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Existence of positive solutions for nonlinear four-point Caputo fractional differential equation with p-Laplacian

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Abstract

In this article, the existence of positive solutions is considered for nonlinear four-point Caputo fractional differential equation with p-Laplacian operator. We use the monotone iterative technique to acquire the existence of positive solutions for the boundary value problem and get iterative schemes for approximating the solutions. An example is also presented to illustrate the effectiveness of the main results.

MSC: 34B15; 34B18

Keywords: Caputo fractional derivative; positive solutions; p-Laplacian; monotone iterative technique

1 Introduction

The differential equation arises in the modeling of different physical and natural phenomena, control system, nonlinear flow laws and many other branches of engineering. Fractional calculus is the extension of integer order calculus to arbitrary order calculus. With the development of fractional calculus, fractional differential equations have wide applications. In these years, there are many papers concerning integer order differential equations with p-Laplacian [1–7] and fractional differential equations with p-Laplacian [8–18].

By means of the monotone iterative technique, Sun *et al.* [1] investigated positive solutions for the following problems for the p-Laplacian operator:

$$\left(\phi_p\big(u'(t)\big)\right)' + q(t)f\big(t, u(t), u'(t)\big) = 0, \quad t \in (0, 1),$$

$$u(0) - \alpha u'(\xi) = 0, \quad u(1) + \beta u'(\eta) = 0.$$

Here $\xi, \eta \in (0, 1)$, $\alpha \geq 0$, $\beta \geq 0$.

In [16], Liu *et al.* discussed the four-point problem for a class of fractional differential equation with mixed fractional derivatives and with p-Laplacian operator

$$D_{0+}^{\alpha}\big(\varphi_p\big({}^c D_{0+}^{\beta} u(t)\big)\big) = f\big(t, u(t), {}^c D_{0+}^{\beta} u(t)\big), \quad t \in (0, 1),$$

$${}^c D_{0+}^{\beta} u(0) = u'(0) = 0, \quad u(1) = r_1 u(\eta), \quad {}^c D_{0+}^{\beta} u(1) = r_2 {}^c D_{0+}^{\beta} u(\xi).$$

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Based on the method of upper and lower solutions, they study the existence of positive solutions of the above boundary problem.

Motivated by the aforementioned work, this work discusses the existence of positive solutions for the following fractional differential equation:

$$
\begin{cases}
D_{0^+}^{\beta}[\phi_p(D_{0^+}^{\alpha}u(t))] + f(t, u(t)) = 0, & t \in (0, 1), \\
[\phi_p(D_{0^+}^{\alpha}u(0))]^{(i)} = 0, & i = 1, 2, \ldots, m - 1, \\
u(0) - aD_{0^+}^{\alpha}u(\xi) = 0, \\
(D_{0^+}^{\alpha}u(0))^{(j)} = 0, & j = 1, 2, \ldots, n - 1, \\
u(1) + bD_{0^+}^{\alpha}u(\eta) = 0,
\end{cases}
\tag{1.1}
$$

where $0 < n - 1 < \alpha \le n, 0 < m - 1 < \beta \le m$ and $m + n - 1 < \alpha + \beta \le m + n, \phi_p(u) = |u|^{p-2}u$, $p > 1$. $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the Caputo fractional derivatives. We use the monotone iterative technique to obtain the existence of positive solutions for the boundary value problem and get iterative schemes for approximating the solutions. A function $u(t)$ is a positive solution of the boundary value problem (1.1) if and only if $u(t)$ satisfies the boundary value problem (1.1), and $u(t) \ge 0$ for $t \in [0, 1]$. We will always suppose the following conditions are satisfied:

(H_1) $a, b \in (0, +\infty)$ are constants, $0 < \xi < \eta < 1$;
(H_2) $f(t, u) : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous, and $f(t, u) \not\equiv 0$ on any subinterval of $t \in (0, 1)$ for fixed $u \in [0, \infty)$.

