Si and Wang *Boundary Value Problems* (2015) 2015:151
DOI 10.1186/s13661-015-0420-7

Boundary Value Problems
a SpringerOpen Journal

RESEARCH | Open Access

Modified characteristics projection finite element method for time-dependent conduction-convection problems

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Abstract
In this paper, we give a modified characteristics projection finite element method for the time-dependent conduction-convection problems, which is gotten by combining the modified characteristics finite element method and the projection method. The stability and the error analysis shows that our method is stable and has optimal convergence order. In order to show the effect of our method, some numerical results are presented. From the numerical results, we can see that the modified characteristics projection finite element method can simulate the fluid field, temperature field, and pressure field very well.

MSC: 76M10; 65N12; 65N30; 35K61

Keywords: time-dependent conduction-convection problems; modified characteristics method; projection method; stability analysis; error estimate

1 Introduction

The conduction-convection problem constitutes an important system of equations in atmospheric dynamics and dissipative nonlinear system of equations. There is a significant amount of literature as regards this problem. An optimizing reduced Petrov-Galerkin least squares mixed finite element (PLSMFE) [1] method for the non-stationary conduction-convection problems was given. An efficient sequential method was developed to estimate the unknown boundary condition on the surface of a body from transient temperature measurements inside the solid [2]. An analysis of conduction natural convection conjugate heat transfer in the gap between concentric cylinders under solar irradiation [3] was carried out by Kim *et al.*, Boland and Layton [4] gave an error analysis for finite element methods for steady natural convection problems. Newton type iterative methods [5–7] and defect-correction methods [8–10] for the conduction-convection equations were presented.

The projection methods, which are efficient methods for solving the incompressible time-dependent fluid flow, were first introduced by Chorin [11] and Temam [12] in the late 1960s. This method is based on a special time-discretization of the Navier-Stokes equations. In this method, the convection-diffusion and the incompressibility are dealt with in two different sub-steps. The velocity obtained in the convection-diffusion sub-step is projected in order to satisfy the weak incompressibility condition. The projection methods can be classified into three families: the pressure-correction method [13, 14], the velocity-correction method [15], and the consistent splitting scheme [16, 17], which is called a gauge method also [18]. The convergence analysis of the semi-discrete projection methods can be found in Shen [19] and Guermond and Quartapelle [20]. In Guermond and Quartapelle [21], the projection method was implemented by the finite element method. It was used to solved the variable density Navier-Stokes equations in [22]. In [23], a gauge-Uzawa projection method was presented. Then it was applied to the conduction-convection equations [24] and incompressible flows with variable density [25].

As we know, the characteristics method is a highly effective method for the advection dominated problems. Douglas and Russell [26] presented the modified method of characteristics first. It was extended to nonlinear coupled systems by Russell [27] in two and three spatial dimensions. A detailed analysis for the Navier-Stokes equations has been done by Dawson *et al.* [28] and numerical tests have been presented by Buscagkia and Dari [29]. A second order MMOC mixed defect-correction finite element method [30] for time-dependent Navier-Stokes problems was given. Notsu *et al.* gave a single-step characteristics finite difference analysis for the convection-diffusion problems [31] and a single-step finite element method for the incompressible Navier-Stokes equations [32]. El-Amrani and Seaid gave the error estimates of the modified method of characteristics finite element methods for the Navier-Stokes [33], natural, and mixed convection flows [34]. In [35], Achdou and Guermond gave the projection/Lagrange-Galerkin method for the incompressible Navier-Stokes equations.

In this paper, we give the modified characteristics projection finite element method (MCPFEM) for the time-dependent conduction-convection problems, which is gotten by combining the modified characteristics finite element method and the projection method. We give stability and error analysis, which show that our method is stable and has optimal convergence order. In order to show the efficiency of our method, some numerical results are presented. At first, the numerical results of Bénard convection problems are given. Then we give some numerical results of the thermal driven cavity flow. From the numerical results, we can see that MCPFEM can simulate the fluid field, temperature field, and pressure field very well.

2 The modified characteristics projection finite element method for the time-dependent conduction-convection problems

In this paper, we consider the time-dependent conduction-convection problem in two dimensions whose coupled equations governing viscous incompressible flow and heat transfer for the incompressible fluid are Boussinesq approximations to the Navier-Stokes equations. For all $t \in (0, t_1]$, find $(u, p, T) \in X \times M \times W$ such that

$$\begin{cases} u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = \kappa\nu^2 gT + f, & x \in \Omega, \\ \operatorname{div} u = 0, & x \in \Omega, \\ T_t - \lambda\nu\Delta T + u \cdot \nabla T = b, & x \in \Omega, \\ u(x,0) = u_0, \qquad T(x,0) = T_0, & x \in \Omega, \\ u = 0, \qquad T = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$. $u = (u_1(x,t), u_2(x,t))^T$ represents the velocity vector, $p(x,t)$ represents the pressure,

$T(x,t)$ represents the temperature, κ represents the Grashoff number, $\lambda = Pr^{-1}$, Pr is the Prandtl number, g represents the vector of gravitational acceleration, $\nu = Re^{-1}$, Re is the Reynolds number, and f and b are the forcing functions.

In this section, we aim to describe some notations and materials which will be frequently used in this paper. For the mathematical setting of the conduction-convection problems (1), we introduce the Hilbert spaces

$$X = H_0^1(\Omega)^2, \qquad W = H^1(\Omega),$$
$$M = L_0^2(\Omega) = \left\{\varphi \in L^2(\Omega); \int_\Omega \varphi\, dx = 0\right\}.$$

$\Im_h$ is a quasi-uniform partition of $\bar{\Omega}_h$ into non-overlapping triangles, indexed by a parameter $h = \max_{K\in\Im_h}\{h_K; h_K = \mathrm{diam}(K)\}$. We introduce the finite element subspace $X_h \subset X$, $M_h \subset M$, $W_h \subset W$ as follows:

$$X_h = \left\{v_h \in X \cap C^0(\bar{\Omega})^2; v_h|_K \in P_\ell(K)^2, \forall K \in \Im_h\right\},$$
$$M_h = \left\{q_h \in M \cap C^0(\bar{\Omega}); q_h|_K \in P_k(K), \forall K \in \Im_h\right\},$$
$$W_h = \left\{\phi_h \in W \cap C^0(\bar{\Omega}); \phi_h|_K \in P_j(K), \forall K \in \Im_h\right\},$$

where $P_\ell(K)$ is the space of piecewise polynomials of degree ℓ on K, $\ell \geq 1$, $k \geq 1$, $j \geq 1$ are three integers. $W_{0h} = W_h \cap H_0^1(\Omega)$, and assume that (X_h, M_h) satisfies the discrete LBB condition, there exists $\beta > 0$ such that

$$\sup_{v_h\in X_h} \frac{d(v_h,\varphi_h)}{\|\nabla v_h\|_0} \geq \beta\|\varphi_h\|_0, \quad \forall \varphi_h \in M_h,$$

where $d(v_h,\varphi_h) = -(\varphi_h, \nabla\cdot v_h)$. Let V_h be the kernel of the discrete divergence operator,

$$V_h = \left\{v_h \in X_h; (q_h, \nabla\cdot v_h) = 0, \forall q_h \in M_h\right\}.$$

For each positive integer N, let $\{\mathcal{J}_n : 1 \leq n \leq N\}$ be a partition of $[0,t_1]$ into subintervals $\mathcal{J}_n = (t_{n-1}, t_n]$, with $t_n = n\tau$, $\tau = T_1/N$. Set $u^n = u(\cdot, t_n)$. The characteristic trace-back along the field u^{n-1} of a point $x \in \Omega$ at time t_n to t_{n-1} is approximately

$$\bar{x}(x, t_{n-1}) = x - u^{n-1}\tau.$$

Consequently, the hyperbolic part in the first equation of (1) at time t_n is approximated by

$$u_t + u^{n-1}\cdot\nabla u^n \approx \frac{u^n - \bar{u}^{n-1}}{\triangle t},$$

where

$$\bar{u}^{n-1} = \begin{cases} u^{n-1}(\bar{x}), & \bar{x} = x - u^{n-1}\tau \in \Omega, \\ 0, & \text{otherwise}, \end{cases}$$

for any function w.

With the previous notations, we get the projection FEM for the time-dependent conduction-convection problem (1), which is a slight transmutation of the projection FEM [13, 19] for the time-dependent Navier-Stokes equations.

Algorithm 2.1 (Projection FEM) Start u_h^0 as a solution of $(u_h^0, v_h) = (u_0, v_h)$ and $(T_h^0, \psi_h) = (T_0, \psi_h)$, $p_h^0 = 0$ for all $v_h \in V_h$, $\psi_h \in M_h$.

Step 1: Find $\hat{u}_h^{n+1} \in X_h$ as the solution of

$$\left(\frac{\hat{u}_h^{n+1} - u_h^n}{\tau}, v_h\right) + B\big(u_h^n, \hat{u}_h^{n+1}, v_h\big) + \nu\big(\nabla \hat{u}_h^{n+1}, \nabla v_h\big) = \kappa\nu^2\big(gT_h^n, v_h\big) + \big(f(t_{n+1}), v_h\big), \quad \forall v_h \in X_h,$$

where $B(u_h, v_h, w_h) = \frac{1}{2}((u_h \cdot \nabla)v_h, w_h) - \frac{1}{2}((u_h \cdot \nabla)w_h, v_h)$.

