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An upper bound for the Z-spectral radius of adjacency tensors

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Abstract

Let $\mathcal{H}$ be a k-uniform hypergraph on n vertices with degree sequence $\Delta = d_1 \geq \cdots \geq d_n = \delta$. In this paper, in terms of degree d_i, we give a new upper bound for the Z-spectral radius of the adjacency tensor of $\mathcal{H}$. Some examples are given to show the efficiency of the bound.

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Keywords: Hypergraph; Z-eigenvalue; Bound; Nonnegative tensor

1 Introduction

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an mth order n-dimensional real square tensor, x be a real n-vector. Then we define the following real n-vector:

$$\mathcal{A}x^{m-1} = \left(\sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \leq i \leq n}, \qquad x^{[m-1]} = \left(x_i^{m-1} \right)_{1 \leq i \leq n}.$$

If there exist a real vector x and a real number λ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an H-eigenvalue of $\mathcal{A}$ and x is called an eigenvector of $\mathcal{A}$ associated with λ [1, 2]. If there exist a real vector x and a real number λ such that

$$\mathcal{A}x^{m-1} = \lambda x, \qquad x^T x = 1,$$

then λ is called a Z-eigenvalue of $\mathcal{A}$ and x is called an eigenvector of $\mathcal{A}$ associated with λ. You can see more about the eigenvalues of tensors in [3–7].

Let $\mathcal{H}$ be a hypergraph with a vertex set $V(\mathcal{H})$ and an edge set $E(\mathcal{H}) = \{e_1, e_2, \ldots, e_t\}$. If every edge of $\mathcal{H}$ contains exactly k distinct vertices, then $\mathcal{H}$ is called a k-uniform hypergraph. The degree of a vertex i in $\mathcal{H}$ is the number of edges incident with i, denoted by d_i. If $d_i = d$ for any $i \in V(\mathcal{H})$, then the hypergraph $\mathcal{H}$ is called a regular hypergraph. Recently, the spectral radii of hypergraphs have been studied in [8, 9].

Let $\{i_1, \ldots, i_k\} \in E(\mathcal{H})$ mean that there is an edge containing k distinct vertices $i_1, \ldots, i_k$. Then the adjacency tensor $\mathcal{A}(\mathcal{H}) = (a_{i_1 \cdots i_k})$ of a hypergraph $\mathcal{H}$ is a kth order n-dimensional tensor with entries:

$$a_{i_1\cdots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1,\ldots,i_k\} \in E(\mathcal{H}), \\ 0, & \text{otherwise.} \end{cases}$$

Let $D(\mathcal{H}) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be the degree diagonal tensor of the graph $\mathcal{H}$. Then the tensor $Q(\mathcal{H}) = D(\mathcal{H}) + \mathcal{A}(\mathcal{H})$ is called the signless Laplacian tensor of the hypergraph $\mathcal{H}$. The largest modulus of the Z-eigenvalues of the adjacency tensor $\mathcal{A}(\mathcal{H})$ is denoted by $\rho_Z(\mathcal{H})$, which is called the Z-spectral radius of the adjacency tensor $\mathcal{A}(\mathcal{H})$.

For a k-uniform hypergraph $\mathcal{H}$, let $\Delta = d_1 \geq \cdots \geq d_n = \delta$ be the degree sequence of the hypergraph $\mathcal{H}$. In 2013, Xie and Chang [8] presented the following upper bound for the largest Z-eigenvalues $\rho_Z(\mathcal{H})$ of adjacency tensors:

$$\rho_Z(\mathcal{H}) \leq \Delta. \tag{1}$$

In this paper, we give a new upper bounds in terms of degree d_i for the Z-spectral radius of hypergraphs, which improves the bound as shown in (1). Then we give some examples to compare these bounds for Z-spectral radius of hypergraphs.

2 Preliminaries

Some basic definitions and useful results are listed as follows.