2 Preliminaries

To show the main result of this work, we give the following some basic definitions, which can be found in [19, 20].

Definition 2.1 The fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$
I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s) \, ds,
$$

provided that the right side is pointwise defined on $(0, +\infty)$, where

$$
\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} \, dx.
$$

Definition 2.2 For a continuous function $y : (0, +\infty) \to \mathbb{R}$, the Caputo derivative of fractional order $\alpha > 0$ is defined as

$$
D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} y^{(n)}(s) \, ds,
$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, +\infty)$.

3 Main result

Lemma 3.1 *The boundary value problem* (1.1) *is equivalent to the following equation*:

$$u(t) = a\phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right)$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}\phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right.$$
$$\left. - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right) ds$$

or

$$u(t) = b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right)$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1}\phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))d\tau}{\Gamma(\beta)}\right.$$
$$\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right) ds$$
$$- \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}\phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))d\tau}{\Gamma(\beta)}\right.$$
$$\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right) ds,$$

where σ is the unique solution of the equation

$$a\phi_q\left(\frac{\int_0^t (t - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right)$$
$$= -\frac{1}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1}\phi_q\left(\frac{\int_0^t (t - \tau)^{\beta-1} f(\tau, u(\tau))d\tau}{\Gamma(\beta)}\right.$$
$$\left. - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right) ds$$
$$- b\phi_q\left(\frac{\int_0^t (t - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)} - \frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right). \tag{3.1}$$

Proof From $D_{0^+}^\beta[\phi_p(D_{0^+}^\alpha u(t))] + h(t)f(t, u(t)) = 0$, we get

$$\phi_p\big(D_{0^+}^\alpha u(t)\big) = c_0 + c_1 t + \cdots + c_{m-1}t^{m-1} - \frac{1}{\Gamma(\beta)}\int_0^t (t - \tau)^{\beta-1} f\big(\tau, u(\tau)\big)\,d\tau.$$

In view of $[\phi_p(D_{0^+}^\alpha u(0))]^{(i)} = 0, i = 1, 2, \ldots, m - 1$, we obtain

$$c_1 = c_2 = \cdots = c_{m-1} = 0,$$

that is,

$$\phi_p\big(D_{0^+}^\alpha u(t)\big) = c_0 - \frac{1}{\Gamma(\beta)}\int_0^t (t - \tau)^{\beta-1} f\big(\tau, u(\tau)\big)\,d\tau. \tag{3.2}$$

By (3.2), we have

$$D_{0^+}^\alpha u(t) = \phi_q\left(c_0 - \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau\right).$$

For $t \in [0,1]$, integrating from 0 to t, we get

$$u(t) = d_0 + d_1 t + \cdots + d_{n-1}t^{n-1}$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(c_0 - \frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau\right)ds.$$

In view of $(D_{0^+}^\alpha u(0))^{(j)} = 0, j = 1, 2, \ldots, n-1$, we obtain

$$d_1 = d_2 = \cdots = d_{n-1} = 0,$$

that is,

$$u(t) = d_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(c_0 - \frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau\right)ds.$$

By the use of $u(0) - aD_{0^+}^\alpha u(\xi) = 0$, we obtain

$$d_0 = a\phi_q\left(c_0 - \frac{1}{\Gamma(\beta)}\int_0^\xi (\xi-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau\right). \tag{3.3}$$

By $u(1) + bD_{0^+}^\alpha u(\eta) = 0$, we obtain

$$d_0 = -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(c_0 - \frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau\right)ds$$
$$- b\phi_q\left(c_0 - \frac{1}{\Gamma(\beta)}\int_0^\eta (\eta-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau\right). \tag{3.4}$$

By (3.3) and (3.4), we get

$$a\phi_q\left(c_0 - \frac{1}{\Gamma(\beta)}\int_0^\xi (\xi-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau\right)$$
$$= -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(c_0 - \frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau\right)ds$$
$$- b\phi_q\left(c_0 - \frac{1}{\Gamma(\beta)}\int_0^\eta (\eta-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau\right). \tag{3.5}$$

