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Extremal functions for Trudinger–Moser inequalities with nonnegative weights

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Abstract

Using blow-up analysis, the author proves the existence of extremal functions for Trudinger–Moser inequalities with nonnegative weights on bounded Euclidean domains or compact Riemannian surfaces. This extends recent results of Yang (J. Differ. Equ. 258:3161–3193, 2015) and Yang–Zhu (Proc. Am. Math. Soc. 145: 3953–3959, 2017).

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1 Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^2$, $W^{1,p}(\Omega)$ be the usual Sobolev space and $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. For $1 \le p < 2$, the classical Sobolev theorem says that

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for any } 1 < q \le 2p/(2-p).$$

As the limit case of the Sobolev inequality, the famous Trudinger–Moser inequality [3, 4] states

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \le 1} \int_\Omega e^{4\pi u^2} \, dx < \infty. \tag{1.1}$$

This inequality is sharp in the sense that, for any $p > 4\pi$, there exists a sequence $\{u_j\} \subset W_0^{1,2}(\Omega)$ with $\|\nabla u_j\|_2 = 1$ such that $\int_\Omega e^{pu_j^2} \, dx \to \infty$ as $j \to \infty$. Furthermore, let $\{u_k\}$ be a sequence of function in $W_0^{1,2}(\Omega)$ with $\|\nabla u_k\|_2 = 1$ such that $u_k \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$. Lions [5] proved that, for any $p < 1/(1 - \|\nabla u\|_2^2)$, we have

$$\limsup_{k \to \infty} \int_\Omega e^{4\pi p u_k^2} \, dx < \infty. \tag{1.2}$$

If $u \not\equiv 0$, the inequality (1.2) gives more information than the Trudinger–Moser inequality (1.1). If $u \equiv 0$, (1.2) is a consequence of (1.1). Motivated by this, Adimurthi and Druet [6]

proved that, for any α, $0 \leq \alpha < \lambda_1(\Omega)$,

$$\sup_{u \in W_0^{1,2}(\Omega), \|\nabla u\|_2 \leq 1} \int_\Omega e^{4\pi u^2 (1 + \alpha \|u\|_2^2)} \, dx < \infty, \tag{1.3}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplace operator with respect to Dirichlet boundary condition. If $\alpha \geq \lambda_1(\Omega)$, then the supremum in (1.3) is infinity. The inequality (1.3) provides valuable supplementary information on (1.2). Note that if $\alpha = 0$, (1.3) becomes the classical Trudinger–Moser inequality. Adimurthi and Druet's result was extended by Yang to high dimensions [7] and compact Riemannian surfaces [8], and by Tintarev to a stronger version [9].

Denote

$$\|u\|_{1,\alpha} = \left(\int_\Omega |\nabla u|^2 \, dx - \alpha \int_\Omega u^2 \, dx \right)^{1/2} \tag{1.4}$$

for any $u \in W_0^{1,2}(\Omega)$ with $\int_\Omega |\nabla u|^2 \, dx - \alpha \int_\Omega u^2 \, dx \geq 0$. In [1], Yang proved that, for any α, $0 \leq \alpha < \lambda_1(\Omega)$, we have

$$\sup_{u \in W_0^{1,2}(\Omega), \|u\|_{1,\alpha} \leq 1} \int_\Omega e^{4\pi u^2} \, dx < \infty \tag{1.5}$$

and the supremum can be attained by some function $u_0 \in W_0^{1,2}(\Omega) \cap C^1(\overline{\Omega})$ with $\|u_0\|_{1,\alpha} = 1$. Let $\lambda_1(\Omega) < \lambda_2(\Omega) < \cdots$ be all distinct eigenvalues of the Laplace operator with respect to Dirichlet boundary condition and $E_{\lambda_j(\Omega)}$ be the eigenfunction space associated to $\lambda_j(\Omega)$. Noting that $W_0^{1,2}(\Omega)$ is a Hilbert space, for any positive integer l, we have

$$W_0^{1,2}(\Omega) = E_l \oplus E_l^\perp,$$

where

$$E_l = E_{\lambda_1(\Omega)} \oplus E_{\lambda_2(\Omega)} \oplus \cdots \oplus E_{\lambda_l(\Omega)} \tag{1.6}$$

and

$$E_l^\perp = \left\{ u \in W_0^{1,2}(\Omega) : \int_\Omega uv \, dx = 0, \forall v \in E_l \right\}. \tag{1.7}$$

It was also proved by Yang [1] that, for any α, $0 \leq \alpha < \lambda_{l+1}(\Omega)$, we have

$$\sup_{u \in E_l^\perp, \|u\|_{1,\alpha} \leq 1} \int_\Omega e^{4\pi u^2} \, dx < \infty \tag{1.8}$$

and the supremum can be attained by some $u_0 \in E_l^\perp \cap C^1(\overline{\Omega})$ with $\|u_0\|_{1,\alpha} = 1$. The analogs of (1.5) and (1.8) still hold on compact Riemannian surfaces.

Our first result is the following.

Theorem 1 *Let Ω be a smooth bounded domain in $\mathbb{R}^2$, $\lambda_1(\Omega)$ be the first eigenvalue of the Laplace operator with Dirichlet boundary condition, and h be in $C^0(\overline{\Omega})$ with $h \geq 0$ and $h \not\equiv 0$. Then we have, for any $0 \leq \alpha < \lambda_1(\Omega)$, the supremum*

$$\sup_{u \in W_0^{1,2}(\Omega), \|u\|_{1,\alpha} \leq 1} \int_\Omega h e^{4\pi u^2} \, dx$$

can be attained by some $u_0 \in W_0^{1,2}(\Omega) \cap C^1(\overline{\Omega})$ satisfying $\|u_0\|_{1,\alpha} = 1$, where $\|\cdot\|_{1,\alpha}$ is defined as in (1.4).

When the high order eigenvalues are involved, we have a similar result.

Theorem 2 *Let Ω be a smooth bounded domain in $\mathbb{R}^2$, $\lambda_{l+1}(\Omega)$ be the $(l+1)$th eigenvalue of the Laplace operator with Dirichlet boundary condition, and h be in $C^0(\overline{\Omega})$ with $h \geq 0$ and $h \not\equiv 0$. Then we see that, for any $0 \leq \alpha < \lambda_{l+1}(\Omega)$, the supremum*

$$\sup_{u \in E_l^\perp, \|u\|_{1,\alpha} \leq 1} \int_\Omega h e^{4\pi u^2} \, dx$$

can be attained by some $u_0 \in E_l^\perp \cap C^1(\overline{\Omega})$ satisfying $\|u_0\|_{1,\alpha} = 1$, where $E_l^\perp$ is defined as in (1.7) and $\|\cdot\|_{1,\alpha}$ defined as in (1.4).