Step 2: Find $u_h^{n+1} \in V_h$, $p_h^{n+1} \in M_h$ as the solution of

$$\begin{aligned} &\left(\frac{u_h^{n+1} - \hat{u}_h^{n+1}}{\tau}, v_h\right) + d\big(p_h^{n+1}, v_h\big) = 0, \quad \forall v_h \in V_h, \\ &d\big(q_h, u_h^{n+1}\big) = 0, \quad \forall q_h \in M_h. \end{aligned} \tag{2}$$

Step 3: Compute $T_h^{n+1} \in W_h$ as the solution of the linear elliptic equation

$$\left(\frac{T_h^{n+1} - T_h^n}{\tau}, \psi_h\right) + \bar{B}\big(u_h^{n+1}, T_h^{n+1}, \psi_h\big) + \lambda\nu\big(\nabla T_h^{n+1}, \nabla \psi_h\big) = \big(b(t_{n+1}), \psi_h\big), \quad \forall \psi_h \in W_{0h}, \tag{3}$$

$$\bar{B}(u_h, T_h, \psi_h) = \tfrac{1}{2}((u_h \cdot \nabla)T_h, \psi_h) - \tfrac{1}{2}((u_h \cdot \nabla)\psi_h, T_h).$$

Remark 2.1 Denote by P_h the orthogonal projector in $(L^2(\Omega))^2$ onto V. We can readily check that (2) is equivalent to [19]

$$u_h^{n+1} = P_h \hat{u}_h^{n+1}. \tag{4}$$

The MC time discretization, combined with the projection finite element method, leads to the following MC projection finite element method.

Algorithm 2.2 (MC projection FEM) Start with u_h^0 as a solution of $(u_h^0, v_h) = (u_0, v_h)$ for all $v_h \in V_h$.

Step 1: Find $\hat{u}_h^{n+1} \in X_h$ as the solution of

$$\left(\frac{\hat{u}_h^{n+1} - \dot{u}_h^n}{\tau}, v_h\right) + \nu\big(\nabla \hat{u}_h^{n+1}, \nabla v_h\big) = \kappa\nu^2\big(gT_h^n, v_h\big) + \big(f(t_{n+1}), v_h\big), \quad \forall v_h \in V_h, \tag{5}$$

where

$$\dot{u}_h^n = \begin{cases} u_h^n(\dot{x}), & \dot{x} = x - u_h^n\tau \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Step 2: Find $u_h^{n+1} \in V_h, p_h^{n+1} \in M_h$ as the solution of

$$\begin{aligned} &\left(\frac{u_h^{n+1} - \hat{u}_h^{n+1}}{\tau}, v_h\right) + b\left(p_h^{n+1}, v_h\right) = 0, \quad \forall v_h \in V_h, \\ &b\left(q_h, u_h^{n+1}\right) = 0, \quad \forall q_h \in M_h. \end{aligned} \tag{6}$$

Step 3: Compute $T_h^{n+1} \in W_h$, the solution of the linear elliptic equation

$$\left(\frac{T_h^{n+1} - \dot{T}_h^n}{\tau}, \psi_h\right) + \lambda\nu\left(\nabla T_h^{n+1}, \nabla\psi_h\right) = \left(b(t_{n+1}), \psi_h\right), \quad \forall\psi_h \in W_{0h}, \tag{7}$$

where

$$\dot{T}_h^n = \begin{cases} T_h^n(\dot{x}), & \dot{x} = x - u_h^n\tau \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.2 Define $\dot{\mathcal{X}}_x^{n+1}(t) = x - (t_{n+1} - t)u_h^n$, $\forall t \in [t_{n-1}, t_{n+1}]$, $2 \leq l \leq N$. Since X_h is a subset of $W^{1,\infty}(\Omega)$, under the condition $\tau \leq \frac{1}{2L_n}$, $L_n = \max_{1\leq i\leq n} \|u_h^n\|_{W^{1,\infty}}$ on the time step it is an easy matter to verify that this mapping has a positive Jacobian, since u_h^n vanishes on $\partial\Omega$; this mapping is one-to-one and this is a change of variables from Ω onto Ω. This yields for any positive function ϕ on Ω the estimate (please see [36] for details)

$$\int_\Omega \phi\left(\dot{\mathcal{X}}_h^{n+1}(t)\right) dx \leq C \int_\Omega \phi(x)\, dx.$$

3 Stability analysis

Theorem 3.1 (Stability) *If* $\tau \leq \frac{1}{2L_n}$, $L_n = \max_{1\leq i\leq n} \|u_h^i\|_{W^{1,\infty}}$, *the MC projection FEM is stable in the sense that*

$$\begin{aligned} &\left\|u_h^{N+1}\right\|_0^2 + \left\|T_h^{N+1}\right\|_0^2 + 2\nu\tau\sum_{n=1}^N \left\|\nabla u_h^{n+1}\right\|_0^2 + \lambda\nu\tau\sum_{n=1}^N \left\|\nabla T_h^{n+1}\right\|_0^2 \\ &\quad\leq C\left\|u_h^0\right\|_0^2 + C\left\|T_h^0\right\|_0^2 + C\frac{\tau}{2\nu}\sum_{n=1}^N \left\|f(t_{n+1})\right\|_{-1}^2 + 2C\tau\sum_{n=1}^N \left\|b(t_{n+1})\right\|_{-1}. \end{aligned}$$

Remark 3.1 We will prove the boundary of $\|u_h^n\|_{W^{1,\infty}}$ in the next section. Here, we use mathematical induction method.

Proof Let $v_h = u_h^{n+1}$ in (5), we obtain

$$\left(\frac{\hat{u}_h^{n+1} - \dot{u}_h^n}{\tau}, u_h^{n+1}\right) + \nu\left(\nabla\hat{u}_h^{n+1}, \nabla u_h^{n+1}\right) = \kappa\nu^2\left(gT_h^n, u_h^{n+1}\right) + \left(f(t_{n+1}), u_h^{n+1}\right).$$

Using (6), we deduce

$$\left(\hat{u}_h^{n+1}, v_h\right) = \left(u_h^{n+1}, v_h\right) + \tau d(p_h, v_h), \quad \forall v_h \in X_h.$$

Noting $\nabla \cdot u_h^{n+1} = 0$, we get

$$\left(\hat{u}_h^{n+1}, u_h^{n+1}\right) = \left(u_h^{n+1}, u_h^{n+1}\right), \tag{8}$$

$$\left(\nabla \hat{u}_h^{n+1}, \nabla u_h^{n+1}\right) = \left(\nabla u_h^{n+1}, \nabla u_h^{n+1}\right). \tag{9}$$

Then we deduce

$$\left(\frac{u_h^{n+1} - \dot{u}_h^n}{\tau}, u_h^{n+1}\right) + \nu \left\| \nabla u_h^{n+1} \right\|_0^2 = \kappa \nu^2 \left(gT_h^n, u_h^{n+1}\right) + \left(f(t_{n+1}), u_h^{n+1}\right).$$

We arrive at

$$\begin{aligned} &\left\| u_h^{n+1} \right\|_0^2 - \left\| u_h^n \right\|_0^2 + 2\nu\tau \left\| \nabla \hat{u}_h^{n+1} \right\|_0^2 \\ &\quad \leq \left\| \dot{u}_h^n \right\|_0^2 - \left\| u_h^n \right\|_0^2 + 2\kappa \nu^2 \tau \left(gT_h^n, \hat{u}_h^{n+1}\right) + 2\tau \left(f(t_{n+1}), \hat{u}_h^{n+1}\right). \end{aligned} \tag{10}$$

Now, we estimate the bound of $\|\dot{u}_h^n\|_0^2 - \|u_h^n\|_0^2$. By the definition of $\dot{\mathcal{X}}_x^n(t_{n-1})$, we have

$$J\left(\dot{\mathcal{X}}_x^n(t_{n-1})\right) = \begin{pmatrix} 1 - \partial_x u_{h1}^{n-1}\tau & -\partial_y u_{h1}^{n-1}\tau \\ -\partial_x u_{h2}^{n-1}\tau & 1 - \partial_y u_{h2}^{n-1}\tau \end{pmatrix}.$$

Hence,

$$\det J\left(\dot{\mathcal{X}}_x^n(t_{n-1})\right) = 1 + \mathcal{O}(\tau).$$

Then we get

$$\begin{aligned} \|\dot{u}_h^n\|_0^2 - \|u_h^n\|_0^2 &= \int_\Omega \left(\dot{u}_h^n\right)^2 dx - \int_\Omega \left(u_h^n\right)^2 dx \\ &= \int_\Omega \left(u_h^n\right)^2 \left(1 + \mathcal{O}(\tau)\right) dx - \int_\Omega \left(u_h^n\right)^2 dx. \end{aligned}$$

We have

$$\left\| \dot{u}_h^n \right\|_0^2 - \left\| u_h^n \right\|_0^2 \leq C\tau \left\| u_h^n \right\|_0^2. \tag{11}$$

On the other hand, by Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} 2\kappa \nu^2 \tau \left(gT_h^n, u_h^{n+1}\right) &\leq 2C\kappa \nu^2 \tau \left\| T_h^n \right\|_0 \left\| u_h^{n+1} \right\|_0 \\ &\leq C\kappa^2 \nu^4 \tau \left\| T_h^n \right\|_0^2 + C\tau \left\| u_h^{n+1} \right\|_0^2. \end{aligned} \tag{12}$$

Combining (10), (11), and (12), we get

$$\begin{aligned} &\left\| u_h^{n+1} \right\|_0^2 - \left\| u_h^n \right\|_0^2 + 2\nu\tau \left\| \nabla u_h^{n+1} \right\|_0^2 \\ &\quad \leq C\tau \left\| u_h^n \right\|_0^2 + \frac{\tau}{2\nu} \left\| f(t_{n+1}) \right\|_{-1}^2 + \frac{3\nu\tau}{2} \left\| \nabla u_h^{n+1} \right\|_0^2 \\ &\qquad + C\kappa^2 \nu^4 \tau \left\| T_h^n \right\|_0^2 + C\tau \left\| u_h^{n+1} \right\|_0^2. \end{aligned} \tag{13}$$