Let

$$F(t) = a\phi_q\left(\frac{\int_0^t (t-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right)$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\frac{\int_0^t (t-\tau)^{\beta-1}f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right.$$

$$- \frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \Bigg) ds$$

$$+ b\phi_q \Bigg(\frac{\int_0^t (t-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} - \frac{\int_0^\eta (\eta-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \Bigg), \qquad (3.6)$$

obviously, $F(t)$ is continuous and nondecreasing for $t \in [0, 1]$. From a direct calculation, we get

$$F(0) = a\phi_q \Bigg(-\frac{\int_0^\xi (\xi-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \Bigg)$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \Bigg(-\frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) d\tau}{\Gamma(\beta)} \Bigg) ds$$

$$+ b\phi_q \Bigg(-\frac{\int_0^\eta (\eta-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \Bigg)$$

$$\leq a\phi_q \Bigg(-\frac{\int_0^\xi (\xi-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \Bigg)$$

$$+ b\phi_q \Bigg(-\frac{\int_0^\eta (\eta-\tau)^{\beta-1} f(\tau, u(\tau)) d\tau}{\Gamma(\beta)} \Bigg)$$

$$< 0$$

and

$$F(1) = a\phi_q \Bigg(\frac{\int_0^1 (1-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \Bigg)$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q \Bigg(\frac{\int_0^1 (1-\tau)^{\beta-1} f(\tau, u(\tau)) d\tau}{\Gamma(\beta)}$$

$$- \frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \Bigg) ds$$

$$+ b\phi_q \Bigg(\frac{\int_0^1 (1-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} - \frac{\int_0^\eta (\eta-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \Bigg)$$

$$\geq a\phi_q \Bigg(\frac{\int_\xi^1 (1-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \Bigg)$$

$$+ b\phi_q \Bigg(\frac{\int_\eta^1 (1-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \Bigg)$$

$$> 0,$$

so we see that the equation $F(t) = 0$ has a unique solution for $t \in (0, 1)$. Let σ be the unique solution of the equation $F(t) = 0$, then by (3.5) and (3.6), we obtain

$$c_0 = \frac{\int_0^\sigma (\sigma-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)}.$$

Consequently, we get

$$
u(t) = a\phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right)
$$
$$
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right.
$$
$$
\left. - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right) ds
$$

or

$$
u(t) = b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right)
$$
$$
+ \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))d\tau}{\Gamma(\beta)}\right.
$$
$$
\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right) ds
$$
$$
- \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))d\tau}{\Gamma(\beta)}\right.
$$
$$
\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right) ds.
$$

The proof is complete. □

Let the Banach space $E = C[0,1]$ be endowed with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define the cone P by $P = \{u \in E, u(t) \geq 0\}$. Then define the operator $T : P \to E$, If $0 \leq t \leq \sigma$,

$$
Tu(t) = a\phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right)
$$
$$
+ \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)} - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right) ds,
$$

and if $\sigma \leq t \leq 1$,

$$
Tu(t) = b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right)
$$
$$
+ \int_0^1 \frac{(1 - s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right) ds
$$
$$
- \int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,d\tau}{\Gamma(\beta)}\right) ds.
$$

Here σ is defined by (3.1). Obviously, $u(t)$ is a solution of problem (1.1) if and only if $u(t)$ is a fixed point of T.

The following theorem is the main result in this paper.