Similar results hold on compact Riemannian surfaces. Denote by (Σ, g) a compact Riemannian surface without boundary, by ∇_g its gradient operator and by Δ_g the Laplace–Beltrami operator, respectively. Let $\lambda_1(\Sigma)$ be the first eigenvalue of Δ_g. Denote

$$\|u\|_{1,\alpha} = \left(\int_\Sigma |\nabla_g u|^2 \, dx - \alpha \int_\Sigma u^2 \, dv_g \right)^{1/2} \tag{1.9}$$

for all $u \in W^{1,2}(\Sigma)$ with $\int_\Sigma |\nabla_g u|^2 \, dx - \alpha \int_\Sigma u^2 \, dv_g \geq 0$. Then we have the following theorem.

Theorem 3 *Let (Σ, g) be a compact Riemannian surface without boundary, h be in $C^0(\Sigma)$ with $h \geq 0$ and $h \not\equiv 0$. Then, for any α, $0 \leq \alpha < \lambda_1(\Sigma)$, the supremum*

$$\sup_{u \in W^{1,2}(\Sigma), \int_\Sigma u \, dv_g = 0, \|u\|_{1,\alpha} \leq 1} \int_\Sigma h e^{4\pi u^2} \, dv_g$$

can be attained by some $u_0 \in W^{1,2}(\Sigma) \cap C^1(\Sigma)$ satisfying $\|u_0\|_{1,\alpha} = 1$ and $\int_\Sigma u_0 \, dv_g = 0$.

Corollary 4 *Let (Σ, g) be a compact Riemannian surface without boundary, h be in $C^0(\Sigma)$ with $h \geq 0$ and $h \not\equiv 0$. For any α, $0 \leq \alpha < \lambda_1(\Sigma)$, $\forall u \in W^{1,2}(\Sigma)$ with $\int_\Sigma u \, dv_g = 0$, define*

$$J(u) = \frac{1}{2} \left(\int_\Sigma |\nabla_g u|^2 \, dv_g - \alpha \int_\Sigma u^2 \, dv_g \right) - 8\pi \log \int_\Sigma h e^u \, dv_g.$$

Then we have the weak Trudinger–Moser inequality $J(u) \geq -C$, where C is a positive constant depending only on (Σ, g) and α.

If h is strictly positive and $J(u)$ has no minimizer on $\mathcal{H} = \{u \in W^{1,2}(\Sigma) : \int_\Sigma u \, dv_g = 0\}$, Yang and Zhu [10] calculated the infimum of $J(u)$ on $\mathcal{H}$ by using the method of blow-up analysis. One may refer to [11] for earlier results on the functional

$$\frac{1}{2} \int_\Sigma |\nabla_g u|^2 \, dv_g + 8\pi \int_\Sigma u \, dv_g - 8\pi \log \int_\Sigma h e^u \, dv_g.$$

Let $\lambda_1(\Sigma) < \lambda_2(\Sigma) < \cdots$ be all distinct eigenvalues of Δ_g and $E_{\lambda_i(\Sigma)}$ be the eigenfunction space associated to $\lambda_i(\Sigma)$. For any positive integer l, denote

$$E_l = E_{\lambda_1(\Sigma)} \oplus E_{\lambda_2(\Sigma)} \oplus \cdots \oplus E_{\lambda_l(\Sigma)}$$

and

$$E_l^\perp = \left\{ u \in W^{1,2}(\Sigma) : \int_\Sigma uv \, dv_g = 0, \forall v \in E_l \right\}.$$

Similar to Theorem 2, we obtain the following.

Theorem 5 *Let (Σ, g) be a compact Riemannian surface without boundary, h be in $C^0(\Sigma)$ with $h \geq 0$ and $h \not\equiv 0$. Then, for any α, $0 \leq \alpha < \lambda_{l+1}(\Sigma)$, the supremum*

$$\sup_{u \in E_l^\perp, \int_\Sigma u \, dv_g = 0, \|u\|_{1,\alpha} \leq 1} \int_\Sigma h e^{4\pi u^2} \, dv_g$$

can be attained by some $u_0 \in E_l^\perp \cap C^1(\Sigma)$ satisfying $\|u_0\|_{1,\alpha} = 1$ and $\int_\Sigma u_0 \, dv_g = 0$.

Existence of extremal functions for Trudinger–Moser inequality can be traced back to Carleson and Chang [12], where the unit ball case was treated. Later contributions in this direction include M. Struwe [13], Flucher [14], Lin [15], Ding–Jost–Li–Wang [11], Adimurthi–Struwe [16], Li [17], Adimurthi–Druet [6], and so on. In our proof, we use the blow-up method. Compared with [1], there are some different key points. First, we derive the different Euler–Lagrange equation on which the analysis is performed. Then we prove that h must be positive at the blow-up point. Hence we use the different scaling when define the maximizing sequences of functions. We also obtain the different upper bound of the subcritical functionals. Finally, when proving the existence of the extremal function, we obtain the different lower bounds for the integrals of test functions constructed in Sects. 2–5. It should be remarked that our analysis on the weight h is essentially different from that of Yang and Zhu [2], where a weak version of Trudinger–Moser inequality was studied.

The rest of the paper is arranged as follows. In Sects. 2 and 3, we prove the main results in the Euclidean case (Theorems 1 and 2). In Sects. 4 and 5, we prove the main results in the Riemannian surface case (Theorems 3 and 5).

2 Proof of Theorem 1
2.1 The subcritical functionals
In this subsection, using the method in the calculus of variations, we prove the existence of maximizers for the subcritical functionals.

Lemma 6 *For any $0 < \epsilon < 4\pi$, any $0 \le \alpha < \lambda_1(\Omega)$, there exists some $u_\epsilon \in W_0^{1,2}(\Omega) \cap C^1(\overline{\Omega})$ with $\|u_\epsilon\|_{1,\alpha} = 1$ such that*

$$\int_\Omega h e^{(4\pi-\epsilon)u_\epsilon^2}\, dx = \sup_{u \in W_0^{1,2}(\Omega), \|u\|_{1,\alpha} \le 1} \int_\Omega h e^{(4\pi-\epsilon)u^2}\, dx, \tag{2.1}$$

where $\|\cdot\|_{1,\alpha}$ is defined as in (1.4).

