$$

Together with (13) this yields that $F(\lambda) \equiv 1$, i.e., $\Delta(\lambda) \equiv \Delta_1(\lambda)$ or

$$\Delta(\lambda) \equiv \Delta_0(\lambda)\exp(-i\lambda\pi). \tag{15}$$

Denote

$$\eta_\nu(x,\lambda) := \frac{1}{\nu!}\frac{\partial\eta(x,\lambda)}{\partial\lambda^\nu}, \quad \nu \geq 0, \qquad \psi_{n+\nu}(x) := \eta_\nu(x,\lambda_n), \quad n \in S, \nu = \overline{m_n - 1}.$$

The functions $\{\psi_n(x)\}_{n\geq 1}$ are eigen and associated functions for the boundary value problem L, and

$$\psi_{n+\nu}(x) = \sum_{j=0}^{\nu} \beta_{n+\nu-j}s_{n+j}(x), \quad n \in S, \nu = \overline{m_n - 1}. \tag{16}$$

The coefficients $\{\beta_n\}_{n\geq 1}$ are called Levinson's weight numbers, and the data $\{\lambda_n, \beta_n\}_{n\geq 1}$ are called the spectral data for the boundary value problem L. We will consider the following inverse problem.

Inverse problem 1 *Given the spectral data* $\{\lambda_n, \beta_n\}_{n\geq 1}$, *construct* R *and* V.

3 The uniqueness theorem

Below we will assume that $R(x) \neq 0$ a.e. on $(0,\pi)$. If this condition does not hold, then the specification of the spectral data does not uniquely determine L (see Example 2).

Let us formulate the uniqueness theorem for this inverse problem. For this purpose, together with L we consider the boundary value problem $\tilde{L} := L(\tilde{R}, \tilde{V})$ of the same form but with different functions $\tilde{R}(x)$, $\tilde{V}(t)$. We agree that in what follows if a certain symbol α denotes an object related to L, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{L}$.

Theorem 1 *Let* $\{\tilde\lambda_n, \tilde\beta_n\}$ *be the spectral data for the problem* $\tilde L = L(\tilde R, \tilde V)$. *If* $\lambda_n = \tilde\lambda_n$, $\beta_n = \tilde\beta_n$ *for all* $n \geq 1$, *then* $R(x) \equiv \tilde R(x)$, $V(x) \equiv \tilde V(x)$, $x \in [0, \pi]$.

Proof Using (14)-(15) and Hadamard's factorization theorem, we get $\Delta_0(\lambda) \equiv \tilde\Delta_0(\lambda)$. Taking (16) into account, we deduce that the functions

$$A_j(x,\lambda) = \big(\Delta_0(\lambda)\big)^{-1} \exp(i\lambda x)\big(\tilde\varphi(x,\lambda)\eta^{(j-1)}(x,\lambda) - \tilde\eta(x,\lambda)\varphi^{(j-1)}(x,\lambda)\big), \quad j = 1, 2,$$

are entire in λ of exponential type. Taking (3), (10) and (13) into account, we obtain for $|\lambda| \to \infty$

$$\begin{aligned} &A_1(x,\lambda) = o(1), \qquad A_2(x,\lambda) = o(\lambda), \quad \lambda \in \Pi_+, \\ &A_1(x,\lambda) = o\big(\lambda^{\gamma+1}\big), \qquad A_2(x,\lambda) = o\big(\lambda^{\gamma+2}\big), \quad \lambda \in \Pi_-^{\delta}, \end{aligned}$$

and consequently,

$$A_1(x,\lambda) \equiv 0, \qquad A_2(x,\lambda) \equiv A(x), \tag{17}$$

where the function $A(x)$ does not depend on λ. In particular, (17) yields

$$\tilde\varphi(x,\lambda)\eta(x,\lambda) \equiv \tilde\eta(x,\lambda)\varphi(x,\lambda), \tag{18}$$

$$\tilde\varphi(x,\lambda)\eta'(x,\lambda) - \tilde\eta(x,\lambda)\varphi'(x,\lambda) \equiv A(x)\Delta_0(\lambda)\exp(-i\lambda x). \tag{19}$$