Theorem 3.1 *Assume there exists $r > 0$ such that:*

(C1) $f(t, u_1) \leq f(t, u_2)$ *for any* $0 \leq t \leq 1, 0 \leq u_1 \leq u_2 \leq r$;

(C2) $\displaystyle \max_{t \in [0,\xi]} f(t, r) \leq \left(\frac{b\Gamma(\alpha + 1)}{\xi^\alpha} \right)^{p-1} \frac{(\eta - \xi)^\beta}{\xi^\beta} \min_{t \in [\xi, \eta]} f(t, 0),$

$\displaystyle \max_{t \in [0,1]} f(t, r) \leq \left[a\Gamma(\alpha + 1) \right]^{p-1} (\eta - \xi)^\beta \min_{t \in [\xi, \eta]} f(t, 0);$

(C3) $\displaystyle \max_{t \in [0,1]} f(t, r) \leq M = \min \left\{ \left(\frac{r}{L_1 + 1} \right)^{p-1}, \left(\frac{r}{L_2 + 1} \right)^{p-1} \right\},$

where

$$L_1 = a \left(\frac{\eta^\beta}{\Gamma(\beta + 1)} \right)^{q-1} + \frac{\eta^\alpha}{\Gamma(\alpha + 1)} \left(\frac{\eta^\beta}{\Gamma(\beta + 1)} \right)^{q-1},$$

$$L_2 = b \left(\frac{\eta^\beta}{\Gamma(\beta + 1)} \right)^{q-1} + \frac{1}{\Gamma(\alpha + 1)} \left(\frac{1}{\Gamma(\beta + 1)} \right)^{q-1}.$$

Then the problem (1.1) *has two positive solutions w^* and v^* such that*

$$0 \leq w^* < r, \quad 0 \leq v^* < r,$$

$$w^* = \lim_{k \to +\infty} w_k, \quad v^* = \lim_{k \to +\infty} v_k,$$

$$w_k = Tw_{k-1}, \quad v_k = Tv_{k-1}, \quad k = 1, 2, \ldots,$$

$$w_0 = \max \left\{ \frac{rL_1}{L_1 + 1}, \frac{rL_2}{L_2 + 1} \right\}, \quad v_0 = 0.$$

Proof We take four steps to prove the theorem.

Step 1. We prove $\xi < \sigma < \eta$.

If $0 < \sigma < \xi$, then

$$
\begin{aligned}
Tu(\sigma) &= a\phi_q \left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^\sigma (\sigma - s)^{\alpha-1} \phi_q \left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right. \\
&\quad \left. - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds \\
&< \frac{1}{\Gamma(\alpha)} \int_0^\sigma (\sigma - s)^{\alpha-1} \phi_q \left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right. \\
&\quad \left. - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-1} \phi_q \left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds \\
&\leq \frac{\xi^\alpha}{\Gamma(\alpha + 1)} \left(\frac{\xi^\beta \max_{t \in [0,\xi]} f(t, r)}{\Gamma(\beta + 1)} \right)^{q-1}
\end{aligned}
$$

and

$$Tu(\sigma) = b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\mathrm{d}\tau}{\Gamma(\beta)}\right)$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\mathrm{d}\tau}{\Gamma(\beta)}\right.$$

$$\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right)\mathrm{d}s$$

$$- \frac{1}{\Gamma(\alpha)} \int_0^\sigma (\sigma - s)^{\alpha-1} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right.$$

$$\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right)\mathrm{d}s$$

$$> b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right.$$

$$\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right)$$

$$\geq b\phi_q\left(\frac{\int_\xi^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right)$$

$$\geq b\left(\frac{(\eta - \xi)^\beta \min_{t\in[\xi,\eta]} f(t,0)}{\Gamma(\beta + 1)}\right)^{q-1}.$$

In view of (C2), we get a contradiction. So we can obtain $\sigma > \xi$.

If $1 > \sigma > \eta$, then

$$Tu(\sigma) = a\phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right)$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^\sigma (\sigma - s)^{\alpha-1} \phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right.$$

$$\left. - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right)\mathrm{d}s$$

$$> a\phi_q\left(\frac{\int_\xi^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right)$$

$$\geq a\left(\frac{(\eta - \xi)^\beta \min_{t\in[\xi,\eta]} f(\tau,0)}{\Gamma(\beta + 1)}\right)^{q-1}$$

and

$$Tu(\sigma) = b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\mathrm{d}\tau}{\Gamma(\beta)}\right)$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\mathrm{d}\tau}{\Gamma(\beta)}\right.$$