Proof For $0 < \epsilon < 4\pi$, we choose a function sequence $u_j \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega |\nabla u_j|^2\, dx - \alpha \int_\Omega u_j^2\, dx \le 1$$

and

$$\int_\Omega h e^{(4\pi-\epsilon)u_j^2}\, dx \to \sup_{u \in W_0^{1,2}(\Omega), \|u\|_{1,\alpha} \le 1} \int_\Omega h e^{(4\pi-\epsilon)u^2}\, dx \tag{2.2}$$

as $j \to \infty$. Then there exists some $u_\epsilon \in W_0^{1,2}(\Omega)$ such that up to a subsequence,

$$u_j \rightharpoonup u_\epsilon \quad \text{weakly in } W_0^{1,2}(\Omega),$$

$$u_j \to u_\epsilon \quad \text{strongly in } L^p(\Omega), \forall p \ge 1,$$

$$u_j \to u_\epsilon \quad \text{a.e. in } \Omega.$$

Using a similar argument in the spirit of the one in [1], we find that $h e^{(4\pi-\epsilon)u_j^2}$ is bounded in $L^q(\Omega)$ for some $q > 1$. Then we get $h e^{(4\pi-\epsilon)u_j^2} \to h e^{(4\pi-\epsilon)u_\epsilon^2}$ strongly in $L^1(\Omega)$. This together with (2.2) immediately yields (2.1). We claim that $\|u_\epsilon\|_{1,\alpha} = 1$. Otherwise $\|u_\epsilon\|_{1,\alpha} < 1$. It follows that

$$\sup_{u \in W_0^{1,2}(\Omega), \|u\|_{1,\alpha} \le 1} \int_\Omega h e^{(4\pi-\epsilon)u^2}\, dx \ge \int_\Omega h e^{(4\pi-\epsilon)\frac{u_\epsilon^2}{\|u_\epsilon\|_{1,\alpha}^2}}\, dx$$

$$> \int_\Omega h e^{(4\pi-\epsilon)u_\epsilon^2}\, dx. \tag{2.3}$$

There is a contradiction between in (2.1) and (2.3). Hence $\|u_\epsilon\|_{1,\alpha} = 1$. □

Moreover, the Euler–Lagrange equation for u_ϵ is

$$\begin{cases} -\Delta u_\epsilon - \alpha u_\epsilon = \frac{1}{\lambda_\epsilon} h u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2} & \text{in } \Omega, \\ u_\epsilon > 0 & \text{in } \Omega, \\ \lambda_\epsilon = \int_\Omega h u_\epsilon^2 e^{(4\pi-\epsilon)u_\epsilon^2}\, dx. \end{cases} \tag{2.4}$$

Using elliptic estimates, we get $u_\epsilon \in C^1(\overline{\Omega})$. Let $c_\epsilon = u_\epsilon(x_\epsilon) = \max_\Omega u_\epsilon$. If c_ϵ is bounded, the existence of the extremal function is trivial by standard elliptic estimates. Thus we assume that $c_\epsilon \to \infty$ and $x_\epsilon \to x_0 \in \overline{\Omega}$. A result of Gidas, Ni and Nirenberg on page 223 of [18] implies $x_0 \notin \partial\Omega$.

Using the same argument as the one in step 2 of [1], we get the energy concentration. For the function sequence u_ϵ, we have $u_\epsilon \rightharpoonup 0$ weakly in $W_0^{1,2}(\Omega)$, $u_\epsilon \to 0$ strongly in $L^q(\Omega)$ for any $q > 1$, and $|\nabla u_\epsilon|^2 \, dx \rightharpoonup \delta_{x_0}$ in the sense of measure as $\epsilon \to 0$, where δ_{x_0} denotes the Dirac measure centered at x_0.

Next we prove that h is positive at the blow-up point x_0. This property plays an important part in our analysis.

Lemma 7 *There holds $h(x_0) > 0$.*

Proof We prove it by contradiction. Suppose that $h(x_0) = 0$. Note that up to a sequence

$$\lim_{\epsilon \to 0} \int_\Omega h\big(e^{(4\pi - \epsilon)u_\epsilon^2} - 1\big) \, dx = \sup_{u \in W_0^{1,2}(\Omega), \|u\|_{1,\alpha} \le 1} \int_\Omega h\big(e^{4\pi u^2} - 1\big) \, dx \ge \eta,$$

where η is a positive constant. Let ϵ be sufficiently small such that

$$\int_\Omega h\big(e^{(4\pi - \epsilon)u_\epsilon^2} - 1\big) \, dx > \frac{\eta}{2}. \tag{2.5}$$

Choose $r > 0$ such that $B_r(x_0) \subset \Omega$. Then

$$
\begin{aligned}
&\int_\Omega h\big(e^{(4\pi - \epsilon)u_\epsilon^2} - 1\big) \, dx \\
&= \int_{B_r(x_0)} h\big(e^{(4\pi - \epsilon)u_\epsilon^2} - 1\big) \, dx + \int_{\Omega \setminus B_r(x_0)} h\big(e^{(4\pi - \epsilon)u_\epsilon^2} - 1\big) \, dx \\
&= o_r(1) \int_{B_r(x_0)} \big(e^{(4\pi - \epsilon)u_\epsilon^2} - 1\big) \, dx + \int_{\Omega \setminus B_r(x_0)} h\big(e^{(4\pi - \epsilon)u_\epsilon^2} - 1\big) \, dx,
\end{aligned}
\tag{2.6}
$$

where $o_r(1) \to 0$ as $r \to 0$.

Choose r sufficiently small such that

$$o_r(1) \int_{B_r(x_0)} \big(e^{(4\pi - \epsilon)u_\epsilon^2} - 1\big) \, dx \le o_r(1) \int_\Omega \big(e^{4\pi u_\epsilon^2} - 1\big) \, dx \le \frac{\eta}{4}. \tag{2.7}$$

Here we have used the Trudinger–Moser inequality (1.5).

Applying elliptic estimates to the Euler–Lagrange equation (2.4), we obtain $u_\epsilon \to 0$ in $C_{\mathrm{loc}}^1(\Omega \setminus \{x_0\})$. Hence

$$\int_{\Omega \setminus B_r(x_0)} h\big(e^{(4\pi - \epsilon)u_\epsilon^2} - 1\big) \, dx = o(\epsilon). \tag{2.8}$$

Combining (2.6), (2.7) and (2.8), we find that if ϵ is sufficiently small,

$$\int_\Omega h\big(e^{(4\pi - \epsilon)u_\epsilon^2} - 1\big) \, dx < \frac{\eta}{2}. \tag{2.9}$$

There is a contradiction between (2.5) and (2.9). Hence $h(x_0) > 0$. □

2.2 Blow-up analysis

We shall analyze the behavior of the maximizers by using a blow-up analysis. Let

$$r_\epsilon = \sqrt{\lambda_\epsilon}\big[h(x_0)\big]^{-1/2} c_\epsilon^{-1} e^{-(2\pi - \epsilon/2)c_\epsilon^2}.$$

Using the Hölder inequality and the classical Trudinger–Moser inequality, we have

$$\int_\Omega h u_\epsilon^2 e^{(4\pi - \epsilon)u_\epsilon^2}\, dx \le e^{\delta c_\epsilon^2} \int_\Omega h u_\epsilon^2 e^{(4\pi - \epsilon - \delta)u_\epsilon^2}\, dx \le C e^{\delta c_\epsilon^2},$$

where $0 < \delta < 4\pi$, C depends only on h and δ. Thus we get

$$r_\epsilon^2 \le C\big[h(x_0)\big]^{-1} c_\epsilon^{-2} e^{-(4\pi - \epsilon - \delta)c_\epsilon^2} \to 0$$

as $\epsilon \to 0$.

