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Orthogonality in smooth countably normed spaces

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Abstract

We generalize the concepts of normalized duality mapping, J-orthogonality and Birkhoff orthogonality from normed spaces to smooth countably normed spaces. We give some basic properties of J-orthogonality in smooth countably normed spaces and show a relation between J-orthogonality and metric projection on smooth uniformly convex complete countably normed spaces. Moreover, we define the J-dual cone and J-orthogonal complement on a nonempty subset S of a smooth countably normed space and prove some basic results about the J-dual cone and the J-orthogonal complement of S.

MSC: 46A04

Keywords: Countably normed space; Normalized duality mapping; J-orthogonality; Uniformly convex countably normed space; Projection theorem in a countably normed space; Metric projection; Birkhoff orthogonality; J-dual cone; J-orthogonal complement

1 Introduction

The concept of duality mapping was introduced by Beurling and Livingston [1] in a geometric form. A slightly extended version of the concept was proposed by Asplund [2], who showed how the duality mappings can be characterized via the subdifferentials of convex functions. It is well known that the geometric properties of a Banach space E correspond to the analytic properties of the duality mapping, and it is recognized that if E is smooth, then the duality mapping is single-valued. Park and Rhee [3] defined J-orthogonality in a smooth Banach space using the normalized duality mapping. In this paper, we define the normalized duality mapping on smooth countably normed spaces, generalize the concepts of J-orthogonality and Birkhoff orthogonality in smooth countably normed spaces, and give some basic properties of J-orthogonality in these spaces. Faried and El-Sharkawy [4] defined real uniformly convex complete countably normed spaces and proved that the metric projection on a nonempty convex and closed proper subset of these spaces is well defined. In this paper, we give a relation between metric projection and J-orthogonality and show fundamental links between metric projection and normalized duality mapping in smooth uniformly convex complete countably normed spaces.

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2 Preliminaries

Definition 2.1 ([5, 6]) A normed linear space E is said to be:

(1) *Strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$;

(2) *Uniformly convex* if for any $\varepsilon \in (0, 2]$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in E$ with $\|x\| = 1$, $\|y\| = 1$, and $\|x - y\| \geq \varepsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$;

(3) *Smooth* if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in S(E)$, where $S(E)$ is the unit sphere of E;

(4) *Uniformly smooth* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = 1$ and $\|y\| \leq \delta$, we have $\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|$.

Definition 2.2 (Metric projection [6]) Let E be a real uniformly convex Banach space, and let K be a nonempty proper subset of E. The operator $P_K : E \to K$ is called a *metric projection operator* if it assigns to each $x \in E$ its nearest point $\bar{x} \in K$, that is, the solution of the minimization problem

$$P_K x = \bar{x} : \|x - \bar{x}\| = \inf_{y \in K} \|x - y\|.$$

Definition 2.3 (The normalized duality mapping [7, 8]) Let E be a real Banach space with norm $\|\cdot\|$, and let E^* be the dual space of E, and let $\langle \cdot, \cdot \rangle$ be the duality pairing. *The normalized duality mapping* J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The Hahn–Banach theorem guarantees that $Jx \neq \emptyset$ for every $x \in E$. It is well known that if E is a smooth Banach space, then the normalized duality mapping is single-valued. In [8], we got the following example of the normalized duality mapping J in the uniformly convex and uniformly smooth Banach space ℓ^p with $p \in (1, \infty)$: $Jx := \|x\|_{\ell^p}^{2-p}\{x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \dots\} \in \ell^q = \ell^{p*}$ for $x = \{x_1, x_2, \dots\} \in \ell^p$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 2.4 ([9]) *Let E be a smooth Banach space, let E^* be the dual space of E, and let J be the normalized duality mapping from E to 2^{E^*}. Then J is a continuous operator in E, and $J(\beta x) = \beta J(x)$ for all $\beta \in \mathbb{R}$.*

Definition 2.5 (Lyapunov functional [7, 8]) Let E be a smooth Banach space, and let E^* be the dual space of E. The *Lyapunov functional* $\varphi : E \times E \to \mathbb{R}$ is defined by

$$\varphi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all $x, y \in E$, where J is the normalized duality mapping from E to 2^{E^*}.