$$\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\,\mathrm{d}\tau}{\Gamma(\beta)}\right)\mathrm{d}s$$

$$
- \frac{1}{\Gamma(\alpha)} \int_0^\sigma (\sigma - s)^{\alpha-1} \phi_q \left(\frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right.
$$

$$
\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
< \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} \phi_q \left(\frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) d\tau}{\Gamma(\beta)} \right.
$$

$$
\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
< \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} \phi_q \left(\frac{\int_0^1 (1-\tau)^{\beta-1} f(\tau, u(\tau)) d\tau}{\Gamma(\beta)} \right) ds
$$

$$
\leq \frac{1}{\Gamma(\alpha+1)} \left(\frac{\max_{t \in [0,1]} f(t, r)}{\Gamma(\beta+1)} \right)^{q-1}.
$$

In view of (C2), we get a contradiction. So we can obtain $\sigma < \eta$.

Therefore we have $\xi < \sigma < \eta$.

Step 2. We prove that $T : P \to P$ is completely continuous.

For any $u \in P$, if $0 \leq t \leq \sigma$,

$$
Tu(t) = a\phi_q \left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right)
$$

$$
+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^\sigma (\sigma-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} - \frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
\geq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^\sigma (\sigma-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} - \frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
\geq 0.
$$

If $\sigma \leq t \leq 1$,

$$
Tu(t) = b\phi_q \left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right)
$$

$$
+ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
\geq \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
\geq \int_t^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^s (s-\tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau)) \, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
\geq 0.
$$

So we get $T : P \to P$. Obviously, T is continuous for the continuity of $f(t, u)$.

Let $\Omega \subset P$ be bounded, that is, there exists a positive constant l for any $u \in \Omega$, and letting $M_1 = \max_{0 \leq t \leq 1, 0 \leq u \leq l} f(t, u) + 1$, then, for any $u \in \Omega$, if $0 \leq t \leq \sigma$,

$$
\begin{aligned}
Tu(t) &= a\phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} \right) \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} \right) \mathrm{d}s \\
&\leq a\phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} \right) \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} \right) \mathrm{d}s \\
&\leq a\phi_q\left(\frac{M_1 \sigma^\beta}{\Gamma(\beta+1)} \right) + \frac{\sigma^\alpha}{\Gamma(\alpha+1)} \phi_q\left(\frac{M_1 \sigma^\beta}{\Gamma(\beta+1)} \right),
\end{aligned}
$$

if $\sigma \leq t \leq 1$,

$$
\begin{aligned}
Tu(t) &= b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} \right. \\
&\quad \left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} \right) \\
&\quad + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\mathrm{d}\tau}{\Gamma(\beta)} \right. \\
&\quad \left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} \right) \mathrm{d}s \\
&\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\mathrm{d}\tau}{\Gamma(\beta)} \right. \\
&\quad \left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} \right) \mathrm{d}s \\
&\leq b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} \right) \\
&\quad + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\mathrm{d}\tau}{\Gamma(\beta)} \right) \mathrm{d}s \\
&\leq b\phi_q\left(\frac{M_1 \eta^\beta}{\Gamma(\beta+1)} \right) + \frac{1}{\Gamma(\alpha+1)} \phi_q\left(\frac{M_1}{\Gamma(\beta+1)} \right).
\end{aligned}
$$

Hence, $T(\Omega)$ is uniformly bounded.

Now, we will prove that $T(\Omega)$ is equicontinuous.