Set

$$\Omega_\epsilon = \big\{ x \in \mathbb{R}^2 : x_\epsilon + r_\epsilon x \in \Omega \big\}.$$

We define two sequences of functions on Ω_ϵ:

$$\psi_\epsilon(x) = c_\epsilon^{-1} u_\epsilon(x_\epsilon + r_\epsilon x), \qquad \varphi_\epsilon(x) = c_\epsilon\big(u_\epsilon(x_\epsilon + r_\epsilon x) - c_\epsilon\big).$$

They satisfy the following equation:

$$-\Delta \psi_\epsilon = \alpha r_\epsilon^2 \psi_\epsilon + \frac{\psi_\epsilon h e^{(4\pi - \epsilon)(u_\epsilon^2 - c_\epsilon^2)}}{c_\epsilon^2 h(x_0)} \quad \text{in } \Omega_\epsilon,$$

$$-\Delta \varphi_\epsilon = \alpha r_\epsilon^2 c_\epsilon^2 \psi_\epsilon + \frac{\psi_\epsilon h e^{(4\pi - \epsilon)(1 + \psi_\epsilon)\varphi_\epsilon}}{h(x_0)} \quad \text{in } \Omega_\epsilon.$$

It is clear that $\Omega_\epsilon \to \mathbb{R}^2$ as $\epsilon \to 0$. Noting that $|\psi_\epsilon| \le 1$ and $\Delta \psi_\epsilon \to 0$ uniformly in Ω_ϵ as $\epsilon \to 0$ and using the elliptic estimates, we get $\psi_\epsilon \to \psi$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where ψ is a bounded harmonic function in $\mathbb{R}^2$. Since $\psi(0) = \lim_{\epsilon \to 0} \psi_\epsilon(0) = 1$, we have by the Liouville theorem

$$\psi_\epsilon \to 1 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2).$$

Similarly, we have by the elliptic estimates

$$\varphi_\epsilon \to \varphi \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2),$$

where φ satisfies

$$-\Delta \varphi = e^{8\pi \varphi} \quad \text{in } \mathbb{R}^2$$

and

$$\varphi(0) = 0.$$

We calculate

$$\int_{\mathbb{B}_R(0)} e^{8\pi\varphi} \, dx \leq \limsup_{\epsilon \to 0} \int_{\mathbb{B}_R(0)} e^{(4\pi-\epsilon)(1+\psi_\epsilon)\varphi_\epsilon} \, dx$$

$$\leq \limsup_{\epsilon \to 0} \lambda_\epsilon^{-1} \int_{\mathbb{B}_{Rr_\epsilon}(x_\epsilon)} h(x_0) c_\epsilon^2 e^{(4\pi-\epsilon)u_\epsilon^2(y)} \, dy$$

$$\leq \limsup_{\epsilon \to 0} \lambda_\epsilon^{-1} \int_{\mathbb{B}_{Rr_\epsilon}(x_\epsilon)} h(y) u_\epsilon^2(y) e^{(4\pi-\epsilon)u_\epsilon^2(y)} \, dy$$

$$\leq 1.$$

A result of Chen and Li [19] implies that

$$\varphi(x) = -\frac{1}{4\pi} \log\left(1 + \pi |x|^2\right)$$

and

$$\int_{\mathbb{R}^2} e^{8\pi\varphi} \, dx = 1.$$

For the convergence behavior away from x_0, we have $c_\epsilon u_\epsilon \rightharpoonup G$ weakly in $W_0^{1,p}(\Omega)$ for any $1 < p < 2$, strongly in $L^q(\Omega)$ for any $q \geq 1$ and in $C_{\text{loc}}^1(\overline{\Omega} \setminus \{x_0\})$, where G is a Green function satisfying

$$\begin{cases} -\Delta G - \alpha G = \delta_{x_0} & \text{in } \Omega, \\ G = 0 & \text{on } \partial\Omega, \end{cases}$$

where δ_{x_0} is the Dirac measure centered at x_0.

G can be represented by

$$G = -\frac{1}{2\pi} \log |x - x_0| + A_0 + \Phi(x),$$

where A_0 is a constant depending on x_0 and α, $\Phi \in C^1(\Omega)$ with $\Phi(x_0) = 0$.

2.3 Upper bound estimates

Let δ be small such that $B_\delta(x_0) \subset \Omega$. Let $s_\epsilon = \sup_{\partial B_\delta(x_0)} u_\epsilon$ and $\bar{u}_\epsilon = (u_\epsilon - s_\epsilon)^+$. Then $\bar{u} \in W_0^{1,2}(B_\delta(x_0))$. Let $\tau_\epsilon = 1 - \frac{1}{c_\epsilon^2}\left(\frac{1}{2\pi} \log \frac{1}{\delta} + A_0 + o_\delta(1) + o_\epsilon(1)\right)$. Then, by the calculation in step 4 of Sect. 3 in [1], we get

$$\limsup_{\epsilon \to 0} \int_{B_\delta(x_0)} \left(e^{4\pi \bar{u}_\epsilon^2/\tau_\epsilon} - 1\right) dx \leq \pi \delta^2 e \tag{2.10}$$

and

$$(4\pi - \epsilon)u_\epsilon^2 \leq 4\pi \bar{u}_\epsilon^2/\tau_\epsilon - 2\log \delta + 4\pi A_0 + o(1).$$

Hence

$$\int_{B_{Rr_\epsilon}(x_\epsilon)} h e^{(4\pi-\epsilon)u_\epsilon^2} \, dx \leq \delta^{-2} e^{4\pi A_0 + o(1)} \int_{B_{Rr_\epsilon}(x_\epsilon)} h e^{4\pi \bar{u}_\epsilon^2/\tau_\epsilon} \, dx$$

$$= \delta^{-2} e^{4\pi A_0 + o(1)} \int_{B_{Rr_\epsilon}(x_\epsilon)} h\big(e^{4\pi \bar{u}_\epsilon^2 / \tau_\epsilon} - 1\big)\, dx + o(1)$$

$$\leq \delta^{-2} e^{4\pi A_0 + o(1)} h(x_0) \int_{B_\delta(x_0)} \big(e^{4\pi \bar{u}_\epsilon^2 / \tau_\epsilon} - 1\big)\, dx.$$