Definition 2.1 ([10]) The tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $\mathbb{J} \subset \{1, 2, \ldots, n\}$ such that $a_{i_1,i_2,\ldots,i_m} = 0$, $\forall i_1 \in \mathbb{J}$, $\forall i_2, \ldots, i_m \notin \mathbb{J}$. If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ to be irreducible.

Definition 2.2 Let $\mathcal{A}$ be an m-order and n-dimensional tensor. We define $\sigma(\mathcal{A})$ the Z-spectrum of $\mathcal{A}$ by the set of all Z-eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq \emptyset$, then the Z-spectral radius of $\mathcal{A}$ is denoted by

$$\rho_Z(\mathcal{A}) = \max\bigl\{|\lambda| : \lambda \in \sigma(\mathcal{A})\bigr\}.$$

The concept of *weakly symmetric* was first introduced and used by Chang, Pearson, and Zhang [11] in order to study the following Perron–Frobenius theorem for the Z-eigenvalue of nonnegative tensors.

Lemma 2.1 ([11]) *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a weakly symmetric nonnegative tensor, then the spectral radius $\rho_Z(\mathcal{A})$ is a positive Z-eigenvalue with a nonnegative Z-eigenvector x. Furthermore, if $\mathcal{A}$ is irreducible, x is positive.*

$|\mathcal{A}|$ means that $(|\mathcal{A}|)_{i_1\cdots i_m} = |a_{i_1\cdots i_m}|$. Two useful lemmas are given as follows.

Lemma 2.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two weakly symmetric and irreducible tensors of order m and dimension n. If $\mathcal{B}$ and $\mathcal{B} - |\mathcal{A}|$ are nonnegative, then $\rho_Z(\mathcal{B}) \geq \rho_Z(|\mathcal{A}|)$.*

Proof Let y be the eigenvector associated with β, where β is a Z-eigenvalue of $\mathcal{A}$. Then we can get

$$|\beta||y| = \bigl|\mathcal{A}y^{[m-1]}\bigr| \leq |\mathcal{A}|\bigl|y^{[m-1]}\bigr| \leq \mathcal{B}\bigl|y^{[m-1]}\bigr|.$$

By Theorem 4.7 of [11], we have

$$\rho_Z(\mathcal{B}) = \max_{y\geq 0}\min_{|y_i|>0}\frac{(\mathcal{B}|y|^{[m-1]})_i}{|y_i|} \geq \min_{|y_i|>0}\frac{(\mathcal{B}|y|^{[m-1]})_i}{|y_i|} \geq |\beta|.$$

Then

$$\rho_Z(\mathcal{B}) \geq \rho_Z(|\mathcal{A}|). \qquad \square$$

Lemma 2.3 *Let $\{\mathcal{A}_k\}$ be a sequence of nonnegative, weakly symmetric tensors of order m and dimension n, and $\mathcal{A}_k - \mathcal{A}_{k+1}$ be nonnegative for each positive integer k. Then*

$$\lim_{k\to\infty}\rho_Z(\mathcal{A}_k) = \rho_Z\Big(\lim_{k\to\infty}\mathcal{A}_k\Big).$$

Proof Let $\mathcal{A} = \lim_{k\to\infty}\mathcal{A}_k$. Since $\mathcal{A}_k - \mathcal{A}_{k+1}$ is nonnegative, by Lemma 2.2, we know that $\{\rho_Z(\mathcal{A}_k)\}$ is a monotone decreasing sequence with a lower bound $\rho_Z(\mathcal{A})$. So $\lim_{k\to\infty}\mathcal{A}_k$ exists and