For each $u \in \Omega$, if $0 \leq t_1 < t_2 \leq 1$,

$$
\begin{aligned}
&\left| (Tu)(t_2) - (Tu)(t_1) \right| \\
&= \left| \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} \right) \mathrm{d}s \right. \\
&\quad \left. - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, u(\tau))\, \mathrm{d}\tau}{\Gamma(\beta)} \right) \mathrm{d}s \right|
\end{aligned}
$$

$$\leq \left| \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha - 1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} \right) ds \right|$$

$$+ \int_0^{t_1} \left[\frac{(t_2 - s)^{\alpha - 1}}{\Gamma(\alpha)} - \frac{(t_1 - s)^{\alpha - 1}}{\Gamma(\alpha)} \right] \phi_q \left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} \right) ds$$

$$\leq \frac{(t_2 - t_1)^\alpha}{\Gamma(\alpha + 1)} \phi_q \left(\frac{M_1 \sigma^\beta}{\Gamma(\beta + 1)} \right) + \frac{t_2^\alpha - t_1^\alpha}{\Gamma(\alpha + 1)} \phi_q \left(\frac{M_1 \sigma^\beta}{\Gamma(\beta + 1)} \right)$$

therefore, $T(\Omega)$ is equicontinuous. Applying the Arzelá-Ascoli theorem, we conclude that T is a completely continuous operator.

Step 3. Let $\overline{P}_r = \{u \in P : 0 \leq \|u\| \leq r\}$, then we prove $T : \overline{P}_r \to \overline{P}_r$.

For any $u \in \overline{P}_r$, if $0 \leq t \leq \sigma$,

$$Tu(t) = a\phi_q \left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} \right)$$

$$+ \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} - \frac{\int_0^s (s - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} \right) ds$$

$$\leq a\phi_q \left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} \right)$$

$$+ \int_0^\sigma \frac{(\sigma - s)^{\alpha - 1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} \right) ds$$

$$\leq a \left(\frac{\sigma^\beta M}{\Gamma(\beta + 1)} \right)^{q - 1} + \frac{\sigma^\alpha}{\Gamma(\alpha + 1)} \left(\frac{\sigma^\beta M}{\Gamma(\beta + 1)} \right)^{q - 1}$$

$$\leq a \left(\frac{\eta^\beta M}{\Gamma(\beta + 1)} \right)^{q - 1} + \frac{\eta^\alpha}{\Gamma(\alpha + 1)} \left(\frac{\eta^\beta M}{\Gamma(\beta + 1)} \right)^{q - 1}$$

$$= M^{q - 1} L_1 \leq r.$$

If $t \leq \sigma \leq 1$,

$$Tu(t) = b\phi_q \left(\frac{\int_0^\eta (\eta - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} \right)$$

$$+ \int_0^1 \frac{(1 - s)^{\alpha - 1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^s (s - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} \right) ds$$

$$- \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^s (s - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} \right) ds$$

$$\leq b\phi_q \left(\frac{\int_0^\eta (\eta - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} \right)$$

$$+ \int_0^1 \frac{(1 - s)^{\alpha - 1}}{\Gamma(\alpha)} \phi_q \left(\frac{\int_0^1 (1 - \tau)^{\beta - 1} f(\tau, u(\tau))\, d\tau}{\Gamma(\beta)} \right) ds$$

$$\leq b \left(\frac{\eta^\beta M}{\Gamma(\beta + 1)} \right)^{q - 1} + \frac{1}{\Gamma(\alpha + 1)} \left(\frac{M}{\Gamma(\beta + 1)} \right)^{q - 1}$$

$$= M^{q - 1} L_2 \leq r.$$

Consequently, we get $T : \overline{P}_r \to \overline{P}_r$.

Step 4. We prove w^* and v^* are two positive solutions of the problem (1.1).

Since $w_0(t) = \max\{\frac{rL_1}{L_1+1}, \frac{rL_2}{L_2+1}\}$, obviously, $w_0(t) \in \overline{P}_r$. Since $w_{k+1}(t) = Tw_k(t), k = 0, 1, 2, \ldots$ and $T : \overline{P}_r \to \overline{P}_r$, we get $w_k(t) \in \overline{P}_r$.