This together with (2.10) leads to

$$\limsup_{\epsilon \to 0} \int_{B_{Rr_\epsilon}(x_\epsilon)} h e^{(4\pi - \epsilon) u_\epsilon^2}\, dx \leq \pi h(x_0) e^{1 + 4\pi A_0}.$$

The argument in the proof of Lemma 3.3 in [20] yields

$$\lim_{\epsilon \to 0} \int_\Omega h e^{(4\pi - \epsilon) u_\epsilon^2}\, dx \leq \gamma + \lim_{R \to +\infty} \limsup_{\epsilon \to 0} \int_{B_{Rr_\epsilon}(x_\epsilon)} h e^{(4\pi - \epsilon) u_\epsilon^2}\, dx$$

$$\leq \gamma + \pi h(x_0) e^{1 + 4\pi A_0},$$

where $\gamma = \int_\Omega h\, dx$. This implies that

$$\sup_{u \in W_0^{1,2}(\Omega), \|u\|_{1,\alpha} \leq 1} \int_\Omega h e^{4\pi u^2}\, dx \leq \gamma + \pi h(x_0) e^{1 + 4\pi A_0}. \tag{2.11}$$

2.4 Existence of extremal functions

Let $r(x) = |x - x_0|$. Define

$$\phi_\epsilon(x) = \begin{cases} c + \frac{1}{c}\big(-\frac{1}{4\pi} \log(1 + \pi \frac{r^2}{\epsilon^2}) + B\big) & \text{for } x \leq R\epsilon, \\ \frac{G - \eta \Phi}{c} & \text{for } R\epsilon \leq r \leq 2R\epsilon, \\ \frac{G}{c} & \text{for } r > 2R\epsilon, \end{cases} \tag{2.12}$$

as in [1], where c and B are constants, $R = -\log \epsilon$, $\eta \in C_0^\infty(B_{2R\epsilon}(x_0))$ with $\eta = 1$ on $B_{R\epsilon}(x_0)$ and $\|\nabla \eta\|_{L^\infty} = O(\frac{1}{R\epsilon})$. Choose

$$c = \left(\frac{-\log \epsilon - 2\pi B + 2\pi A_0 + \frac{1}{2} \log \pi + O(\frac{1}{R^2})}{2\pi} \right)^{1/2},$$

$$B = \frac{1}{4\pi} + O\left(\frac{1}{R^2}\right) + O\big(R\epsilon \log(R\epsilon)\big),$$

as in [1] such that $\phi_\epsilon \in W_0^{1,2}(\Omega)$ and $\|\phi_\epsilon\|_{1,\alpha} = 1$. Then we get

$$\int_{B_{R\epsilon}(x_0)} h e^{4\pi \phi_\epsilon^2}\, dx \geq \pi h(x_0) e^{1 + 4\pi A_0} + O\left(\frac{1}{R^2}\right)$$

and

$$\int_{\Omega \setminus B_{R\epsilon}(x_0)} h e^{4\pi \phi_\epsilon^2}\, dx \geq \int_{\Omega \setminus 2B_{R\epsilon}(x_0)} h\big(1 + 4\pi \phi_\epsilon^2\big)\, dx \geq \gamma + 4\pi \frac{\|\sqrt{h}G\|_2^2}{c^2} + o\left(\frac{1}{c^2}\right).$$

Finally, we obtain

$$\int_\Omega h e^{4\pi \phi_\epsilon^2}\, dx > \gamma + \pi h(x_0) e^{1 + 4\pi A_0}.$$

This contradicts (2.11). Hence c_ϵ must be bounded and the elliptic estimates imply the existence of extremal functions. This completes the proof of Theorem 1.

3 Proof of Theorem 1

Let l be a positive integer and $0 \leq \alpha < \lambda_{l+1}(\Omega)$. Following the same steps as in the proof of Theorem 1, we see that, for any ϵ, $0 < \epsilon < 4\pi$, there exists some $u_\epsilon \in E_l^\perp \cap C^1(\overline{\Omega})$ with $\|u_\epsilon\|_{1,\alpha} = 1$ such that

$$\int_\Omega h e^{(4\pi-\epsilon)u_\epsilon^2}\, dx = \sup_{u \in E_l^\perp, \|u\|_{1,\alpha} \leq 1} \int_\Omega h e^{(4\pi-\epsilon)u^2}\, dx,$$

where $\|\cdot\|_{1,\alpha}$ is defined as in (1.4). Moreover, the Euler–Lagrange equation for u_ϵ is

$$\begin{cases} -\Delta u_\epsilon - \alpha u_\epsilon = \frac{1}{\lambda_\epsilon} h u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2} & \text{in } \Omega, \\ u_\epsilon \in E_l^\perp \cap C^1(\overline{\Omega}), \\ \lambda_\epsilon = \int_\Omega h u_\epsilon^2 e^{(4\pi-\epsilon)u_\epsilon^2}\, dx. \end{cases}$$

Let $c_\epsilon = |u_\epsilon(x_\epsilon)| = \max_{\overline{\Omega}} |u_\epsilon|$. We assume that $c_\epsilon \to \infty$ and $x_\epsilon \to x_0 \in \overline{\Omega}$. Similar to (2.11), we obtain

$$\sup_{u \in E_l^\perp, \|u\|_{1,\alpha} \leq 1} \int_\Omega h e^{4\pi u^2}\, dx \leq \gamma + \pi h(x_0) e^{1+4\pi A_0}, \tag{3.1}$$

where $\gamma = \int_\Omega h\, dx$.