$$\lambda = \lim_{k\to\infty}\rho_Z(\mathcal{A}_k) \geq \rho_Z(\mathcal{A}).$$

Since $\{\mathcal{A}_k\}$ is nonnegative, weakly symmetric, then there exists a nonnegative vector $x^{(k)}$ such that $\mathcal{A}_k(x^{(k)})^{m-1} = \rho_Z(\mathcal{A}_k)x^{(k)}$ and $(x^{(k)})^T x^{(k)} = 1$. Then $\{x^{(k)}\}$ is a bounded sequence, it has a convergent subsequence $\{y_t\}$. Suppose that $y = \lim_{k\to\infty} y_t$. By $\mathcal{A}_k y_t^{m-1} = \rho_Z(\mathcal{A}_k)y_t$, we get $\mathcal{A}y^{m-1} = \lambda y$. So λ is an eigenvalue of $\mathcal{A}$. Since $\lambda \leq \rho_Z(\mathcal{A})$, we have $\rho_Z(\mathcal{A}) = \lambda$. $\square$

3 The Z-spectral radius of tensors and hypergraphs

In this section, let $r_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m=1}^{n}|a_{ii_2\cdots i_m}| - |a_{ii\cdots i}|$, we give some bounds on the Z-spectral radius of tensors and hypergraphs.

Theorem 3.1 *Let $\mathcal{A}$ be weakly symmetric nonnegative tensors of order m and dimension n. Then*

$$\rho_Z(\mathcal{A}) \leq \max_{a_{i_1\cdots i_m}\neq 0}\left\{\prod_{j=1}^{m} r_{i_j}^{\frac{1}{m}}(\mathcal{A})\right\}.$$

Proof Case 1. If $\mathcal{A}$ is irreducible, by Lemma 2.1, let $u = (u_i)$ be the positive eigenvector associated with the largest Z-eigenvalues $\rho_Z(\mathcal{A})$ of $\mathcal{A}$. Then

$$\mathcal{A}u^{m-1} = \rho_Z(\mathcal{A})u.$$

Let $u_\alpha = \max\{u_{i_1}\cdots u_{i_m} : a_{i_1\cdots i_m}\neq 0, 1\leq i_1,\ldots,i_m\leq n\}$, then

$$\begin{aligned}\rho_Z(\mathcal{A})u_i^2 &= \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m}u_i u_{i_2}\cdots u_{i_m}\\ &= \sum_{a_{ii_2\cdots i_k}\neq 0} a_{ii_2\cdots i_k}u_i u_{i_2}\cdots u_{i_m}\\ &\leq r_i(\mathcal{A})u_\alpha. \end{aligned} \tag{2}$$

Suppose that $u_\alpha = u_{j_1} \cdots u_{j_m}$. Then, from (2), we can get

$$\rho_Z(\mathcal{A})u_{j_1}^2 \leq r_{j_1}(\mathcal{A})u_\alpha,$$

$$\vdots$$

$$\rho_Z(\mathcal{A})u_{j_m}^2 \leq r_{j_m}(\mathcal{A})u_\alpha.$$

Then, by $u_\alpha^m \leq u_\alpha^2$, we have

$$\prod_{l=1}^{m} \rho_Z^m(\mathcal{A})u_{j_l}^2 \leq u_\alpha^m \prod_{l=1}^{m} r_{i_l}(\mathcal{A}) \leq u_\alpha^2 \prod_{l=1}^{m} r_{i_l}(\mathcal{A}).$$

Therefore,

$$\rho_Z(\mathcal{A}) \leq \max_{a_{i_1\cdots i_m}\neq 0} \left\{ \prod_{j=1}^{m} r_{i_j}^{\frac{1}{m}}(\mathcal{A}) \right\}.$$

Case 2. If $\mathcal{A}$ is reducible. Let $\mathcal{T} = (t_{i_1 i_2\cdots i_m})$, $t_{i_1 i_2\cdots i_m} = 1$ for all $1 \leq i_1, i_2, \ldots, i_m \leq n$. Then $\mathcal{A} + \epsilon\mathcal{T}$ is an irreducible nonnegative tensor for any chosen positive real number ϵ. Now we substitute $\mathcal{A} + \epsilon\mathcal{T}$ for $\mathcal{A}$, respectively, in the previous case. When $\epsilon \to 0$, the result follows by the continuity of $\rho_Z(\mathcal{A} + \epsilon\mathcal{T})$. □

By Theorem 3.1, a bound on the Z-spectral radius of a uniform hypergraph is obtained, we also compare the bound with the result in (1).