If $0 \leq t \leq \sigma$,

$$
\begin{aligned}
Tw_0(t) &= a\phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, w_0(\tau))\, d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta-1} f(\tau, w_0(\tau)) d\tau}{\Gamma(\beta)} \right) \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, w_0(\tau))\, d\tau}{\Gamma(\beta)} \right. \\
&\quad \left. - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, w_0(\tau))\, d\tau}{\Gamma(\beta)} \right) ds \\
&\leq a\left(\frac{\eta^\beta M}{\Gamma(\beta+1)} \right)^{q-1} + \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left(\frac{\eta^\beta M}{\Gamma(\beta+1)} \right)^{q-1} \\
&= M^{q-1} L_1 \leq \frac{rL_1}{L_1+1} \leq w_0(t).
\end{aligned}
$$

If $t \leq \sigma \leq 1$,

$$
\begin{aligned}
Tw_0(t) &= b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, w_0(\tau))\, d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, w_0(\tau))\, d\tau}{\Gamma(\beta)} \right) \\
&\quad + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, w_0(\tau))\, d\tau}{\Gamma(\beta)} \right. \\
&\quad \left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, w_0(\tau))\, d\tau}{\Gamma(\beta)} \right) ds \\
&\quad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, w_0(\tau))\, d\tau}{\Gamma(\beta)} \right. \\
&\quad \left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, w_0(\tau))\, d\tau}{\Gamma(\beta)} \right) ds \\
&\leq b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, w_0(\tau))\, d\tau}{\Gamma(\beta)} \right) \\
&\quad + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^1 (1 - \tau)^{\beta-1} f(\tau, w_0(\tau))\, d\tau}{\Gamma(\beta)} \right) ds \\
&\leq b\left(\frac{\eta^\beta M}{\Gamma(\beta+1)} \right)^{q-1} + \frac{1}{\Gamma(\alpha+1)} \left(\frac{M}{\Gamma(\beta+1)} \right)^{q-1} \\
&= M^{q-1} L_2 \leq \frac{L_2 r}{L_2+1} \leq w_0(t).
\end{aligned}
$$

So we get $w_1(t) \leq w_0(t)$, and on the basis of the definition of T, we obtain

$$
w_2(t) = Tw_1(t) \leq Tw_0(t) = w_1(t).
$$

Hence by induction we obtain

$$
w_{k+1}(t) \leq w_k(t), \quad k = 0, 1, 2, \ldots.
$$

Thus, we get $w^* \in \overline{P}_r$ such that $w_k \to w^*$. Applying the continuity of T and $w_{k+1}(t) = Tw_k(t)$, we get $w^*(t) = Tw^*(t)$, hence $w^*(t)$ is a positive solution of problem (1.1).

Since $v_0(t) = 0$, obviously, $v_0(t) \in \overline{P}_r$. Since $v_{k+1}(t) = Tv_k(t)$, $k = 0, 1, 2, \ldots$ and $T : \overline{P}_r \to \overline{P}_r$, we get $v_k(t) \in \overline{P}_r$.

If $0 \le t \le \sigma$,

$$
Tv_0(t) = a\phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta-1} f(\tau, v_0(\tau)) d\tau}{\Gamma(\beta)} \right)
$$

$$
+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} \right.
$$

$$
\left. - \frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
\ge a\phi_q\left(\frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} - \frac{\int_0^\xi (\xi - \tau)^{\beta-1} f(\tau, v_0(\tau)) d\tau}{\Gamma(\beta)} \right)
$$

$$
\ge 0 = v_0(t).
$$

If $t \le \sigma \le 1$,

$$
Tv_0(t) = b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} \right)
$$

$$
+ \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} \right.
$$

$$
\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_q\left(\frac{\int_0^s (s - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} \right.
$$

$$
\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} \right) ds
$$

$$
\ge b\phi_q\left(\frac{\int_0^\eta (\eta - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} \right.
$$

$$
\left. - \frac{\int_0^\sigma (\sigma - \tau)^{\beta-1} f(\tau, v_0(\tau))\, d\tau}{\Gamma(\beta)} \right)
$$