Let $r(x) = |x - x_0|$. Define the same function

$$\phi_\epsilon(x) = \begin{cases} c + \frac{1}{c}\left(-\frac{1}{4\pi}\log(1 + \pi \frac{r^2}{\epsilon^2}) + B\right) & \text{for } x \leq R\epsilon, \\ \frac{G - \eta\Phi}{c} & \text{for } R\epsilon \leq r \leq 2R\epsilon, \\ \frac{G}{c} & \text{for } r > 2R\epsilon, \end{cases}$$

as in (2.12). Set

$$\widetilde{\phi}_\epsilon = \phi_\epsilon - \sum_{i=1}^{l} \sum_{j=1}^{n_i} (\phi_\epsilon, e_{ij}) e_{ij},$$

where (e_{ij}) $(1 \leq i \leq l, 1 \leq j \leq n_i)$ is the basis of E_l. Then, by (75) and (76) of [1], we have

$$\widetilde{\phi}_\epsilon = \phi_\epsilon + o\left(\frac{1}{\log^2 \epsilon}\right)$$

and

$$\|\widetilde{\phi}_\epsilon\|_{1,\alpha}^2 = 1 + o\left(\frac{1}{\log^2 \epsilon}\right).$$

Thus

$$\int_\Omega h e^{4\pi \frac{\widetilde{\phi}_\epsilon^2}{\|\widetilde{\phi}_\epsilon\|_{1,\alpha}^2}}\, dx = \int_\Omega h e^{4\pi \phi_\epsilon^2 + o(\frac{1}{\log \epsilon})}\, dx$$

$$\geq \left(1 + o\left(\frac{1}{\log \epsilon}\right)\right)\left(\gamma + \pi h(x_0)e^{1+4\pi A_0} + 4\pi \frac{\|\sqrt{h}G\|_2^2}{c^2} + o\left(\frac{1}{c^2}\right)\right)$$

$$\geq \gamma + \pi h(x_0)e^{1+4\pi A_0} + 4\pi \frac{\|\sqrt{h}G\|_2^2}{c^2} + o\left(\frac{1}{c^2}\right).$$

Set $\widehat{\phi}_\epsilon = \frac{\widetilde{\phi}_\epsilon}{\|\phi\|_{1,\alpha}}$. Then $\int_\Omega h e^{4\pi \widehat{\phi}_\epsilon^2} \, dx > \gamma + \pi h(x_0)e^{1+4\pi A_0}$. This contradicts (3.1). Hence c_ϵ must be bounded and the extremal function exists. We finish the proof of Theorem 2.

4 Proof of Theorem 3

First, we prove that, for any $0 < \epsilon < 4\pi$, there exists some $u_\epsilon \in C^1(\Sigma)$ such that

$$\int_\Sigma h e^{(4\pi-\epsilon)u_\epsilon^2} \, dv_g = \sup_{u \in W^{1,2}(\Sigma), \int_\Sigma u \, dv_g = 0, \|u\|_{1,\alpha} \leq 1} \int_\Sigma h e^{(4\pi-\epsilon)u^2} \, dv_g \qquad (4.1)$$

with $\|u_\epsilon\|_{1,\alpha} = 1$ and $\int_\Sigma u_\epsilon \, dv_g = 0$.

The main procedure of the proof is as follows. Since $0 \leq \alpha < \lambda_1(\Sigma)$, we may choose a bounded sequence u_j in $W^{1,2}(\Sigma)$ such that

$$\int_\Sigma h e^{(4\pi-\epsilon)u_j^2} \, dv_g \to \sup_{u \in W^{1,2}(\Sigma), \int_\Sigma u \, dv_g = 0, \|u\|_{1,\alpha} \leq 1} \int_\Sigma h e^{(4\pi-\epsilon)u^2} \, dv_g.$$

There exists some $u_\epsilon \in W^{1,2}(\Sigma)$ such that up to a subsequence,

$$u_j \rightharpoonup u_\epsilon \quad \text{weakly in } W^{1,2}(\Sigma),$$

$$u_j \to u_\epsilon \quad \text{strongly in } L^2(\Sigma),$$

$$u_j \to u_\epsilon \quad \text{a.e. in } \Sigma.$$

Using the same argument as in the proof of Theorem 3 in [1], we get $h e^{(4\pi-\epsilon)u_j^2}$ is bounded in L^q for some $q > 1$. Hence $h e^{(4\pi-\epsilon)u_j^2} \to h e^{(4\pi-\epsilon)u_\epsilon^2}$ strongly in $L^1(\Sigma)$. Hence (4.1) holds. The fact that $\int_\Sigma u_j \, dv_g = 0$ implies $\int_\Sigma u_\epsilon \, dv_g = 0$. We also have $\|u_\epsilon\|_{1,\alpha} = 1$ by contradiction as in the proof of Lemma 6.

Moreover, u_ϵ satisfies the Euler–Lagrange equation

$$\begin{cases} \Delta_g u_\epsilon - \alpha u_\epsilon = \frac{1}{\lambda_\epsilon} h u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2} - \frac{\mu_\epsilon}{\lambda_\epsilon}, \\ \lambda_\epsilon = \int_\Sigma h u_\epsilon^2 e^{(4\pi-\epsilon)u_\epsilon^2} \, dv_g, \\ \mu_\epsilon = \frac{1}{\text{Vol}_g(\Sigma)} \int_\Sigma h u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2} \, dv_g, \end{cases}$$

where Δ_g denotes the Laplace–Beltrami operator.

Denote $c_\epsilon = |u_\epsilon(x_\epsilon)| = \max_\Sigma |u_\epsilon|$. If c_ϵ is bounded, the existence of the extremal function follows from the elliptic estimates. We assume that $c_\epsilon \to +\infty$ and $x_\epsilon \to p \in \Sigma$. Similar to Lemma 7, we have $h(p) > 0$. Choosing an isothermal coordinate system (U, ϕ) near p such that the metric g can be written as $g = e^f(dx_1^2 + dx_2^2)$, where $f \in C^1(\phi(U), \mathbb{R})$ and $f(0) = 0$. Denote $\Omega = \phi(U)$, $\widetilde{u}_\epsilon = u_\epsilon \circ \phi^{-1}$ and $\widetilde{x}_\epsilon = \phi(x_\epsilon)$. Let

$$r_\epsilon = \sqrt{\lambda_\epsilon}\big[h(p)\big]^{-1/2} c_\epsilon^{-1} e^{-(2\pi-\epsilon/2)c_\epsilon^2}.$$

Define

$$\psi_\epsilon(x) = c_\epsilon^{-1}\widetilde{u}_\epsilon(\widetilde{x}_\epsilon + r_\epsilon x)$$

and

$$\varphi_\epsilon(x) = c_\epsilon\big(\widetilde{u}(\widetilde{x}_\epsilon + r_\epsilon x) - c_\epsilon\big)$$

for $x \in \Omega_\epsilon = \{x \in \mathbb{R}^2 : \widetilde{x}_\epsilon + r_\epsilon x \in \Omega\}$. Then we get

$$-\Delta_{\mathbb{R}^2}\psi_\epsilon = e^{f(\widetilde{x}_\epsilon + r_\epsilon x)}\left(\alpha r_\epsilon^2 \psi_\epsilon + \frac{h\psi_\epsilon e^{(4\pi-\epsilon)(\widetilde{u}_\epsilon^2 - c_\epsilon^2)}}{c_\epsilon^2 h(p)} - \frac{\mu_\epsilon}{c_\epsilon^3 e^{(4\pi-\epsilon)c_\epsilon^2}h(p)}\right),$$