Definition 2.6 (Compatible norms [10, 11]) Two norms in a linear space E are said to be *compatible* if every Cauchy sequence $\{x_n\}$ in E with respect to both norms that converges to a limit $x \in E$ with respect to one of them also converges to the same limit x with respect to the other norm.

Definition 2.7 (Countably normed space [10, 11]) A linear space E equipped with a countable family of pairwise compatible norms $\{\|\cdot\|_n, n \in \mathbb{N}\}$ is said to be a *countably normed space*. An example of a countably normed space is the space $\ell^{p+0} := \bigcap_n \ell^{p_n}$ $(1 < p < \infty)$ for any choice of a decreasing sequence p_n converging to p.

Remark 2.8 ([11]) For a countably normed space $(E, \{\|\cdot\|_n, n \in \mathbb{N}\})$, let E_n be the completion of E with respect to the norm $\|\cdot\|_n$. We may assume that $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \|\cdot\|_3 \leq \cdots$ in any countably normed space; we also have $E \subset \cdots \subset E_{n+1} \subset E_n \subset \cdots \subset E_1$.

Proposition 2.9 ([10]) *Let $(E, \{\|\cdot\|_n, n \in \mathbb{N}\})$ be a countably normed space. Then E is complete if and only if $E = \bigcap_{n\in\mathbb{N}} E_n$. Each Banach space E_n has a dual E_n^*, which is a Banach space, and the dual of the countably normed space E is given by $E^* = \bigcup_{n\in\mathbb{N}} E_n^*$. We have the following inclusions*:

$$E_1^* \subset \cdots \subset E_n^* \subset E_{n+1}^* \subset \cdots \subset E^*.$$

Moreover, for $f \in E_n^*$, we have $\|f\|_n \geq \|f\|_{n+1}$ for all $n \in \mathbb{N}$.

Definition 2.10 (Uniformly convex countably normed space [4]) A countably normed space $(E, \{\|\cdot\|_n, n \in \mathbb{N}\})$ is said to be *uniformly convex* if $(E_n, \|\cdot\|_n)$ is uniformly convex for all $n \in \mathbb{N}$.

Theorem 2.11 ([4]) *Let $(E, \{\|\cdot\|_n, n \in \mathbb{N}\})$ be a real uniformly convex complete countably normed space, and let K be a nonempty convex proper subset of E such that K is closed in each E_n. Then there exists a unique $\bar{x} \in K$ such that $\|x - \bar{x}\|_n = \inf_{y\in K} \|x - y\|_n$ for all $n \in \mathbb{N}$, and the metric projection $P : E \to K$ is defined by $P(x) = \bar{x}$.*

Definition 2.12 (J-orthogonality [3]) Let E be a smooth Banach space. Two elements $x, y \in E$ are said to be *J-orthogonal*, written "x is *J-orthogonal* to y" or $x \perp^J y$, if $\langle y, Jx\rangle = 0$.

Definition 2.13 (Gauge function [8]) A *gauge function* is a continuous strictly increasing function $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\vartheta(0) = 0$ and $\lim_{t\to\infty} \vartheta(t) = \infty$.

3 Main results

Now we introduce the concept of the normalized duality mapping in smooth countably normed (SCN) spaces.

Definition 3.1 (The normalized duality mapping in SCN spaces) Let $(E, \{\|\cdot\|_n, n \in \mathbb{N}\})$ be a smooth countably normed space such that E_n is the completion of E in $\|\cdot\|_n$ and $(E_n, \|\cdot\|_n)$ is a smooth Banach space for all $n \in \mathbb{N}$, so that there exists a single-valued normalized duality mapping $J_n : E_n \to E_n^*$ with respect to $\|\cdot\|_n$ for all $n \in \mathbb{N}$. Without being confused, we understand that $\|J_n x\|_n$ is the E_n^*-norm and $\|x\|_n$ is the E_n-norm, for all $n \in \mathbb{N}$.