Similarly, we obtain

$$\varphi(x,\lambda)\eta'(x,\lambda) - \eta(x,\lambda)\varphi'(x,\lambda) \equiv A^*(x)\Delta_0(\lambda)\exp(-i\lambda x), \tag{20}$$

where $A^*(x)$ does not depend on λ. Using (18) we calculate

$$\tilde\varphi(x,\lambda)\big(\varphi(x,\lambda)\eta'(x,\lambda) - \eta(x,\lambda)\varphi'(x,\lambda)\big) = \varphi(x,\lambda)\big(\tilde\varphi(x,\lambda)\eta'(x,\lambda) - \tilde\eta(x,\lambda)\varphi'(x,\lambda)\big).$$

Together with (19)-(20) this yields

$$\tilde\varphi(x,\lambda)A^*(x) \equiv \varphi(x,\lambda)A(x).$$

Taking (3) into account, we conclude that $A(x) \equiv A^*(x)$, and

$$\big(\tilde\varphi(x,\lambda) - \varphi(x,\lambda)\big)A(x) \equiv 0. \tag{21}$$

Furthermore, using (18), (19) and Eqs. (1) and (4), we infer

$$\begin{aligned} &iA(x)\Delta_0(\lambda)\exp(-i\lambda x) \\ &\quad = i\tilde\varphi(x,\lambda)\eta'(x,\lambda) - i\tilde\eta(x,\lambda)\varphi'(x,\lambda) \\ &\quad = \tilde\varphi(x,\lambda)\left(R(x)\int_x^{\pi} V(t)\eta(t,\lambda)\,dt - R(x)\right) + \tilde\eta(x,\lambda)R(x)\int_0^x V(t)\varphi(t,\lambda)\,dt. \end{aligned}$$

Hence, for $|\lambda| \to \infty, \lambda \in \Pi_+$, we get $A(x) \equiv iR(x)$. In view of (21), one has

$$\big(\tilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\big)R(x) \equiv 0. \tag{22}$$

Since $R(x) \not\equiv 0$ a.e. on $(0,\pi)$, it follows from (22) that

$$\tilde{\varphi}(x,\lambda) \equiv \varphi(x,\lambda).$$

By virtue of (18),

$$\tilde{\eta}(x,\lambda) \equiv \eta(x,\lambda).$$

Then, according to (4),

$$R(x) - \tilde{R}(x) = R(x)\int_x^\pi V(t)\eta(t,\lambda)\,dt - \tilde{R}(x)\int_x^\pi \tilde{V}(t)\eta(t,\lambda)\,dt.$$

For $|\lambda| \to \infty$ this yields $R(x) \equiv \tilde{R}(x)$, $x \in [0,\pi]$, and consequently $V(x) \equiv \tilde{V}(x)$, $x \in [0,\pi]$. Theorem 1 is proved. □

Example 2 Fix $a \in (0,\pi)$. Let $R(x) \equiv 0$ for $x \in [0,a]$ and $R(x) \not\equiv 0$ for $x \in (a,\pi)$. Put $\tilde{R}(x) \equiv R(x)$ for $x \in [0,\pi]$, and choose $V(t)$, $\tilde{V}(t)$ such that $V(t) \equiv \tilde{V}(t)$ for $t \in (a,\pi)$, and $V(t) \not\equiv \tilde{V}(t)$ for $t \in [0,a]$. Then $\tilde{\varphi}(x,\lambda) \equiv \varphi(x,\lambda)$ and $\tilde{\eta}(x,\lambda) \equiv \eta(x,\lambda)$; hence $\tilde{\lambda}_n = \lambda_n$, $\tilde{\beta}_n = \beta_n$ for all $n \geq 1$.

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