Let $\psi_h = 2\tau T_h^{n+1}$ in (7), we obtain

$$2\left(T_h^{n+1} - \dot{T}_h^n, T_h^{n+1}\right) + 2\tau\lambda\nu \left(\nabla T_h^{n+1}, \nabla T_h^{n+1}\right) = 2\tau \left(b(t_{n+1}), T_h^{n+1}\right). \tag{14}$$

We deduce

$$\|T_h^{n+1}\|_0^2 - \|T_h^n\|_0^2 + \tau\lambda\nu\|\nabla T_h^{n+1}\|_0^2 \le 2\tau\|b(t_{n+1})\|_{-1} + \|\dot{T}_h^n\|_0^2 - \|T_h^n\|_0^2.$$

Similar to (11), we have

$$\|\dot{T}_h^n\|_0^2 - \|T_h^n\|_0^2 \le C\tau\|u_h^n\|_0^2.$$

Then we can get

$$\|T_h^{n+1}\|_0^2 - \|T_h^n\|_0^2 + \tau\lambda\nu\|\nabla T_h^{n+1}\|_0^2 \le 2\tau\|b(t_{n+1})\|_{-1} + C\tau\|u_h^n\|_0^2. \tag{15}$$

Adding (13) and (15), summing over all n from 0 to N, we can get

$$\begin{aligned}
&\|u_h^{N+1}\|_0^2 + \|T_h^{N+1}\|_0^2 + 2\nu\tau\sum_{n=1}^{N}\|\nabla u_h^{n+1}\|_0^2 + \lambda\nu\tau\sum_{n=1}^{N}\|\nabla T_h^{n+1}\|_0^2 \\
&\quad\le \|u_h^0\|_0^2 + \|T_h^0\|_0^2 + C\tau\sum_{n=1}^{N}\|u_h^{n+1}\|_0^2 \\
&\qquad + \frac{\tau}{2\nu}\sum_{n=1}^{N}\|f(t_{n+1})\|_{-1}^2 + C\kappa^2\nu^4\tau\sum_{n=1}^{N}\|T_h^n\|_0^2 + 2\tau\sum_{n=1}^{N}\|b(t_{n+1})\|_{-1}.
\end{aligned}$$

Using Gronwall lemma, we deduce

$$\begin{aligned}
&\|u_h^{N+1}\|_0^2 + \|T_h^{N+1}\|_0^2 + 2\nu\tau\sum_{n=1}^{N}\|\nabla u_h^{n+1}\|_0^2 + \lambda\nu\tau\sum_{n=1}^{N}\|\nabla T_h^{n+1}\|_0^2 \\
&\quad\le C\|u_h^0\|_0^2 + C\|T_h^0\|_0^2 + C\frac{\tau}{2\nu}\sum_{n=1}^{N}\|f(t_{n+1})\|_{-1}^2 + 2C\tau\sum_{n=1}^{N}\|b(t_{n+1})\|_{-1}.
\end{aligned}$$

□

4 Error analysis

In order to get the error analysis, we give some lemmas first.

Lemma 4.1 [37, 38] *Let* $e(x,n) = [\frac{u^n(x)-\bar{u}^{n-1}(x)}{\tau} - (\frac{\partial u}{\partial t}(x,t_n) + u^n(x)\nabla u^n(x))]$ *and let* $\tau > 0$ *be such that* $u \in \mathscr{C}^4([\tau,T];H^3(\Omega)^2)$. *For* $t_n > \tau$, *we have*

$$e(x,n) = -\tau\left(\frac{1}{2}\frac{d^2 g_x^n}{dt^2} + \frac{\partial u}{\partial t}\cdot\nabla u(x,t_n)\right) + O(\tau^2), \tag{16}$$

where $g_x^n(t) = u(x-(t_n-t)u^{n-1},t)$, $u^n(x) = u(x,t_n)$.

Lemma 4.2 *Let*

$$\zeta(x,n) = \left(\frac{T^n(x) - \dot{T}^{n-1}(x)}{\tau} - T_t(x,t) - u\cdot\nabla T\right),$$

and let $\tau > 0$ *be such that* $T \in \mathscr{C}^4([\tau,T];H^3(\Omega))$. *For* $t_n > \tau$, *we have*

$$\zeta(x,n) = -\tau\left(\frac{1}{2}\frac{d^2\gamma_x^n}{dt^2} + \frac{\partial u}{\partial t}\cdot\nabla T(x,t_n)\right) + O(\tau^2),$$

where $\gamma_x^n(t) = T(x-(t_n-t)u_h^{n-1},t)$, $u^n(x) = u(x,t_n)$.

Lemma 4.3 *There exists $r_h : W \to W_h$; for all $\psi \in W$ we have*

$$\big(\nabla(\psi - r_h\psi), \phi_h\big) = 0, \quad \forall \phi_h \in W_h, \tag{17}$$

$$\int_\Omega (\psi - r_h\psi)\,dx = 0, \quad \|\nabla r_h\psi\|_0 \le \|\nabla\psi\|_0. \tag{18}$$

When $\psi \in W^{r,q}(\Omega)$ $(1 \le q \le \infty)$, we have

$$\|\psi - r_h\psi\|_{-s,q} \le Ch^{r+s}|\psi|_{r,q}, \quad -1 \le s \le m, 0 \le r \le m+1. \tag{19}$$

There exists $\bar{r}_h : W_0 \to W_{0h}$; for all $\psi \in W_0$ we have

$$\big(\nabla(\psi - \bar{r}_h\psi), \phi_h\big) = 0, \quad \forall \phi_h \in W_{0h}, \|\nabla\bar{r}_h\psi\|_0 \le \|\nabla\psi\|_0. \tag{20}$$

When $\psi \in W^{r,q}(\Omega)$ $(1 \le q \le \infty)$, we have

$$\|\psi - \bar{r}_h\psi\|_{-s,q} \le Ch^{r+s}|\psi|_{r,q}, \quad -1 \le s \le m, 0 \le r \le m+1. \tag{21}$$

Then we define the Galerkin projection $(R_h, Q_h) = (R_h(u,p), Q_h(u,p)) : (X, M) \to (X_h, M_h)$, such that

$$\begin{aligned} &\nu a(R_h - u, v_h) - d(Q_h - p, v_h) + d(\varphi_h, R_h - u) = 0, \\ &\quad \forall (u,p) \in (X, M), (v_h, \varphi_h) \in (X_h, M_h). \end{aligned} \tag{22}$$

Lemma 4.4 [39, 40] *The Galerkin projection (R_h, Q_h) satisfies*

$$\begin{aligned} &\|R_h - u\|_0 + h\big(\big\|\nabla(R_h - u)\big\|_0 + \|Q_h - p\|_0\big) \le Ch^{k+1}\big(\nu\|u\|_{k+1} + \|p\|_k\big), \\ &\quad k = 1, 2. \end{aligned} \tag{23}$$

4.1 Error estimate for velocity and temperature

Lemma 4.5 *If $\tau \le \frac{1}{2L_n}$, $L_n = \max_{1\le i\le n}\|u_h^i\|_{W^{1,\infty}}$, u, p, u_t, and p_t are sufficiently smooth, we have*

$$\begin{aligned} &\big\|u_h^{n+1}\big\|_{W^{1,\infty}} < +\infty, \\ &\big\|\xi_h^{N+1}\big\|_0^2 + \sum_{n=0}^N \big\|\hat{\xi}_h^{n+1} - \xi_h^n\big\|_0^2 + \frac{1}{2}\nu\tau\sum_{n=0}^N \big\|\nabla\hat{\xi}_h^{n+1}\big\|_0^2 + \big\|\varepsilon_h^{N+1}\big\|_0^2 \\ &\quad + \sum_{n=0}^N \big\|\varepsilon_h^{n+1} - \xi_h^n\big\|_0^2 + \lambda\nu\tau\sum_{n=0}^N \big\|\nabla\varepsilon_h^{n+1}\big\|_0^2 \le C\big(\tau^2 + h^{2(k+1)}\big), \\ &\big\|\xi_h^{N+1}\big\|_0^2 + \sum_{n=0}^N \big\|\hat{\xi}_h^{n+1} - \xi_h^n\big\|_0^2 + \frac{1}{2}\nu\tau\sum_{n=0}^N \big\|\nabla\xi_h^{n+1}\big\|_0^2 + \big\|\varepsilon_h^{N+1}\big\|_0^2 \\ &\quad + \sum_{n=0}^N \big\|\varepsilon_h^{n+1} - \xi_h^n\big\|_0^2 + \lambda\nu\tau\sum_{n=0}^N \big\|\nabla\varepsilon_h^{n+1}\big\|_0^2 \le C\big(\tau^2 + h^{2(k+1)}\big), \end{aligned}$$

where $\hat{\xi}_h^n = \hat{u}_h^n - R_h^n$, $\xi_h^n = u_h^n - R_h^n$, $\varepsilon_h^{n+1} = T_h^{n+1} - r_h T^{n+1}$, $R_h^n = R_h(u^n, p^n)$, *C is a positive constant independent of* τ *and h*.