We define the following multivalued mapping $J : E \to 2^{E^*}$ to be the *normalized duality mapping* of a smooth countably normed space: $J(x) = \{J_n x\}_{n=1}^\infty \subseteq E^* = \bigcup_{n\in\mathbb{N}} E_n^*$, $\|J_n x\|_n = \|x\|_n$, $\langle J_n x, x\rangle = \|x\|_n^2$ for $n \in \mathbb{N}$.

Remark 3.2 Let $(E, \{\|\cdot\|_n, n \in \mathbb{N}\})$ be a smooth countably normed space. The sequence of norms is increasing in E, and from the definition of normalized duality mappings J_n for each E_n with respect to $\|\cdot\|_n$ we have

$$\big(\|x\|_1 = \|J_1 x\|_1\big) \leq \big(\|x\|_2 = \|J_2 x\|_2\big) \leq \cdots \leq \big(\|x\|_n = \|J_n x\|_n\big) \leq \cdots,$$

and thus $\langle J_1x,x\rangle \leq \langle J_2x,x\rangle \leq \cdots \leq \langle J_nx,x\rangle \leq \dots$, and using the properties of countably normed spaces, we have $\|J_ix\|_n \geq \|J_ix\|_{n+1}$ for all i and n.

Remark 3.3 The multivalued normalized duality mapping of a smooth countably normed space cannot be a single-valued mapping, unlike the case of a smooth Banach space. Indeed, if it were a single-valued mapping, then it would be the same single-valued normalized duality mapping for each E_n with respect to $\|\cdot\|_n$, which would imply that $\langle Jx,x\rangle = \|x\|_n^2$ for all n. Then we would get $\|x\|_1 = \|x\|_2 = \cdots = \|x\|_n = \cdots$, which would mean that we are back to a normed vector space, and this ruins the construction of the countably normed space.

Proposition 3.4 *If $(E,\{\|\cdot\|_n, n\in\mathbb{N}\})$ is a smooth countably normed space, then $J_m|_{E_n}$ is the single-valued normalized duality mapping of E_n with respect to $\|\cdot\|_n$ for all $m = 1,2,\dots,n-1$ and $n\geq 2$.*

Proof Let J_{n-1} be the normalized duality mapping of E_{n-1} with respect to $\|\cdot\|_{n-1}$. We have $J_{n-1}: E_{n-1}\to E_n^*$, $E_{n-1}^*\subseteq E_n^*$, $E_n\subseteq E_{n-1}$, so $J_{n-1}|_{E_n}: E_n\to E_n^*$ and $\|J_{n-1}|_{E_n}x\|_n = \|x\|_{n-1}$, $\langle J_{n-1}|_{E_n}x,x\rangle = \|x\|_{n-1}^2$ for all $x\in E_n\subseteq E_{n-1}$. So $J_{n-1}|_{E_n}$ is the single-valued normalized duality mapping of E_n with respect to $\|\cdot\|_{n-1}$. The same holds for all $m = 1,2,\dots,n-1$, and hence $J_m|_{E_n}$ is the single-valued normalized duality mapping of E_n with respect to $\|\cdot\|_n$ for all $n\geq 2$. □

Corollary 3.5 *If $(E,\{\|\cdot\|_n, n\in\mathbb{N}\})$ is a smooth countably normed space, then E_n is a smooth Banach space with respect to $\|\cdot\|_m$, $m = 1,2,\dots,n-1$, $n\geq 2$.*

Proof Since $J_m|_{E_n}$ is the single-valued normalized duality mapping of E_n with respect to $\|\cdot\|_n$ for all $m = 1,2,\dots,n-1$, then E_n is a smooth Banach space with respect to $\|\cdot\|_m$ for all $n\geq 2$. □

Proposition 3.6 *Let E be a smooth countably normed space, let E^* be its dual space, and let J_n be the normalized duality mapping of E_n with respect to $\|\cdot\|_n$ relative to the gauge function ϑ_n, where $\vartheta_n(\|x\|_n) = \|x\|_n = \|J_nx\|_n$. Define $\psi_n(r) = \int_0^r \vartheta_n(\sigma)\,d\sigma$. Then $\psi_n(\|y\|_n) - \psi_n(\|x\|_n)\geq \langle J_nx, y-x\rangle$ for all $y\in E$ and $n\in\mathbb{N}$.*