$$-\Delta_{\mathbb{R}^2}\varphi_\epsilon = e^{f(\widetilde{x}_\epsilon + r_\epsilon x)}\left(\alpha r_\epsilon^2 c_\epsilon^2 \psi_\epsilon + \frac{h\psi_\epsilon e^{(4\pi-\epsilon)(1+\psi_\epsilon)\varphi_\epsilon}}{h(p)} - \frac{\mu_\epsilon}{c_\epsilon e^{(4\pi-\epsilon)c_\epsilon^2}h(p)}\right),$$

where $-\Delta_{\mathbb{R}^2}$ is the usual Laplace operator in $\mathbb{R}^2$. By the same argument as in Sect. 2.2, we obtain

$$\psi_\epsilon \to 1 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2)$$

and

$$\varphi_\epsilon \to \varphi \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^2),$$

where

$$\varphi(x) = -\frac{1}{4\pi}\log\big(1 + \pi|x|^2\big)$$

and

$$\int_{\mathbb{R}^2} e^{8\pi\varphi}\,dx = 1.$$

We also have $c_\epsilon u_\epsilon \rightharpoonup G$ weakly in $W^{1,q}(\Sigma)$ for all $1 < q < 2$, and $c_\epsilon u_\epsilon \to G$ in $C^1_{\mathrm{loc}}(\Sigma\setminus\{p\})\cap L^2(\Sigma)$, where G is Green function satisfying

$$\Delta_g G - \alpha G = \delta_p - \frac{1}{\mathrm{Vol}_g(\Sigma)} \quad \text{in } \Sigma$$

and $\int_\Sigma G\,dv_g = 0$. As before, G can be represented by

$$G = -\frac{1}{2\pi}\log r + A_p + \Phi_p,$$

where r is the geodesic distance from p, A_p is a constant and $\Phi_p \in C^1(\Sigma)$ with $\Phi_p(p) = 0$.

Similar to (2.11), we can get

$$\sup_{u\in W^{1,2}(\Sigma),\int_\Sigma u\,dv_g=0,\|u\|_{1,\alpha}\leq 1}\int_\Sigma h e^{4\pi u^2}\,dv_g \leq \gamma_1 + \pi h(p)e^{1+4\pi A_p}, \tag{4.2}$$

where $\gamma_1 = \int_\Sigma h\,dv_g$.

For the extremal function, define

$$
\phi_\epsilon(x) = \begin{cases} c + \frac{1}{c}\left(-\frac{1}{4\pi}\log(1 + \pi\frac{r^2}{\epsilon^2}) + B\right) & \text{for } x \le R\epsilon, \\ \frac{G - \eta\Phi_p}{c} & \text{for } R\epsilon \le r \le 2R\epsilon, \\ \frac{G}{c} & \text{for } r > 2R\epsilon, \end{cases} \tag{4.3}
$$

as in [1], where c and B are constants, $R = -\log\epsilon$, $\eta \in C_0^\infty(B_{2R\epsilon}(p))$ with $\eta = 1$ on $B_{R\epsilon}(p)$ and $\|\nabla_g\eta\|_{L^\infty} = O(\frac{1}{R\epsilon})$. Choose

$$
c = \left(\frac{-\log\epsilon - 2\pi B + 2\pi A_p + \frac{1}{2}\log\pi + O(\frac{1}{R^2})}{2\pi}\right)^{1/2},
$$

$$
B = \frac{1}{4\pi} + O\left(\frac{1}{R^2}\right) + O\big(R\epsilon\log(R\epsilon)\big),
$$

as in [1] such that $\phi_\epsilon \in W^{1,2}(\Sigma)$ and $\|\phi_\epsilon - \bar{\phi}_\epsilon\|_{1,\alpha} = 1$, where

$$
\bar{\phi}_\epsilon = \frac{1}{\text{Vol}_g(\Sigma)}\int_\Sigma \phi_\epsilon \, dv_g.
$$

Then we have on $B_{R\epsilon}(p)$

$$
4\pi(\phi_\epsilon - \bar{\phi}_\epsilon)^2 \ge 4\pi c^2 - 2\log\left(1 + \pi\frac{r^2}{\epsilon^2}\right) + 8\pi B + O\big(R\epsilon\log(R\epsilon)\big).
$$

It follows that

$$
\int_{B_{R\epsilon}(p)} h e^{4\pi(\phi_\epsilon - \bar{\phi}_\epsilon)^2} \, dv_g \ge \pi h(p) e^{1 + 4\pi A_p} + O\left(\frac{1}{(\log\epsilon)^2}\right) \tag{4.4}
$$

and

$$
\int_{\Sigma\setminus B_{R\epsilon}(p)} h e^{4\pi(\phi_\epsilon - \bar{\phi}_\epsilon)^2} \, dv_g \ge \int_{\Sigma\setminus B_{2R\epsilon}(p)} \left(1 + 4\pi\phi_\epsilon^2\right) dv_g + O\big(R\epsilon\log(R\epsilon)\big)
$$

$$
\ge \gamma_1 + 4\pi\frac{\|\sqrt{h}G\|_2^2}{c^2} + o\left(\frac{1}{c^2}\right). \tag{4.5}
$$

Combining (4.4) and (4.5), we find a contradiction with (4.2). Hence c_ϵ must be bounded. Using the elliptic estimates, we have the existence of the extremal function.

5 Proof of Theorem 5

Let l be a positive integer and $0 \le \alpha < \lambda_{l+1}(\Sigma)$. First, by the same arguments, we obtain for any ϵ, $0 < \epsilon < 4\pi$, there exists some $u_\epsilon \in E_l^\perp \cap C^1(\Sigma)$ satisfying $\|u_\epsilon\|_{1,\alpha} = 1$ and

$$
\int_\Sigma h e^{(4\pi - \epsilon)u_\epsilon^2} \, dv_g = \sup_{u \in E_l^\perp, \|u\|_{1,\alpha} \le 1} \int_\Sigma h e^{(4\pi - \epsilon)u^2} \, dv_g. \tag{5.1}
$$

Moreover, u_ϵ satisfies the Euler–Lagrange equation

$$
\begin{cases}
\Delta_g u_\epsilon - \alpha u_\epsilon = \frac{1}{\lambda_\epsilon} h u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2} - \frac{\mu_\epsilon}{\lambda_\epsilon}, \\
u_\epsilon \in E_l^\perp \cap C^1(\Sigma), \\
\lambda_\epsilon = \int_\Sigma h u_\epsilon^2 e^{(4\pi-\epsilon)u_\epsilon^2} \, dv_g, \\
\mu_\epsilon = \frac{1}{\mathrm{Vol}_g(\Sigma)} \int_\Sigma h u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2} \, dv_g.
\end{cases}
$$

Let $c_\epsilon = \max_\Sigma |u_\epsilon|$. We assume that $c_\epsilon = |u_\epsilon(x_\epsilon)| \to +\infty$ and $x_\epsilon \to p \in \Sigma$. We also get the upper bound estimate

$$
\sup_{u \in E_l^\perp, \int_\Sigma u \, dv_g = 0, \|u\|_{1,\alpha} \le 1} \int_\Sigma h e^{4\pi u^2} \, dv_g \le \gamma_1 + \pi h(p) e^{1+4\pi A_p}, \tag{5.2}
$$

where $\gamma_1 = \int_\Sigma h \, dv_g$.