Proof Subtracting $(\frac{R_h^{n+1} - \dot{R}_h^n}{\tau}, v_h) + \nu(\nabla R_h^{n+1}, \nabla v_h)$ from both sides of (5), we can get

$$\begin{aligned}
&\left(\frac{(\hat{u}_h^{n+1} - R_h^{n+1}) - (\dot{u}_h^n - \dot{R}_h^n)}{\tau}, v_h\right) + \nu\big(\nabla\big(\hat{u}_h^{n+1} - R_h^{n+1}\big), \nabla v_h\big) \\
&\quad = k\nu^2 g\big(T_h^n, v_h\big) + \big(f^n, v_h\big) - \left(\frac{R_h^{n+1} - \dot{R}_h^n}{\tau}, v_h\right) - \nu\big(\nabla R_h^{n+1}, \nabla v_h\big).
\end{aligned} \tag{24}$$

Defining $\eta^n = u^n - R_h^n$, we can get

$$\begin{aligned}
&\left(\frac{\hat{\xi}_h^{n+1} - \xi_h^n}{\tau}, v_h\right) + \nu\big(\nabla\hat{\xi}_h^{n+1}, \nabla v_h\big) \\
&\quad = -\left(\frac{u^{n+1} - \bar{u}^n}{\tau} - \nu\triangle u^{n+1} + \nabla p^{n+1} - k\nu^2 g T^{n+1} - f^{n+1}, v_h\right) \\
&\qquad + \left(\frac{\eta^{n+1} - \dot{\eta}^n}{\tau}, v_h\right) + \left(\frac{\dot{u}^n - \bar{u}^n}{\tau}, v_h\right) + \big(\nabla p^{n+1}, v_h\big) + \left(\frac{\dot{\xi}_h^n - \xi_h^n}{\tau}, v_h\right) \\
&\qquad + \nu\big(\nabla\big(u^{n+1} - R_h^{n+1}\big), \nabla v_h\big) + k\nu^2 g\big(T^{n+1} - T_h^n, v_h\big) \qquad (25) \\
&\quad = -\left(\frac{u^{n+1} - \bar{u}^n}{\tau} - \nu\triangle u^{n+1} + \nabla p^{n+1} - k\nu^2 g T^{n+1} - f^{n+1}, v_h\right) \\
&\qquad + \left(\frac{\dot{u}^n - \bar{u}^n}{\tau}, v_h\right) + d\big(Q_h^{n+1} - p^{n+1}, v_h\big) + \left(\frac{\eta^{n+1} - \dot{\eta}^n}{\tau}, v_h\right) \\
&\qquad + \left(\frac{\dot{\xi}_h^n - \xi_h^n}{\tau}, v_h\right) + \nu\big(\nabla\big(u^{n+1} - R_h^{n+1}\big), \nabla v_h\big) + k\nu^2 g\big(T^{n+1} - T_h^n, v_h\big) \\
&\quad = -\left(\frac{u^{n+1} - \bar{u}^n}{\tau} - \nu\triangle u^{n+1} + \nabla p^{n+1} - k\nu^2 g T^{n+1} - f^{n+1}, v_h\right) \\
&\qquad + \left(\frac{\eta^{n+1} - \dot{\eta}^n}{\tau}, v_h\right) + \left(\frac{\dot{u}^n - \bar{u}^n}{\tau}, v_h\right) + \left(\frac{\dot{\xi}_h^n - \xi_h^n}{\tau}, v_h\right) \\
&\qquad + k\nu^2 g\big(T^{n+1} - T_h^n, v_h\big). \qquad (26)
\end{aligned}$$

Let $v_h = 2\tau\hat{\xi}_h^{n+1}$ in (26), we can get

$$\begin{aligned}
&\big\|\hat{\xi}_h^{n+1}\big\|_0^2 - \big\|\xi_h^n\big\|_0^2 + \big\|\hat{\xi}_h^{n+1} - \xi_h^n\big\|_0^2 + 2\nu\tau\big\|\nabla\hat{\xi}_h^{n+1}\big\|_0^2 \\
&\quad = -2\tau\left(\frac{u^{n+1} - \bar{u}^n}{\tau} - \nu\triangle u^{n+1} + \nabla p^{n+1} - k\nu^2 g T^{n+1} - f^{n+1}, \hat{\xi}_h^{n+1}\right) \\
&\qquad + 2\big(\eta^{n+1} - \dot{\eta}^n, \hat{\xi}_h^{n+1}\big) + 2\big(\dot{u}^n - \bar{u}^n, \hat{\xi}_h^{n+1}\big) + 2\big(\dot{\xi}_h^n - \xi_h^n, \hat{\xi}_h^{n+1}\big) \\
&\qquad + k\nu^2\tau g\big(T^{n+1} - T_h^n, \hat{\xi}_h^{n+1}\big) \\
&\quad \equiv \sum_{i=1}^{5}\mathcal{A}_i,
\end{aligned} \tag{27}$$

where

$$\mathcal{A}_1 = -2\tau\left(\frac{u^{n+1}-\bar{u}^n}{\tau} - \nu\triangle u^{n+1} + \nabla p^{n+1} - k\nu^2 gT^{n+1} - f^{n+1} - f^{n+1}, \hat{\xi}_h^{n+1}\right),$$
$$\mathcal{A}_2 = 2\left(\dot{u}^n - \bar{u}^n, \hat{\xi}_h^{n+1}\right),$$
$$\mathcal{A}_3 = 2\left(\eta^{n+1} - \dot{\eta}^n, \hat{\xi}_h^{n+1}\right),$$
$$\mathcal{A}_4 = 2\left(\dot{u}^n - \bar{u}^n, \hat{\xi}_h^{n+1}\right) + 2\left(\dot{\xi}_h^n - \xi_h^n, \hat{\xi}_h^{n+1}\right),$$
$$\mathcal{A}_5 = k\nu^2 g\tau\left(T^{n+1} - T_h^n, \hat{\xi}_h^{n+1}\right).$$

Now, we estimate each term $\mathcal{A}_i$, respectively. By Hölder inequality, we get

$$\mathcal{A}_1 \le C\tau\|\bar{e}^{n+1}\|_0^2 + \frac{\nu\tau}{8}\|\nabla\hat{\xi}_h^{n+1}\|_0^2. \tag{28}$$

By the definition of $\dot{x}$ and $\bar{x}$, we can get

$$\dot{x}(x,t_n) - \bar{x}(x,t_n) = \left(u_h^n - u^n\right)\tau.$$

Using Taylor's formula, we obtain

$$\begin{aligned}
\left|\dot{u}^n - \bar{u}^n\right| &= \left|u^n(\dot{x}) - u^n(\bar{x})\right| \\
&\le \tau\|\nabla u^n\|_\infty\left|u_h^n - u^n\right| \\
&\le \tau\|\nabla u^n\|_\infty\left(\left|u_h^n - R_h^n\right| + \left|R_h^n - u^n\right|\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|\dot{u}^n - \bar{u}^n\|_0 &\le \tau\|\nabla u^n\|_\infty\left(\|u^n - R_h^n\|_0 + \|R_h^n - u_h^n\|_0\right) \\
&\le C\tau\left(h^{k+1} + \|\xi_h^n\|_0\right).
\end{aligned}$$

Then we deduce

$$\begin{aligned}
\mathcal{A}_2 &\le C\|\dot{u}^n - \bar{u}^n\|_0\|\nabla\hat{\xi}_h^{n+1}\|_0 \\
&\le C\tau\left(h^{2(k+1)} + \|\xi_h^n\|_0^2\right) + \frac{\nu\tau}{8}\|\nabla\hat{\xi}_h^{n+1}\|_0^2.
\end{aligned} \tag{29}$$

Now, we estimate the boundedness of $\mathcal{A}_3$. We have

$$\begin{aligned}
\|\eta^{n+1} - \eta^n\|_0 &= \left(\int_\Omega\left(\eta^{n+1} - \eta^n\right)^2 dx\right)^{\frac{1}{2}} = \left(\int_\Omega\left|\int_{t_n}^{t_{n+1}}\frac{\partial\eta}{\partial t}(x,\theta)\,d\theta\right|^2 dx\right)^{\frac{1}{2}} \\
&\le \sqrt{\tau}\left(\int_\Omega\int_{t_n}^{t_{n+1}}\left|\frac{\partial\eta}{\partial t}\right|^2(x,\theta)\,d\theta\,dx\right)^{\frac{1}{2}} \\
&\le \sqrt{\tau}\left\|\frac{\partial\eta}{\partial t}\right\|_{L^2([t_n,t_{n+1}]:L^2(\Omega))}.
\end{aligned} \tag{30}$$

By the definition of $\hat{\mathcal{X}}_x^{n+1}(t_n)$, we can get

$$J(\hat{\mathcal{X}}_x^{n+1}(t_n)) = \begin{pmatrix} 1-\partial_x u_{h1}^{n-1}\tau & -\partial_y u_{h1}^{n-1}\tau \\ -\partial_x u_{h2}^{n-1}\tau & 1-\partial_y u_{h2}^{n-1}\tau \end{pmatrix}.$$

Hence,

$$\det J(\hat{\mathcal{X}}_x^{n+1}(t_n)) = 1 + \mathcal{O}(\tau).$$

Then we get

$$\begin{aligned}
\|\eta^n - \dot{\eta}^n\|_{-1} &= \sup_{v\in V}\big(\|\nabla v\|_0^{-1}(\eta^n - \hat{\eta}^n, v)\big) \\
&= \sup_{v\in V}\left[\|\nabla v\|_0^{-1}\left(\int_\Omega \eta^n(x)v(x)\,dx - \int_\Omega \eta^n(z)v(\hat{\mathcal{X}}_x^n(t_n)^{-1})(1+\mathcal{O}(\triangle t^2))\,dz\right)\right] \\
&\le \sup_{v\in V}\left(\|\nabla v\|_0^{-1}\int_\Omega \eta^{n-1}(x)\big(v(x) - v(\hat{\mathcal{X}}_x^n(t_n)^{-1})\big)\,dx\right) \\
&\quad + \sup_{v\in V}\left(C\tau^2\|\nabla v\|_0^{-1}\int_\Omega \eta^{n-1}(z)v(\hat{\mathcal{X}}_x^n(t_n)^{-1})\,dz\right).
\end{aligned}$$