Proof We have

$$\psi_n\big(\|y\|_n\big) - \psi_n\big(\|x\|_n\big) = \int_{\|x\|_n}^{\|y\|_n} \vartheta_n(t)\,dt \geq \vartheta_n\big(\|x\|_n\big)\big(\|y\|_n - \|x\|_n\big), \quad \forall n,$$

that is, $\psi_n(\|y\|_n) - \psi_n(\|x\|_n) = \vartheta_n(\|x\|_n)\|y\|_n - \langle J_nx,x\rangle \geq \langle J_nx, y-x\rangle$ for all $y\in E$ and $n\in\mathbb{N}$. □

Proposition 3.7 *Let $(E,\{\|\cdot\|_n, n\in\mathbb{N}\})$ be a real smooth uniformly convex complete countably normed space, and let K be a nonempty proper convex subset of E such that K is closed in each E_n. Then $\bar{x} = P_K(x)$ is the metric projection of an arbitrary element $x\in E$ if and only if $\langle J(x-\bar{x}), \bar{x}-y\rangle \geq 0$ for all $y\in K$, where J is the normalized duality mapping on E.*

Proof "⇒" By the definition of the metric projection and the convexity of K we have

$$\|x-\bar{x}\|_n \leq \big\|x - \big(\mu y + (1-\mu)\bar{x}\big)\big\|_n, \quad \forall y\in K, \mu\in[0,1], \forall n. \tag{*}$$

Consider $\psi_n(r) = \int_0^r \vartheta_n(\sigma)\,d\sigma$. If J_n is the normalized duality mapping relative to the gauge function ϑ_n with respect to $\|\cdot\|_n$, then (*) is equivalent to

$$\psi_n\big(\|x-\bar{x}\|_n\big) \leq \psi_n\big(\big\|x-\big[\mu y+(1-\mu)\bar{x}\big]\big\|_n\big). \tag{**}$$

By Proposition 3.6 and (**) we get

$$0 \geq \psi_n\big(\|x-\bar{x}\|_n\big) - \psi_n\big(\big\|x-\big(\mu y+(1-\mu)\bar{x}\big)\big\|_n\big)\big) \geq \big\langle J_n\big(x-\bar{x}-\mu(y-\bar{x})\big), \mu(y-\bar{x})\big\rangle.$$

As μ tends to 0, we get $\langle J_n(x-\bar{x}), y-\bar{x}\rangle \leq 0$ for all $y \in K$ and n, that is, $\langle J_n(x-\bar{x}), \bar{x}-y\rangle \geq 0$ for all $y \in K$ and n.

"$\Leftarrow$" If $\langle J_n(x-\bar{x}), \bar{x}-y\rangle \geq 0$ for all $y \in K$ and n, then using Proposition 3.6, we get

$$\psi_n\big(\|x-y\|_n\big) - \psi_n\big(\|x-\bar{x}\|_n\big) \geq \big\langle J_n(x-\bar{x}), \bar{x}-y\big\rangle \geq 0.$$

Thus $\|x-\bar{x}\|_n \leq \|x-y\|_n$ for all $y \in K$ and n, and so $\bar{x} = P_K(x)$. □

Theorem 3.8 *Let $(E, \{\|\cdot\|_n, n \in \mathbb{N}\})$ be a real smooth uniformly convex complete countably normed space, and let K be a nonempty proper convex subset of E such that K is closed in each E_n.*

Then $\bar{x} = P_K(x)$ is the metric projection of an arbitrary element $x \in E$ if and only if $\langle J_n(x-\bar{x}), x-y\rangle \geq \|x-\bar{x}\|_n^2$ for all $y \in K$ and n.

Proof "$\Rightarrow$" By Proposition 3.6 we have $\langle J_n(x-\bar{x}), \bar{x}-y\rangle \geq 0$ for all $y \in K$ and n. Besides,

$$\begin{aligned}\big\langle J_n(x-\bar{x}), \bar{x}-y\big\rangle &= J_n(x-\bar{x})(\bar{x}-y) \\ &= J_n(x-\bar{x})(\bar{x}-x) + J_n(x-\bar{x})(x-y) \\ &= -\|x-\bar{x}\|_n^2 + J_n(x-\bar{x})(x-y),\end{aligned}$$

and therefore $\langle J_n(x-\bar{x}), x-y\rangle \geq \|x-\bar{x}\|_n^2$ for all $y \in K$ and n.