Theorem 3.2 *Let $\mathcal{H}$ be a k-uniform hypergraph on n vertices with the degree sequence $\Delta = d_1 \geq \cdots \geq d_n = \delta$. Then*

$$\rho_Z(\mathcal{H}) \leq \max_{\{i_1,\ldots,i_k\}\in E(H)} \left\{ \prod_{j=1}^{k} d_{i_j}^{\frac{1}{k}}(\mathcal{A}) \right\}. \tag{3}$$

Proof Case 1. $\mathcal{A}(\mathcal{H})$ is irreducible. In this case, by Lemma 2.1, there exists a positive eigenvector corresponding to the spectral radius $\rho_Z(\mathcal{H})$. Then, by Theorem 3.1, we have

$$\rho_Z(\mathcal{H}) \leq \max_{\{i_1,\ldots,i_k\}\in E(H)} \left\{ \prod_{j=1}^{k} d_{i_j}^{\frac{1}{k}}(\mathcal{A}) \right\}.$$

Case 2. If $\mathcal{A}(\mathcal{H})$ is reducible. Let $\mathcal{T} = (t_{i_1 i_2\cdots i_k})$, $t_{i_1 i_2\cdots i_k} = 1$, for all $1 \leq i_1, i_2, \ldots, i_k \leq n$. Then $\mathcal{A}(\mathcal{H}) + \epsilon\mathcal{T}$ is an irreducible nonnegative tensor for any chosen positive real number ϵ. Now we substitute $\mathcal{A}(\mathcal{H}) + \epsilon\mathcal{T}$ for $\mathcal{A}(\mathcal{H})$, respectively, in the previous case. When $\epsilon \to 0$, the result follows by the continuity of $\rho_Z(\mathcal{A}(\mathcal{H}) + \epsilon\mathcal{T})$. □

Remark Obviously, we can get

$$\max_{\{i_1,\ldots,i_k\}\in E(H)} \left\{ \prod_{j=1}^{k} d_{i_j}^{\frac{1}{k}}(\mathcal{A}) \right\} \leq \Delta.$$

That is to say, our bound in Theorem 3.2 is always better than the bound in (1).

Table 1 Upper bounds for the hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$

	(1)	(3)
$\mathcal{H}_1$	3	$3^{\frac{1}{3}}$
$\mathcal{H}_2$	3	$3^{\frac{2}{3}}$

We now show the efficiency of the new upper bound in Theorem 3.2 by the following examples.

Example 1 Consider 3-uniform hypergraph $\mathcal{H}_1$ with a vertex set $V(\mathcal{H}_1) = \{1,2,3,4,5,6,7\}$ and an edge set $E(\mathcal{H}_1) = \{e_1, e_2, e_3\}$, where $e_1 = \{1,2,3\}$, $e_2 = \{1,4,5\}$, $e_3 = \{1,6,7\}$.

Example 2 Consider 3-uniform hypergraph $\mathcal{H}_2$ with a vertex set $V(\mathcal{H}_2) = \{1,2,3,4,5,6,7\}$ and an edge set $E(\mathcal{H}_2) = \{e_1, e_2, e_3\}$, where $e_1 = \{1,6,7\}$, $e_2 = \{2,6,7\}$, $e_3 = \{3,6,7\}$.

From Table 1, we can find that bound (3) is always better than (1).

4 Conclusion

In this paper, we get a new bound for the Z-spectral radius of tensors. As applications, in terms of the degree sequence d_i, we obtain a new bound for the Z-spectral radius of hypergraphs, which is always better than the bound in [8]. We list two examples to show the efficiency of our new bound.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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