For the existence of the extremal function, we define ϕ_ϵ as in (4.3). Then we have

$$
\int_\Sigma h e^{4\pi(\phi_\epsilon - \overline{\phi}_\epsilon)^2} \, dv_g \ge \gamma_1 + \pi h(p) e^{1+4\pi A_p} + 4\pi \frac{\|\sqrt{h}G\|_2^2}{c^2} + o\left(\frac{1}{c^2}\right).
$$

Set

$$
\widetilde{\phi}_\epsilon = \phi_\epsilon - \overline{\phi}_\epsilon - \sum_{i=1}^{l} \sum_{j=1}^{n_i} (\phi_\epsilon - \overline{\phi}_\epsilon, e_{ij}) e_{ij},
$$

where

$$
\overline{\phi}_\epsilon = \frac{1}{\mathrm{Vol}_g(\Sigma)} \int_\Sigma \phi_\epsilon \, dv_g
$$

and (e_{ij}) $(1 \le i \le l, 1 \le j \le n_i)$ is the basis of E_l.

Then, by (102) and (103) in [1], we have

$$
\widetilde{\phi}_\epsilon = \phi_\epsilon - \overline{\phi}_\epsilon + o\left(\frac{1}{\log^2 \epsilon}\right)
$$

and

$$
\|\widetilde{\phi}_\epsilon\|_{1,\alpha}^2 = 1 + o\left(\frac{1}{\log^2 \epsilon}\right).
$$

Thus

$$
\begin{aligned}
\int_\Sigma h e^{4\pi \frac{\widetilde{\phi}_\epsilon^2}{\|\widetilde{\phi}_\epsilon\|_{1,\alpha}^2}} \, dv_g &= \int_\Sigma h e^{4\pi(\phi_\epsilon - \overline{\phi}_\epsilon)^2 + o\left(\frac{1}{\log \epsilon}\right)} \, dv_g \\
&\ge \left(1 + o\left(\frac{1}{\log \epsilon}\right)\right) \left(\gamma_1 + \pi h(p) e^{1+4\pi A_p} + 4\pi \frac{\|\sqrt{h}G\|_2^2}{c^2} + o\left(\frac{1}{c^2}\right)\right) \\
&\ge \gamma_1 + \pi h(p) e^{1+4\pi A_p} + 4\pi \frac{\|\sqrt{h}G\|_2^2}{c^2} + o\left(\frac{1}{c^2}\right).
\end{aligned}
$$

Set $\widehat{\phi}_\epsilon = \frac{\widetilde{\phi}_\epsilon}{\|\phi_\epsilon\|_{1,\alpha}}$. We have

$$\int_\Sigma h e^{\widehat{\phi}_\epsilon^2}\, dv_g > \gamma_1 + \pi h(p) e^{1+4\pi A_p}.$$

This contradicts (5.2). Hence c_ϵ must be bounded and the extremal function exists. We finish the proof.

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References
1. Yang, Y.: Extremal functions for Trudinger–Moser inequalities of Adimurthi–Druet type in dimension two. J. Differ. Equ. **258**, 3161–3193 (2015)
2. Yang, Y., Zhu, X.: A remark on a result of Ding–Jost–Li–Wang. Proc. Am. Math. Soc. **145**, 3953–3959 (2017)
3. Trudinger, N.: On embeddings into Orlicz spaces and some applications. J. Math. Mech. **17**, 473–484 (1967)
4. Moser, J.: A sharp form of an inequality by N. Trudinger. Indiana Univ. Math. J. **20**, 1077–1091 (1971)
5. Lions, P.L.: The concentration-compactness principle in the calculus of variation, the limit case, part I. Rev. Mat. Iberoam. **1**, 145–201 (1985)
6. Adimurthi, Druet, O.: Blow-up analysis in dimension 2 and a sharp form of Trudinger–Moser inequality. Commun. Partial Differ. Equ. **29**, 295–322 (2004)
7. Yang, Y.: A sharp form of Moser–Trudinger inequality in high dimension. J. Funct. Anal. **239**, 100–126 (2006)
8. Yang, Y.: A sharp form of the Moser–Trudinger inequality on a compact Riemannian surface. Trans. Am. Math. Soc. **359**, 5761–5776 (2007)
9. Tintarev, C.: Trudinger–Moser inequality with remainder terms. J. Funct. Anal. **266**, 55–66 (2014)
10. Yang, Y., Zhu, X.: Existence of solutions to a class of Kazdan–Warner equations on compact Riemannian surface. Sci. China Math. **61**, 1109–1128 (2018)
11. Ding, W., Jost, J., Li, J., Wang, G.: The differential equation $-\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface. Asian J. Math. **1**, 230–248 (1997)
12. Carleson, L., Chang, A.: On the existence of an extremal function for an inequality of J. Moser. Bull. Sci. Math. **110**, 113–127 (1986)
13. Struwe, M.: Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **5**, 425–464 (1988)
14. Flucher, M.: Extremal functions for Trudinger–Moser inequality in 2 dimensions. Comment. Math. Helv. **67**, 471–497 (1992)
15. Lin, K.: Extremal functions for Moser's inequality. Trans. Am. Math. Soc. **348**, 2663–2671 (1996)
16. Adimurthi, Struwe, M.: Global compactness properties of semilinear elliptic equation with critical exponential growth. J. Funct. Anal. **175**, 125–167 (2000)
17. Li, Y.: Moser-Trudinger inequality on compact Riemannian manifolds of dimension two. J. Partial Differ. Equ. **14**, 163–192 (2001)
18. Gidas, B., Ni, W., Nirenberg, L.: Symmetry and related properties via the maximum principle. Commun. Math. Phys. **68**, 209–243 (1979)
19. Chen, W., Li, C.: Classification of solutions of some nonlinear elliptic equations. Duke Math. J. **63**, 615–622 (1991)
20. Lu, G., Yang, Y.: The sharp constant and extremal functions for Moser–Trudinger inequalities involving L^p norm. Discrete Contin. Dyn. Syst. **25**, 963–979 (2009)