"$\Leftarrow$" If $\|x-\bar{x}\|_n = 0$, then we are done. So, let us assume that $\|x-\bar{x}\|_n \neq 0$. Then

$$\begin{aligned}\|x-\bar{x}\|_n &\leq \frac{1}{\|x-\bar{x}\|_n}\big\langle J_n(x-\bar{x}), x-y\big\rangle \\ &\leq \frac{1}{\|x-\bar{x}\|_n}\big\|J_n(x-\bar{x})\big\|_n\|x-y\|_n \\ &= \|x-y\|_n, \quad \forall y \in K, \forall n,\end{aligned}$$

that is, $\bar{x} = P_K(x)$. □

Definition 3.9 (J-orthogonality in smooth countably normed spaces) Let $(E, \{\|\cdot\|_n, n \in \mathbb{N}\})$ be a smooth countably normed space. We say that an element $x \in E$ is *J-orthogonal* to an element $y \in E$ and write $x \perp^J y$ if $\langle y, J_n x\rangle = 0$ for all n, that is, $\langle y, Jx\rangle = 0$, where J is the normalized duality mapping of E.

Definition 3.10 Let $(E,\{\|\cdot\|_n, n\in\mathbb{N}\})$ be a smooth countably normed space, and let $x_1,x_2,\dots,x_n\in E\setminus\{0\}$. Then the set $\{x_1,x_2,\dots,x_n\}$ is called a *J-orthogonal set* if $x_i\perp x_j$ for all $i,j\in\{1,2,\dots,n\}$ with $i\neq j$.

Definition 3.11 Let $(E,\{\|\cdot\|_n, n\in\mathbb{N}\})$ be a smooth countably normed space. We say that an element $x\in E$ is *orthogonal* to an element $y\in E$ in the *Birkhoff sense* if $\|x+\alpha y\|_i^2\geq\|x\|_i^2$ for all $i=1,2,\dots,n,\dots$ and $\alpha\in\mathbb{R}$.

Proposition 3.12 *Let* $(E,\{\|\cdot\|_n, n\in\mathbb{N}\})$ *be a smooth countably normed space, and let* $x_1,x_2,\dots,x_n\in E\setminus\{0\}$. *Then*:

(1) *If* $\{x_1,x_2,\dots,x_n\}$ *is a J-orthogonal set, then* $x_1,x_2,\dots,x_n$ *are linearly independent*;

(2) *Let* $x,y\in E$. *Then* $x\perp^J y$ *if and only if* $x\perp y$ *in the Birkhoff sense.*

Proof (1) Let $\alpha_1x_1+\alpha_2x_2+\cdots+\alpha_nx_n=0$ for some scalars $\alpha_1,\alpha_2,\dots,\alpha_n\in\mathbb{R}$.

For all $m\in\{1,\dots,n\}$ and i, we have:

$$\begin{aligned}\langle\alpha_1x_1+\cdots+\alpha_nx_n,J_ix_m\rangle &= \alpha_1\langle x_1,J_ix_m\rangle+\cdots+\alpha_n\langle x_n,J_ix_m\rangle\\ &=\alpha_m\|x_m\|_i^2\\ &=0,\end{aligned}$$

and so $\alpha_m=0$ for all m. Thus $x_1,x_2,\dots,x_n$ are linearly independent.

(2) If $x\perp^J y$, then $\langle y,J_ix\rangle=0$ for all i. Besides, using the Lyapunov functional, we have

$$\begin{aligned}\varphi_i(x+\alpha y,x) &= \|x+\alpha y\|_i^2-2\langle x+\alpha y,J_ix\rangle+\|x\|_i^2,\quad\forall i\\ &=\|x+\alpha y\|_i^2-\|x\|_i^2-2\alpha\langle y,J_ix\rangle\\ &\geq 0,\quad\forall i,\forall\alpha\in\mathbb{R}.\end{aligned}$$

Thus $\|x+\alpha y\|_i^2\geq\|x\|_i^2$ for all i and $\alpha\in\mathbb{R}$. Hence $x\perp y$ in the Birkhoff sense.