$$
\ge 0 = v_0(t).
$$

So we get $v_1(t) \ge v_0(t)$, and on the basis of the definition of T, we obtain

$$
v_2(t) = Tv_1(t) \ge Tv_0(t) = v_1(t).
$$

Hence by induction we obtain

$$
v_{k+1}(t) \ge v_k(t), \quad k = 0, 1, 2, \ldots.
$$

Thus, we get $v^* \in \overline{P}_r$ such that $v_k \to v^*$. Applying the continuity of T and $v_{k+1}(t) = Tv_k(t)$, we get $v^*(t) = Tv^*(t)$, hence $v^*(t)$ is a positive solution of problem (1.1). □

Remark 3.1 The positive solutions $w^*(t)$ and $v^*(t)$ of problem (1.1) in Theorem 3.1 may coincide, and in this case the problem (1.1) has a positive solution in $\overline{P}_r$.

4 Example

In this section, we give a simple example to explain the main theorem.

Example 4.1 For the problem (1.1), let $\alpha = 1.6, \beta = 1.5, a = 12, b = 3, p = 3, \xi = 0.6, \eta = 0.9, r = 80. f(t,u) = 5t + \sin t + 0.5u$.

From a direct calculation, we get $q = 1.5$,

$$\max_{t\in[0,\xi]} f(t,r) = 5 \times 0.6 + \sin 0.6 + 0.5 \times 80 = 43 + \sin 0.6 < 44,$$

$$\max_{t\in[0,1]} f(t,r) = 5 \times 1 + \sin 1 + 0.5 \times 80 = 85 + \sin 1 < 86,$$

$$\left(\frac{b\Gamma(\alpha+1)}{\xi^\alpha}\right)^{p-1} \frac{(\eta-\xi)^\beta}{\xi^\beta} \min_{t\in[\xi,\eta]} f(t,0) \approx 33.3471 \min_{t\in[\xi,\eta]} f(t,0) > 33 \times 3 = 99,$$

$$\left[a\Gamma(\alpha+1)\right]^{p-1} (\eta-\xi)^\beta \min_{t\in[\xi,\eta]} f(t,0) \approx 48.3603 \min_{t\in[\xi,\eta]} f(t,0) > 48 \times 3 = 144,$$

$$L_1 = a\left(\frac{\eta^\beta}{\Gamma(\beta+1)}\right)^{q-1} + \frac{\eta^\alpha}{\Gamma(\alpha+1)}\left(\frac{\eta^\beta}{\Gamma(\beta+1)}\right)^{q-1} \approx 10.0908,$$

$$L_2 = b\left(\frac{\eta^\beta}{\Gamma(\beta+1)}\right)^{q-1} + \frac{1}{\Gamma(\alpha+1)}\left(\frac{1}{\Gamma(\beta+1)}\right)^{q-1} \approx 3.0109,$$

$$M \approx 52.0366.$$

So we get

$$\max_{t\in[0,\xi]} f(t,r) \leq \left(\frac{b\Gamma(\alpha+1)}{\xi^\alpha}\right)^{p-1} \frac{(\eta-\xi)^\beta}{\xi^\beta} \min_{t\in[\xi,\eta]} f(t,0),$$

$$\max_{t\in[0,1]} f(t,r) \leq \left[a\Gamma(\alpha+1)\right]^{p-1} (\eta-\xi)^\beta \min_{t\in[\xi,\eta]} f(t,0),$$

$$\max_{t\in[0,1]} f(t,r) \leq M.$$

Then all the conditions of Theorem 3.1 are satisfied. Hence, by Theorem 3.1, we see that the aforementioned problem has two positive solutions w^* and v^*.

5 Conclusions

The monotone iterative technique is used to solve the problem of a kind of nonlinear four-point Caputo fractional differential equation with p-Laplacian operator. Under certain nonlinear growth conditions of the nonlinearity, we acquire the existence of positive solutions for the boundary value problem and get iterative schemes for approximating the solutions.

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