Let $G(x) = x - \hat{\mathcal{X}}_x^{n+1}(t_n)^{-1}$, then $|G(x)| \le C\tau$, and

$$\begin{aligned}
\|v(x) - v(\hat{\mathcal{X}}_x^{n+1}(t_n)^{-1})\|_0^2 &\le \int_\Omega\left(\int_{t_n}^{t_{n+1}} \frac{d}{dt}v(\hat{\mathcal{X}}_x^{n+1}(t)^{-1})\,dt\right)^2 dx \\
&\le C\tau^2\|\nabla v\|_0^2.
\end{aligned}$$

Similarly, we have

$$\|v(\hat{\mathcal{X}}_x^{n+1}(t_n)^{-1})\| \le \|v\|_0^2(1+C\tau).$$

Then we deduce

$$\|\eta^n - \dot{\eta}^n\|_{-1} \le C\tau\|\eta^n\|_0. \tag{31}$$

By (30) and (31), we have

$$\|\eta^{n+1} - \dot{\eta}^n\|_{-1} \le C\tau h^{k+1}\|u_h^n\|_\infty + Ch^{k+1}\sqrt{\tau}. \tag{32}$$

Therefore, we get

$$\begin{aligned}
\mathcal{A}_3 &\le \|\eta^{n+1} - \dot{\eta}^n\|_{-1}\|\nabla\hat{\xi}_h^{n+1}\|_0 \\
&\le C\tau^2 h^{2(k+1)} + C\tau h^{2(k+1)} + \frac{\nu\tau}{16}\|\nabla\hat{\xi}_h^{n+1}\|_0^2.
\end{aligned} \tag{33}$$

Similarly, we obtain

$$\mathcal{A}_4 \le C\tau\|\xi_h^n\|_0^2 + \frac{\nu\tau}{16}\|\nabla\hat{\xi}_h^{n+1}\|_0^2.$$

For the term $\mathcal{A}_5$, by Taylor's formula, we can get $\|T^{n+1} - T^n\|_0 \le C\tau$, then

$$\begin{aligned}\mathcal{A}_5 &= k\nu^2 g\tau\big(T^{n+1} - T_h^n, \hat{\xi}_h^{n+1}\big)\\ &= k\nu^2 g\tau\big(T^{n+1} - T^n, \hat{\xi}_h^{n+1}\big) + k\nu^2 g\tau\big(T^n - T_h^n, \hat{\xi}_h^{n+1}\big)\\ &\le Ck^2\nu^4\tau^3 + Ck^2\nu^4\tau\big\|T^n - T_h^n\big\|_0^2 + \frac{\nu\tau}{8}\big\|\nabla\xi_h^{n+1}\big\|_0^2. \end{aligned} \tag{34}$$

Combining (27), (28), (20), and (34), we arrive at

$$\begin{aligned}&\big\|\hat{\xi}_h^{n+1}\big\|_0^2 - \big\|\xi_h^n\big\|_0^2 + \big\|\hat{\xi}_h^{n+1} - \xi_h^n\big\|_0^2 + \frac{1}{2}\nu\tau\big\|\nabla\hat{\xi}_h^{n+1}\big\|_0^2 + \nu\tau\big(\big\|s_h^{n+1}\big\|_0^2 - \big\|s_h^n\big\|_0^2\big)\\ &\quad\le C\tau\big(\tau^2 + h^{2(k+1)}\big) + C\tau\big\|\xi_h^n\big\|_0^2 + Ck^2\nu^4\tau\big\|T^n - T_h^n\big\|_0^2. \end{aligned} \tag{35}$$

Subtracting $\tau^{-1}(r_h T^{n+1} - r_h\dot{T}^n, \psi_h) + \lambda\nu(\nabla r_h T^{n+1}, \nabla\psi_h)$ from both sides of (7), we can get

$$\begin{aligned}&\left(\frac{(T_h^{n+1} - r_h T^{n+1}) - (\dot{T}_h^n - r_h\dot{T}^n)}{\tau}, \psi_h\right) + \lambda\nu\big(\nabla\big(T_h^{n+1} - r_h T^{n+1}\big), \nabla\psi_h\big)\\ &\quad= -\left(\frac{r_h T^{n+1} - r_h\dot{T}^n}{\tau}, \psi_h\right) - \lambda\nu\big(\nabla r_h T^{n+1}, \nabla\psi_h\big) + \big(b(t_{n+1}), \nabla\psi_h\big)\\ &\quad= \left(\frac{(T^{n+1} - r_h T^{n+1}) - (\dot{T}^n - r_h\dot{T}^n)}{\tau}, \psi_h\right) + \lambda\nu\big(\nabla\big(T^{n+1} - r_h T^{n+1}\big), \nabla\psi_h\big)\\ &\qquad+ \big(\zeta(t_{n+1}), \psi_h\big). \end{aligned} \tag{36}$$

Letting $\dot{\varepsilon}_h^{n+1} = \dot{T}_h^{n+1} - r_h\dot{T}^{n+1}$, $\theta^{n+1} = T^{n+1} - r_h T^{n+1}$, $\dot{\theta}^n = \dot{T}^n - r_h\dot{T}^n$, and $\psi_h = 2\tau\varepsilon_h^{n+1}$ in (36), we can get

$$\begin{aligned}&2\big(\varepsilon_h^{n+1} - \varepsilon_h^n, \varepsilon_h^{n+1}\big) + 2\lambda\nu\tau\big(\nabla\varepsilon_h^{n+1}, \nabla\varepsilon_h^{n+1}\big)\\ &\quad= 2\big(\theta^{n+1} - \dot{\theta}^n, \varepsilon_h^{n+1}\big) + 2\big(\zeta(t_{n+1}), \varepsilon_h^{n+1}\big) + 2\big(\varepsilon_h^n - \dot{\varepsilon}_h^n, \varepsilon_h^{n+1}\big). \end{aligned}$$

Similarly to (32), we get

$$\begin{aligned}&\big\|\varepsilon_h^n - \dot{\varepsilon}_h^n\big\|_0 \le C\tau\big\|\varepsilon_h^n\big\|_0,\\ &\big\|\theta^{n+1} - \dot{\theta}^n\big\|_{-1} \le C\tau h^{k+1}\big\|u_h^n\big\|_\infty + Ch^{k+1}\sqrt{\tau}. \end{aligned}$$

Then we deduce

$$\begin{aligned}&\big\|\varepsilon_h^{n+1}\big\|_0^2 - \big\|\varepsilon_h^n\big\|_0^2 + \big\|\varepsilon_h^{n+1} - \varepsilon_h^n\big\|_0 + 2\lambda\nu\tau\big\|\nabla\varepsilon_h^{n+1}\big\|_0^2\\ &\quad\le C\tau h^{r+1} + C\tau^3 + \lambda\nu\tau\big\|\nabla\varepsilon_h^{n+1}\big\|_0^2 + C\tau\big\|\varepsilon_h^n\big\|_0^2. \end{aligned}$$

Namely,

$$\begin{aligned}&\big\|\varepsilon_h^{n+1}\big\|_0^2 - \big\|\varepsilon_h^n\big\|_0^2 + \big\|\varepsilon_h^{n+1} - \varepsilon_h^n\big\|_0 + \lambda\nu\tau\big\|\nabla\varepsilon_h^{n+1}\big\|_0^2\\ &\quad\le C\tau h^{2(r+1)} + C\tau^3 + C\tau\big\|\varepsilon_h^n\big\|_0^2. \end{aligned} \tag{37}$$

Adding (35) to (37), we get

$$\begin{aligned}&\|\xi_h^{n+1}\|_0^2-\|\xi_h^n\|_0^2+\|\xi_h^{n+1}-\xi_h^n\|_0^2+\frac{1}{2}\nu\tau\|\nabla\hat{\xi}_h^{n+1}\|_0^2\\&\quad+\|\varepsilon_h^{n+1}\|_0^2-\|\varepsilon_h^n\|_0^2+\|\varepsilon_h^{n+1}-\varepsilon_h^n\|_0+\lambda\nu\tau\|\nabla\varepsilon_h^{n+1}\|_0^2\\&\leq C\tau h^{2(r+1)}+C\tau^3+C\tau\|\xi_h^n\|_0^2+C\tau\|\varepsilon_h^n\|_0^2.\end{aligned}$$

Summing over n from 0 to N gives

$$\begin{aligned}&\|\hat{\xi}_h^{N+1}\|_0^2+\sum_{n=0}^N\|\hat{\xi}_h^{n+1}-\xi_h^n\|_0^2+2\sum_{n=0}^N\|\nabla\rho_h^{n+1}\|_0^2\\&\quad+\frac{1}{2}\nu\tau\sum_{n=0}^N\|\nabla\hat{\xi}_h^{n+1}\|_0^2+\|\varepsilon_h^{N+1}\|_0^2\\&\quad+\sum_{n=0}^N\|\varepsilon_h^{n+1}-\varepsilon_h^n\|_0^2+\lambda\nu\tau\sum_{n=0}^N\|\nabla\varepsilon_h^{n+1}\|_0^2\\&\leq C\big(\tau^2+h^{2(k+1)}\big)+C\tau\sum_{n=0}^N\big(\|\varepsilon_h^n\|_0^2+\|\xi_h^n\|_0^2\big).\end{aligned}$$