On the other hand, let $x\perp y$ in the Birkhoff sense, that is, $\|x+\alpha y\|_i^2\geq\|x\|_i^2$ for all i and $\alpha\in\mathbb{R}$. If $\langle y,J_ix\rangle\neq 0$ for some i, then by taking $\alpha_0=\frac{\|x+\alpha y\|_i^2-\|x\|_i^2}{\langle y,J_ix\rangle}$ we get that the Lyapunov functional $\varphi_i(x+\alpha_0y,x)<0$. This contradicts that $\varphi_i(x,y)>0$ for all i. □

Proposition 3.13 *Let* $\{x_1,x_2,\dots,x_n\}$ *be a J-orthogonal set in a smooth countably normed space E with dual space* E^*. *The set* $\{J_ix_1,\dots,J_ix_n\}$ *is linearly independent in the dual space* E^* *for all i.*

Proof If $\alpha_1J_ix_1+\cdots+\alpha_nJ_ix_n=0$ for some scalars $\alpha_1,\dots,\alpha_n\in\mathbb{R}$, then for each $m\in\{1,2,\dots,n\}$, we get $\langle x_m,\alpha_1J_ix_1+\cdots+\alpha_nJ_ix_n\rangle=\alpha_m\|x\|_i^2=0$ for all i. Hence $\alpha_m=0$ for all m. Thus, for all i, the set $\{J_ix_1,\dots,J_ix_n\}$ is linearly independent in the dual space E^*. □

The following theorem gives a relation between metric projection and orthogonality in real uniformly convex complete countably normed spaces.

Theorem 3.14 *Let* $(E,\{\|\cdot\|_n, n\in\mathbb{N}\})$ *be a real smooth uniformly convex complete countably normed space, and let M be a nonempty proper subspace of E such that M is closed in* *each E_i. Then*

$$\forall x \in E \setminus M, \exists! \bar{x} \in M: \quad \|x - \bar{x}\|_i = \inf_{y \in M} \|x - y\|_i$$

for all i if and only if $x - \bar{x} \perp^J M$.

Proof Assume that

$$\forall x \in E \setminus M, \exists! \bar{x} \in M: \quad \|x - \bar{x}\|_i = \inf_{y \in M} \|x - y\|_i, \quad \forall i.$$

If $z \in M$, then $\bar{x} - \alpha z \in M$ for all $\alpha \in \mathbb{R}$, and $\|x - \bar{x}\|_i \le \|x - (\bar{x} - \alpha z)\|_i = \|(x - \bar{x}) + \alpha z\|_i$ for all i. Therefore $x - \bar{x}$ is orthogonal to M in the Birkhoff sense. Consequently, $x - \bar{x} \perp^J M$.

On the other hand, if $x - \bar{x} \perp^J M$, then $x - \bar{x}$ is orthogonal to M in the Birkhoff sense, that is, $\|x - \bar{x}\|_i \le \|x - \bar{x} + \alpha y\|_i$ for all $\alpha \in \mathbb{R}$, $y \in M$, and i.

Since $\bar{x} - y \in M$, for all $y \in M$ and i, we get

$$\|x - \bar{x}\|_i \le \bigl\|x - \bar{x} + \alpha(\bar{x} - y)\bigr\|_i$$

for all $\alpha \in \mathbb{R}$.

Taking $\alpha = 1$, we get $\|x - \bar{x}\|_i \le \|x - y\|_i$ for all $y \in M$ and i. Thus $\|x - \bar{x}\|_i = \inf_{y \in M} \|x - y\|_i$ for all i. □

Example 3.15 $\ell_{2+0} := \bigcap_{n \in \mathbb{N}} \ell_{2+\frac{1}{n}}$ is a uniformly convex uniformly smooth complete countably normed space with the norms

$$\|\cdot\|_3 \le \|\cdot\|_{2.5} \le \cdots \le \|\cdot\|_{2+\frac{1}{n}} \le \cdots$$

for each $x = \{x_i\} \in \ell_{2+0}$, and

$$J_n(x) = \|x\|_{2+\frac{1}{n}}^{-\frac{1}{n}} \bigl\{x_i |x_i|^{\frac{1}{n}}\bigr\} \in \ell_{\frac{2n+1}{n+1}}, \quad \forall n.$$

Consider the closed subspace M of ℓ_{2+0} generated by $\{1, 0, 0, 0, \ldots\}$. Using the previous theorem, we get

$$\begin{aligned}
&P_M(x) = \bar{x} = \{\bar{x}_1, 0, 0, \ldots\} \\
&\quad\Leftrightarrow \quad \bigl\langle \{t, 0, 0, \ldots\}, J_n(x - \bar{x}) \bigr\rangle = \{0, 0, \ldots\}, \quad \forall t \in \mathbb{R}, \forall n \\
&\quad\Leftrightarrow \quad \bigl\langle \{t, 0, 0, \ldots\}, J_n\{x_1 - \bar{x}_1, x_2, x_3, \ldots, x_n, \ldots\} \bigr\rangle = \{0, 0, \ldots\} \\
&\quad\Leftrightarrow \quad \Bigl\langle \{t, 0, 0, \ldots\}, \|x - \bar{x}\|_{2+\frac{1}{n}}^{-\frac{1}{n}} \Bigl\{ |x_1 - \bar{x}_1|^{-\frac{1}{n}} (x_1 - \bar{x}_1), \ldots, x_i |x_i|^{\frac{1}{n}}, \ldots \Bigr\} \Bigr\rangle = \{0, 0, \ldots\} \\
&\quad\Leftrightarrow \quad \|x - \bar{x}\|_{2+\frac{1}{n}}^{-\frac{1}{n}} |x_1 - \bar{x}_1|^{-\frac{1}{n}} (x_1 - \bar{x}_1) t = 0, \quad \forall t \in \mathbb{R}, \forall n \\
&\quad\Leftrightarrow \quad \bar{x}_1 = x_1, \qquad P_M(x) = \bar{x} = \{x_1, 0, 0, \ldots\}.
\end{aligned}$$

Definition 3.16 The *J-dual cone* of a nonempty subset S of a smooth countably normed space $(E, \{\|\cdot\|_n, n \in \mathbb{N}\})$ is the set

$$S_J^o = \{x \in E : \langle y, J_i x\rangle \leq 0, \forall y \in S, \forall i\}.$$

In addition, the *J-orthogonal complement* of S is the set

$$S_J^\perp = S_J^o \cap (-S)_J^o = \{x \in E : \langle y, J_i x\rangle = 0, \forall y \in S, \forall i\}.$$

Theorem 3.17 *Let S be a nonempty subset of a smooth countably normed space $(E, \{\|\cdot\|_n, n \in \mathbb{N}\})$. Then*:

(1) S_J^o *and* $S_J^\perp$ *are closed cones*;

(2) $S_J^o = (\bar{S})_J^o$ *and* $S_J^\perp = (\bar{S})_J^\perp$;

(3) $S_J^o = [\operatorname{conv}(S)]_J^o = \overline{[\operatorname{conv}(S)]}_J^o$ *and* $S_J^\perp = [\operatorname{span}(S)]_J^\perp = \overline{[\operatorname{span}(S)]}_J^\perp$, *where* $\operatorname{conv}(S)$ *is the convex hull of* S, *and* $\operatorname{span}(S)$ *is the subspace generated by* S;

(4) $\bar{S} \subset (S_J^o)^o$ *and* $\bar{S} \subset (S_J^\perp)^\perp$;

(5) *If C is a cone, then* $(C - y)_J^o = C_J^o \cap y_J^\perp$ *for all* $y \in C$;

(6) *If M is a subspace, then* $M_J^o = M_J^\perp$.