By Gronwall lemma, we obtain

$$\begin{aligned}&\|\hat{\xi}_h^{N+1}\|_0^2+\sum_{n=0}^N\|\hat{\xi}_h^{n+1}-\xi_h^n\|_0^2+\frac{1}{2}\nu\tau\sum_{n=0}^N\|\nabla\hat{\xi}_h^{n+1}\|_0^2+\|\varepsilon_h^{N+1}\|_0^2\\&\quad+\sum_{n=0}^N\|\varepsilon_h^{n+1}-\xi_h^n\|_0^2+\lambda\nu\tau\sum_{n=0}^N\|\nabla\varepsilon_h^{n+1}\|_0^2\leq C\big(\tau^2+h^{2(k+1)}\big).\end{aligned}$$

Using (6), we get

$$\left(\frac{\xi_h^{n+1}-\hat{\xi}_h^{n+1}}{\tau},v_h\right)-b\big(p_h^{n+1},v_h\big)=0.$$

Let $v_h=2\tau\xi_h^{n+1}$, we can get

$$\|\xi_h^{n+1}\|_0^2-\|\hat{\xi}_h^{n+1}\|_0^2+\|\xi_h^{n+1}-\hat{\xi}_h^{n+1}\|_0^2=0.$$

Then we have

$$\begin{aligned}&\|\xi_h^{N+1}\|_0^2+\sum_{n=0}^N\|\hat{\xi}_h^{n+1}-\xi_h^n\|_0^2+\frac{1}{2}\nu\tau\sum_{n=0}^N\|\nabla\hat{\xi}_h^{n+1}\|_0^2+\|\varepsilon_h^{N+1}\|_0^2\\&\quad+\sum_{n=0}^N\|\varepsilon_h^{n+1}-\xi_h^n\|_0^2+\lambda\nu\tau\sum_{n=0}^N\|\nabla\varepsilon_h^{n+1}\|_0^2\leq C\big(\tau^2+h^{2(k+1)}\big).\end{aligned}$$

Using the inequality (see [41], Remark 1.6 and [19]),

$$\|\nabla P_h u\|_0\leq C\|\nabla u\|_0$$

we have

$$\|\xi_h^{N+1}\|_0^2+\sum_{n=0}^N\|\hat{\xi}_h^{n+1}-\xi_h^n\|_0^2+\frac{1}{2}\nu\tau\sum_{n=0}^N\|\nabla\xi_h^{n+1}\|_0^2+\|\varepsilon_h^{N+1}\|_0^2$$
$$+\sum_{n=0}^N\|\varepsilon_h^{n+1}-\xi_h^n\|_0^2+\lambda\nu\tau\sum_{n=0}^N\|\nabla\varepsilon_h^{n+1}\|_0^2\le C(\tau^2+h^{2(k+1)}). \tag{38}$$

Using the triangle inequality, we deduce

$$\|u_h^{n+1}\|_{W^{1,\infty}}\le\|u_h^{n+1}-R_h^{n+1}\|_{W^{1,\infty}}+\|R_h^{n+1}\|_{W^{1,\infty}}.$$

Via the inverse inequality, $\|v_h\|_{W^{1,\infty}}\le Ch^{-1}\|\nabla v_h\|_0$ (see [36]), we can get

$$\|u_h^{n+1}\|_{W^{1,\infty}}\le Ch^{-1}\|\nabla(u_h^{n+1}-R_h^{n+1})\|_0+\|R_h^{n+1}\|_\infty.$$

We thus finish the proof. □

Theorem 4.6 (Error estimates for the velocity and temperature) *If* $\tau\le\frac{1}{2L_n}$, u, p, u_t, *and* p_t *are sufficiently smooth, we have*

$$\tau\sum_{n=0}^N\|u^{N+1}-u_h^{N+1}\|_0^2\le C(\tau^2+h^{2(k+1)}), \tag{39}$$

$$\frac{1}{2}\nu\tau\sum_{n=0}^N\|\nabla(u^{n+1}-u_h^{n+1})\|_0^2\le C(\tau^2+h^{2k}), \tag{40}$$

$$\tau\sum_{n=0}^N\|T^{N+1}-T_h^{N+1}\|_0^2\le C(\tau^2+h^{2(k+1)}), \tag{41}$$

$$\frac{1}{2}\nu\tau\sum_{n=0}^N\|\nabla(T^{n+1}-T_h^{n+1})\|_0^2\le C(\tau^2+h^{2k}). \tag{42}$$

Proof Using triangle inequality, (23), and Lemma 4.3, we can get this theorem. □

4.2 Error estimates for the pressure

The following theorem on the pressure is a consequence of the previous theorem on the velocity.

Theorem 4.7 (Error estimate for pressure) *If* $\tau\le\frac{1}{2L_n}$, u, p, u_t, *and* p_t *are sufficiently smooth, we have for all* $1\le n\le N$,

$$\|p^{n+1}-p_h^{n+1}\|_0\le C(\tau+h^k).$$

Proof By (6), we deduce

$$\begin{aligned}&(p^{n+1}-p_h^{n+1},\nabla v_h)\\&\quad=\left(\frac{u^{n+1}-\bar{u}^n}{\tau}-\nu\triangle u^{n+1}+\nabla p^{n+1}-k\nu^2gT^{n+1}-f^{n+1},v_h\right)-\left(\frac{\xi_h^{n+1}-\xi_h^n}{\tau},v_h\right)\end{aligned}$$

$$
\begin{aligned}
&- \nu\big(\nabla\hat{\xi}_h^{n+1}, \nabla v_h\big) + \left(\frac{\eta^{n+1} - \dot{\eta}^n}{\tau}, v_h\right) \\
&+ \left(\frac{\dot{u}^n - \bar{u}^n}{\tau}, v_h\right) + \left(\frac{\dot{\xi}_h^n - \xi_h^n}{\tau}, v_h\right) \\
&+ \nu\big(\nabla\big(u^{n+1} - R_h^{n+1}\big), \nabla v_h\big) + k\nu^2 g\big(T^{n+1} - T_h^n, v_h\big).
\end{aligned}
$$

By the LBB condition and Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
&\big\|p^{n+1} - p_h^{n+1}\big\|_0 \\
&\quad\le \left\| \frac{u^{n+1} - \bar{u}^n}{\tau} - \nu\triangle u^{n+1} + \nabla p^{n+1} - k\nu^2 g T^{n+1} - f^{n+1} \right\|_0 \\
&\quad+ C\tau^{-1}\big\|\xi_h^{n+1} - \xi_h^n\big\|_0 + \nu\big\|\hat{\xi}_h^{n+1}\big\|_0 + C\left\|\frac{\dot{u}^n - \bar{u}^n}{\tau}\right\|_0 \\
&\quad+ C\left\|\frac{\eta^{n+1} - \dot{\eta}^n}{\tau}\right\|_0 + C\left\|\frac{\dot{\xi}_h^n - \xi_h^n}{\tau}\right\|_0 - \nu\big\|\nabla\big(u^{n+1} - R_h^{n+1}\big)\big\|_0 \\
&\quad+ k\nu^2 g\big\|T^{n+1} - T_h^n\big\|_0.
\end{aligned}
$$

Using (27), (28), (20), and (34), we arrive at

$$
\big\|p^{n+1} - p_h^{n+1}\big\|_0 \le C\big(\tau + h^k\big).
$$

Thus, we finish the proof. □

5 Numerical experiments

In order to show the effect of our method, we give some numerical results in this section.

5.1 Bénard convection problem

The first experiments is Bénard convection problem in the domain $\Omega = [0,5] \times [0,1]$ with the forcing $f = 0$ and $b = 0$. Figure 1 displays the initial and boundary conditions for velocity u and temperature T. It means that the boundary conditions for the velocity are the no-slip boundary condition $u = 0$ on $\partial\Omega$, thermal insulation $\partial_\upsilon T = 0$ on the lateral boundaries, and a fixed temperature difference between top and bottom boundaries. Here, we choose $h = 1/16$, $\tau = 0.01$, and the finite element space is a Taylor-Hood finite element space. Here, we use the software package FreeFEM++ [42] for our program.

First, we set $\kappa = 10^4$, $\lambda = 1.0$, $\nu = 1.0$. Figure 2 gives the numerical temperature at $t = 0.05, 0.1, 0.15,$ and 1.0. Figure 3 gives the numerical pressure at $t = 0.05, 0.1, 0.15,$ and 1.0. Figure 4 gives the numerical streamline at $t = 0.05, 0.1, 0.15,$ and 1.0.

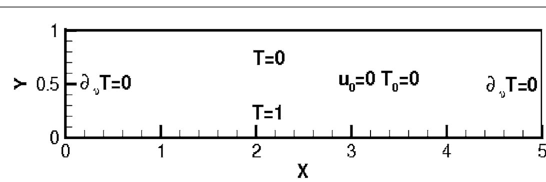


Figure 1 Physics model of Bénard convection problem.