Proof (1) If $x_n \in S_J^o$ and $x_n \to x$, then for all $y \in S$, $\langle y, J_i x\rangle = \lim_{n\to\infty}\langle y, J_i x_n\rangle \leq 0$ $\forall i$ implies that $x \in S_J^o$, and thus S_J^o is closed. If $x \in S_J^o$ and $\alpha \geq 0$, then for all $y \in S$ and i, we get

$$\langle y, J_i(\alpha x)\rangle = \langle y, \alpha J_i x\rangle = \alpha\langle y, J_i x\rangle \leq 0.$$

Hence $\alpha x \in S_J^o$, and thus S_J^o is a cone. Since $S_J^\perp = S_J^o \cap (-S)_J^o$, $S_J^\perp$ is a closed cone.

(2) Since $S \subseteq \bar{S}$, we have $(\bar{S})_J^o \subseteq S_J^o$. If $x \in S_J^o$ and $y \in \bar{S}$, choose $y_n \in S$ such that $y_n \to y$. Then $\langle y, J_i x\rangle = \lim_{n\to\infty}\langle y_n, J_i x\rangle \leq 0$ for all i implies $x \in (\bar{S})_J^o$. Thus $S_J^o = (\bar{S})_J^o$. Moreover, $S_J^\perp = (\bar{S})_J^\perp$.

(3) Since $S \subseteq \operatorname{conv}(S)$, $[\operatorname{conv}(S)]_J^o \subseteq S_J^o$. Let $x \in S_J^o$ and $y \in \operatorname{conv}(S)$. By the definition of $\operatorname{conv}(S)$, $y = \sum_{m=1}^n \rho_m y_m$ for some $y_i \in S$ and $\rho_i \geq 0$ with $\sum_{m=1}^n \rho_m = 1$, $i = 1, 2, \ldots, n$.

Then $\langle y, J_i x\rangle = \sum_{m=1}^n \rho_m\langle y_m, J_i x\rangle \leq 0$ for all i implies $x \in [\operatorname{conv}(S)]_J^o$, so $S_J^o \subseteq [\operatorname{conv}(S)]_J^o$. Thus $S_J^o = [\operatorname{conv}(S)]_J^o$. Moreover, $S_J^\perp = [\operatorname{span}(S)]_J^\perp = \overline{[\operatorname{span}(S)]}_J^\perp$.

(4) If $x \in S$, then for all $y \in S_J^o$, $\langle x, J_i y\rangle \leq 0$ for all i. Hence $x \in (S_J^o)^o$. Thus $S \subseteq (S_J^o)^o$. Since $(S_J^o)^o$ is closed, $\bar{S} \subseteq (S_J^o)^o$.

(5) Now $x \in (C - y)_J^o$ if and only if $\langle c - y, J_i x\rangle \leq 0$ for all i and $c \in C$. Let $x \in (C - y)_J^o$. Taking $c = 0$ and $c = 2y$, we have $\langle y, J_i x\rangle = 0$, and $\langle c, J_i x\rangle \leq 0$ for all i and $c \in C$. Thus $x \in C_J^o \cap y_J^\perp$. Moreover, if $x \in C_J^o \cap y_J^\perp$, then $\langle c, J_i x\rangle \leq 0$ and $\langle y, J_i x\rangle = 0$ for all i and $c \in C$. Thus $x \in (C - y)_J^o$. Therefore $(C - y)_J^o = C_J^o \cap y_J^\perp$ for all $y \in C$.

(6) If M is a subspace of E, then $-M = M$ implies $M_J^\perp = M_J^o \cap (-M)_J^o = M_J^o$. □

4 Conclusion

In this paper, we defined J-orthogonality and Birkhoff orthogonality in smooth countably normed spaces and showed that these two types of orthogonality coincide in these spaces. Besides, we proved some basic properties of J-orthogonality in smooth countably normed spaces and gave a relation between J-orthogonality and metric projection on smooth uniformly convex complete countably normed spaces. Moreover, we gave fundamental links between J-orthogonality and metric projection in smooth uniformly convex complete countably normed spaces. In addition, we defined the J-dual cone and J-orthogonal complement on a nonempty subset $\boldsymbol{S}$ of a smooth countably normed space and proved some basic results about the J-dual cone and J-orthogonal complement of $\boldsymbol{S}$.

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Abbreviations
SCN, smooth countably normed (space).

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