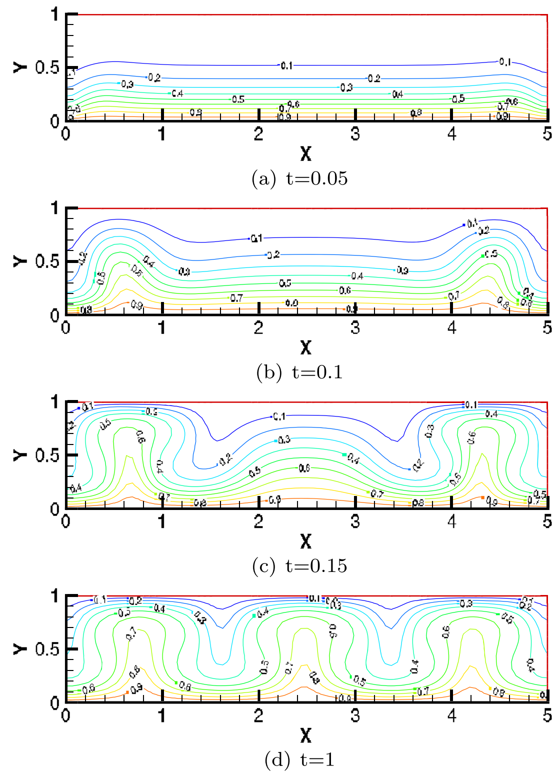


Figure 2 Numerical temperatures of Bénard convection problem with $\kappa = 1 \times 10^4$ at different times.

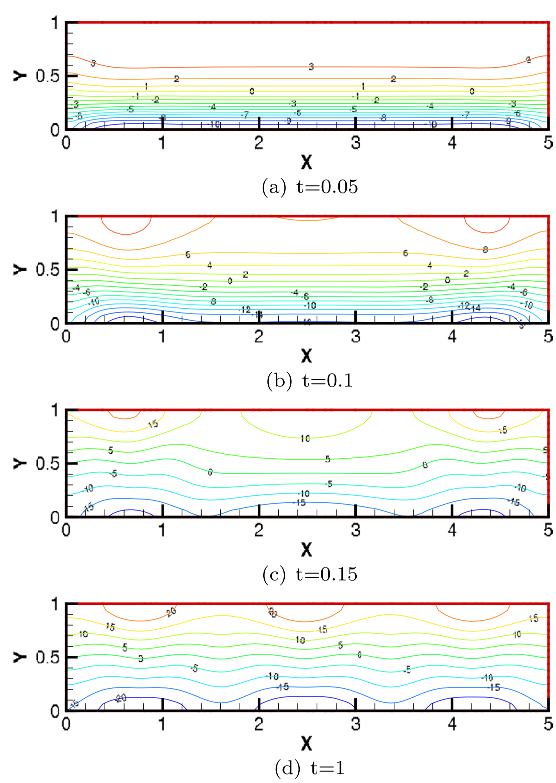


Figure 3 Numerical pressures of Bénard convection problem with $\kappa = 1 \times 10^4$ at different times.

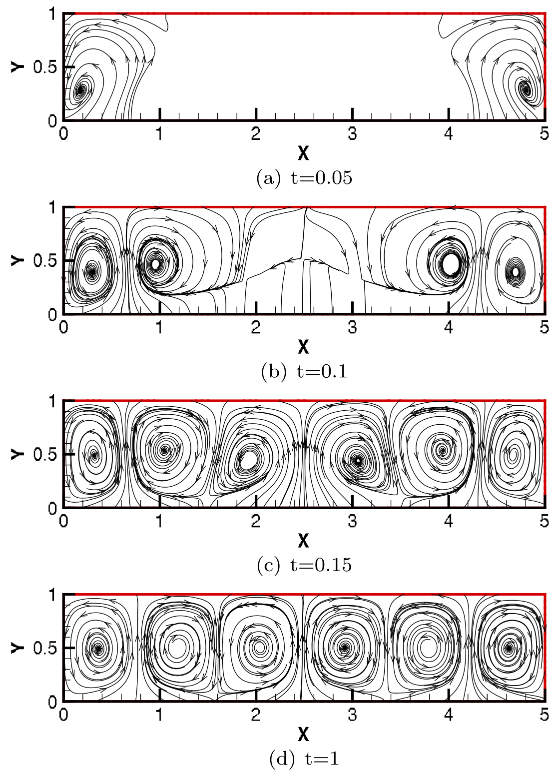


Figure 4 Numerical streamlines of Bénard convection problem with $\kappa = 1 \times 10^4$ at different times.

Then we set $\kappa = 10^5$, $\lambda = 1.0$, $\nu = 1.0$. Figure 5 gives the numerical temperature at $t = 0.05, 0.1, 0.15$, and 1.0. Figure 6 gives the numerical pressure at $t = 0.05, 0.1, 0.15$, and 1. Figure 7 gives the numerical streamline at $t = 0.05, 0.1, 0.15$, and 1. From the numerical results, we can see that MCPFEM can simulate the fluid field, temperature field and pressure field very well, and it works well for a high Grashoff number κ.

5.2 Thermal driven cavity flow problem

Here, we consider the thermal driven flow in an enclosed square $\Omega = [0,1]^2$ with the forcing $f = 0$ and $b = 0$, and the initial and boundary conditions are given by Figure 8. It means that the boundary conditions for velocity is no-slip boundary condition $u = 0$ on $\partial\Omega$, and thermal insulation $\partial_\upsilon T = 0$ on the top and bottom boundaries, and a fixed temperature difference between left and right boundaries. Here, we choose $h = 1/32$, $\tau = 10^{-4}$, and the finite element space is a Taylor-Hood finite element space.

We choose $\lambda = 1$, $\nu = 1$, $\kappa = 10^5$ and 10^6 respectively. Figures 9 and 10 give the numerical results for $\kappa = 10^5$ and 10^6, respectively. From the numerical results, we can see that MCPFEM can simulate the fluid field, temperature field, and pressure field very well. The numerical experiments confirm our theoretical analysis and demonstrate the efficiency of our method.

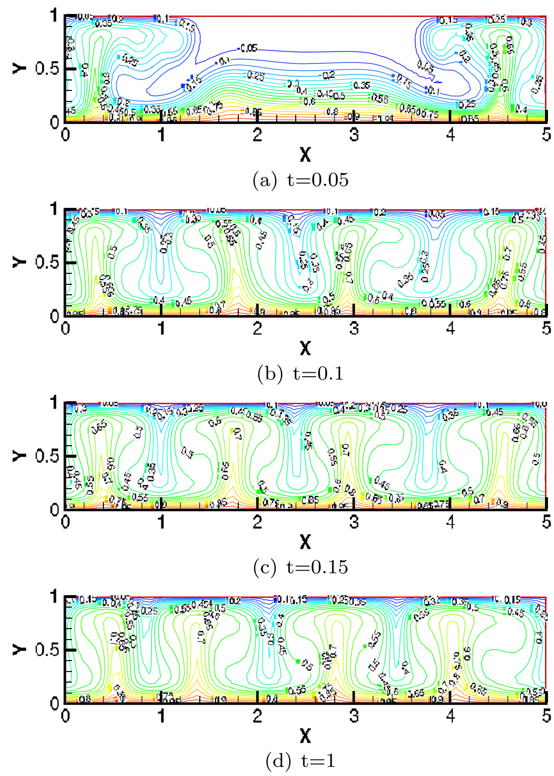


Figure 5 Numerical temperatures of Bénard convection problem with $\kappa = 1 \times 10^5$ at different times.

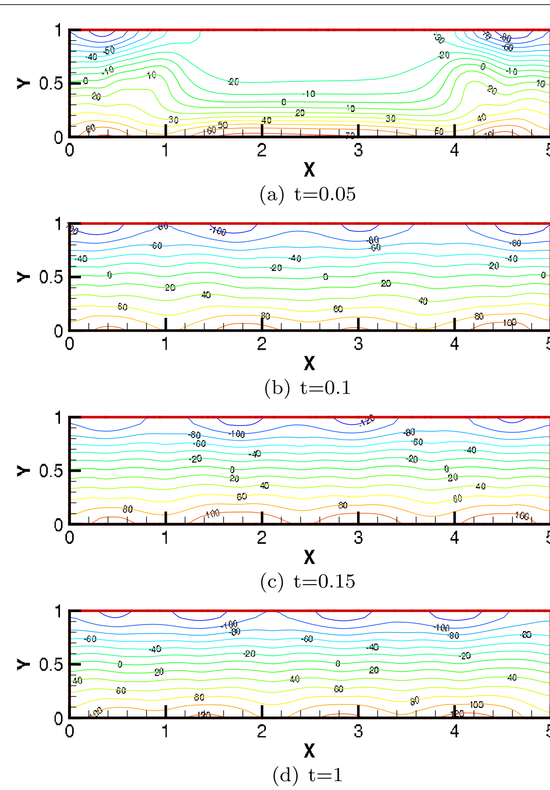


Figure 6 Numerical pressures of Bénard convection problem with $\kappa = 1 \times 10^5$ at different times.

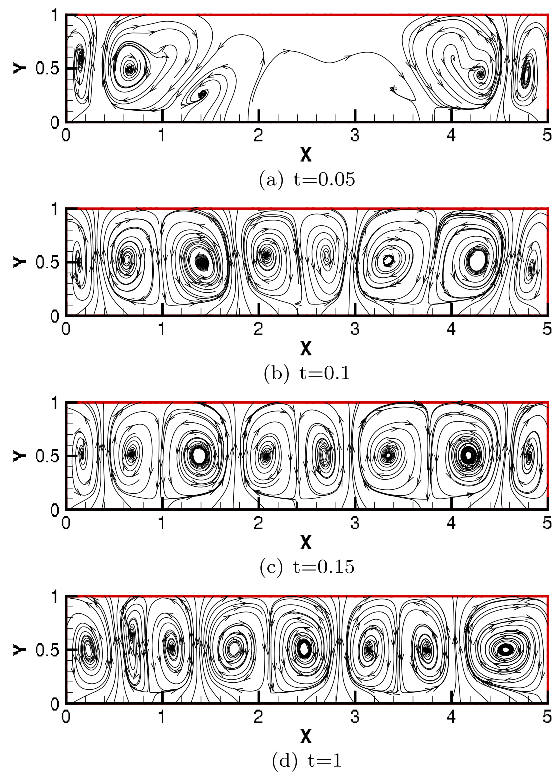
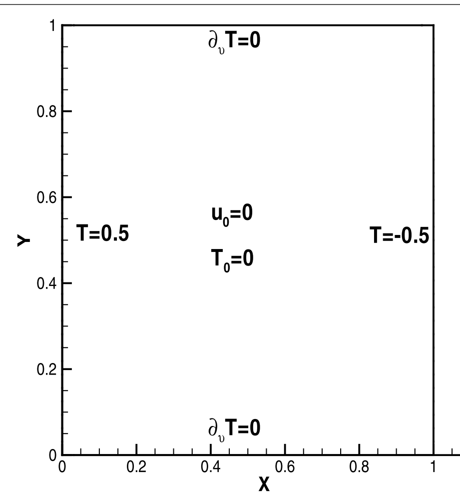


Figure 7 Numerical streamline of Bénard convection problem with $\kappa = 1 \times 10^5$ at different times.

Figure 8 Physics model of thermal driven cavity flow.



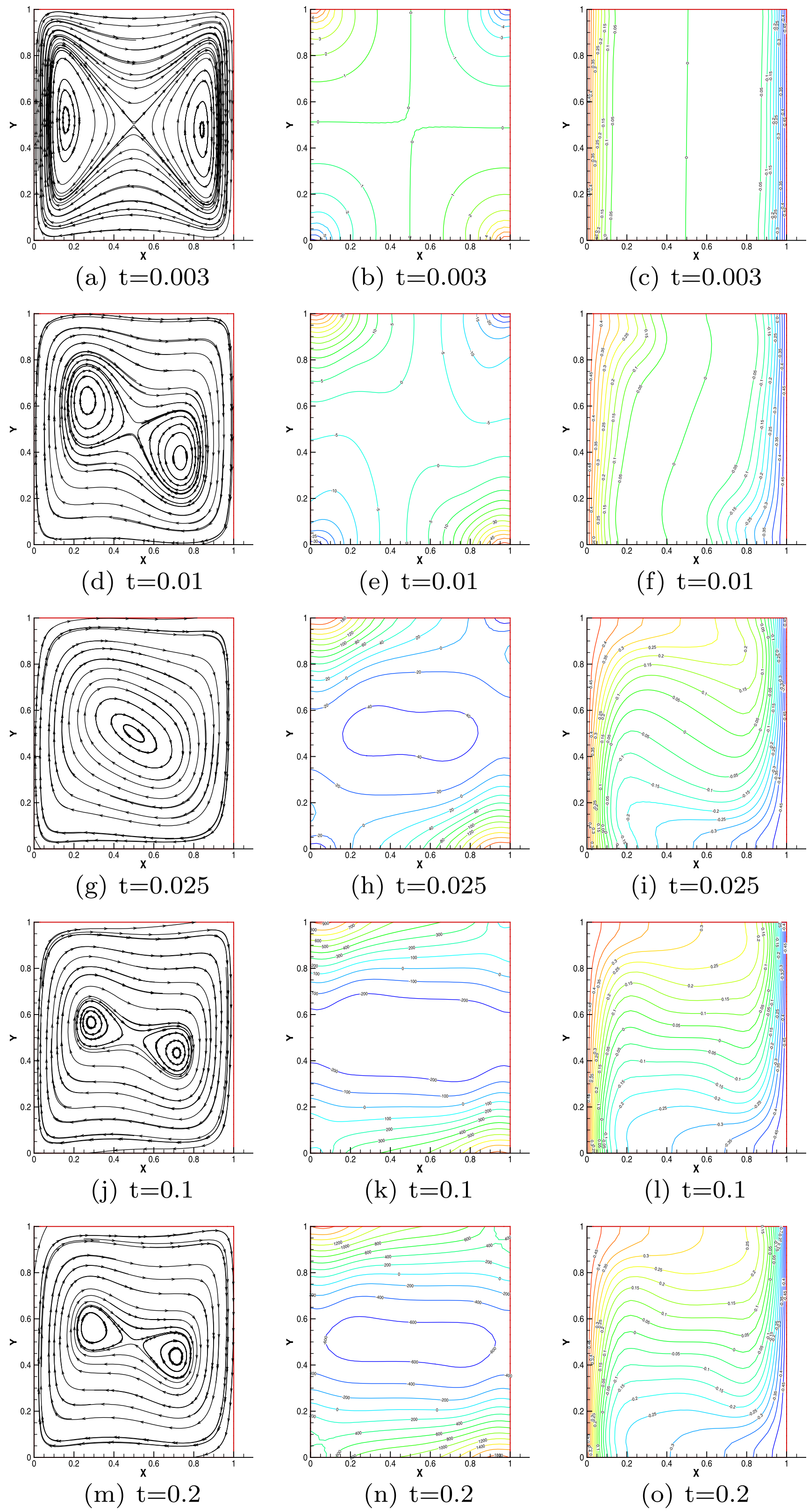


Figure 9 Numerical results of thermal driven cavity flow with $\kappa = 1 \times 10^5$ at different time, left panels the numerical streamlines, middle panels the numerical pressures, and right panels the numerical temperatures.

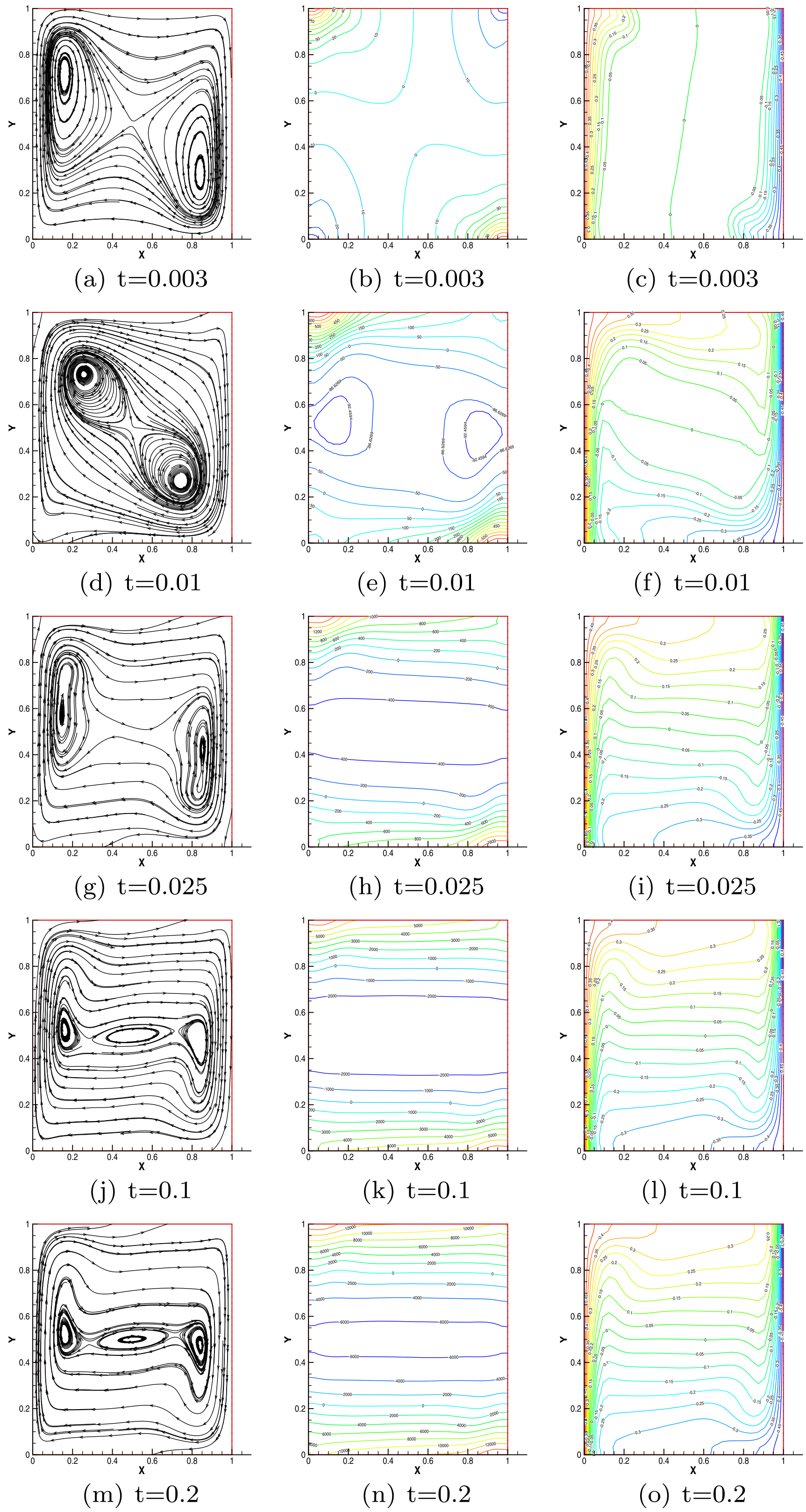


Figure 10 Numerical results of thermal driven cavity flow with $\kappa = 1 \times 10^6$ at different times, left panels the numerical streamlines, middle panels the numerical pressures, and right panels the numerical temperatures.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in this article. They read and approved the final manuscript.

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Acknowledgements
The authors would like to thank the anonymous referees for their valuable suggestions and comments, which helped to improve the quality of the paper. This work of the authors was supported in part by Chinese NSF (Grant Nos. 11301156 and 11401177).

Received: 4 March 2015 Accepted: 19 August 2015 Published online: 01 September 2015

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