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Remoteness and distance, distance (signless) Laplacian eigenvalues of a graph

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Abstract

Let G be a connected graph of order n . The remoteness of G , denoted by ρ , is the maximum average distance from a vertex to all other vertices. Let $\partial_1 \geq \dots \geq \partial_n$, $\partial_1^L \geq \dots \geq \partial_n^L$ and $\partial_1^Q \geq \dots \geq \partial_n^Q$ be the distance, distance Laplacian and distance signless Laplacian eigenvalues of G , respectively. In this paper, we give lower bounds on $\rho + \partial_1$, $\rho - \partial_n$, $\rho + \partial_1^L$, $\partial_1^L - \rho$, $2\rho + \partial_1^Q$ and $\partial_1^Q - 2\rho$ and the corresponding extremal graphs are also characterized.

MSC: 05C50

Keywords: Remoteness; Distance eigenvalues; Distance (signless) Laplacian eigenvalues

1 Introduction

In this paper, we consider simple, undirected and connected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)| = n$, $|E(G)| = m$. Let δ be the minimum degree of the graph G . The distance between vertices v_i and v_j is the length of a shortest path connecting them in G , denoted by d_{ij} . The diameter of a graph is the maximum distance between any two vertices of G , denoted by d . The transmission $\text{Tr}(v_i)$ of vertex v_i is defined to be the sum of distances from v_i to all other vertices and G is transmission regular if $\text{Tr}(v_1) = \dots = \text{Tr}(v_n)$. The remoteness ρ of G is denoted by

$$\rho = \rho(G) = \max_{v \in V(G)} \frac{\text{Tr}(v)}{n-1}.$$

The distance matrix of G , denoted by $D(G)$, is the symmetric real matrix with (i, j) -entry being d_{ij} . Let $\text{Tr}(G) = \text{diag}(\text{Tr}(v_1), \text{Tr}(v_2), \dots, \text{Tr}(v_n))$ be the diagonal matrix of the vertex transmissions in G . The distance Laplacian matrix and the distance signless Laplacian matrix of G are defined as $D^L(G) = \text{Tr}(G) - D(G)$ and $D^Q(G) = \text{Tr}(G) + D(G)$, respectively. Let $\partial_1 \geq \dots \geq \partial_n$, $\partial_1^L \geq \dots \geq \partial_n^L$ and $\partial_1^Q \geq \dots \geq \partial_n^Q$ are the distance eigenvalues (see [1–3]), distance Laplacian eigenvalues (see [4]) and distance signless Laplacian eigenvalues (see [5]) of G , respectively. In particular, the eigenvalues ∂_1 , ∂_1^L and ∂_1^Q are called the distance spectral radius, the distance Laplacian spectral radius and the distance signless Laplacian spectral radius of G , respectively.

Recently, remoteness, which is one of the most important distance graph parameters, has attracted much attention of many graph theory researchers. In [6], Sedlar et al. proved

two AutoGraphiX (a software package devoted to conjecture-making in graph theory) conjectures on remoteness, vertex connectivity and algebraic connectivity. Sedlar [7] also studied AutoGraphiX conjectures involving remoteness and other distance invariants. Aouchiche and Hansen [8] gave Nordhaus–Gaddum-type inequalities for remoteness in graphs and the extremal graphs were also characterized. Hua et al. [9, 10] solved several conjectures related to remoteness and used remoteness to give a new sufficient condition for a connected bipartite graph to be Hamiltonian. Aouchiche and Hansen [11] provided the lower bounds on $\partial_1 - \rho$ and $\rho + \partial_2$. Furthermore, they also proposed two conjectures. Lin et al. [12] confirmed these two conjectures. They also gave lower bounds on $\rho + \partial_n$ and $\partial_1 - \rho$ when $G \not\cong K_n$ and the extremal graphs were characterized. Inspired by these two papers, we continue to study the relations between remoteness and distance, distance (signless) Laplacian eigenvalues. In particular, we give lower bounds on $\rho + \partial_1$, $\rho - \partial_n$, $\rho + \partial_1^L$, $\partial_1^L - \rho$, $2\rho + \partial_1^Q$ and $\partial_1^Q - 2\rho$ and the corresponding extremal graphs are characterized.

2 Preliminaries

Before giving the proof of our theorems, we introduce some fundamental lemmas and properties in this section.

Lemma 2.1 ([12]) *Let G be a connected graph of order n with diameter d and remoteness ρ . Then*

$$\rho > \frac{d}{2}.$$

Denote by H_{n-d} ($n > d$) a graph of order $n - d$ such that $V(H_{n-d}) = V(\overline{K}_{n-d})$ and $E(H_{n-d}) \supseteq E(\overline{K}_{n-d})$, where \overline{K}_{n-d} is a null graph of order $n - d$. Let $H_{n,d}$ be a graph of order n with diameter d obtained by joining $n - d$ edges between one end of the path P_d with each vertex of H_{n-d} .

Lemma 2.2 ([12]) *Let G be a connected graph of order n with diameter d and remoteness ρ . Then*

$$\rho \leq d - \frac{d^2 - d}{2(n - 1)}$$

with equality holding if and only if $G \cong H_{n,d}$.

Lemma 2.3 ([12]) *Let G be a connected graph of order n with diameter $d \geq 3$. Then*

$$\partial_1 > n - 2 + d.$$

Lemma 2.4 ([13]) *Let G be a connected graph of order n with Wiener index W . Then*

$$\partial_1 \geq \frac{2W}{n},$$

and the equality holds if and only if G is transmission regular.

Lemma 2.5 ([14]) *Let $G = K_{n_1, \dots, n_k}$ be a complete k -partite graph, where $\sum_{i=1}^k n_i = n$ and $2 \leq k \leq n - 1$. Then the characteristic polynomial of $D(G)$ is*

$$P_D(\lambda) = (\lambda + 2)^{n-k} \left[\prod_{i=1}^k (\lambda - n_i + 2) - \sum_{i=1}^k n_i \prod_{j=1, j \neq i}^k (\lambda - n_j + 2) \right].$$

Lemma 2.6 ([4]) *Let G be a connected graph of order n . Then $\partial_1^L \geq n$, with equality if and only if $G \cong K_n$.*

Lemma 2.7 ([15]) *Let G be a connected graph of order n . Then*

$$\partial_1^L \geq D_1 + \frac{D_1}{n - 1},$$

where D_1 is the maximum transmission of G . Hence

$$\partial_1^L \geq D_1 + 1,$$

and the equality holds if and only if $G \cong K_n$.

Lemma 2.8 ([5]) *Let $G = K_{n_1, \dots, n_k}$ be a complete k -partite graph, where $\sum_{i=1}^k n_i = n$ and $2 \leq k \leq n - 1$. Then the characteristic polynomial of $D^L(G)$ is*

$$P_{D^L}(\mu) = \mu(\mu - n)^{k-1} \prod_{i=1}^k (\mu - n + n_i)^{n_i-1}.$$

Lemma 2.9 ([16]) *If G is a connected graph on $n \geq 2$ vertices. Then $\partial_1^Q \geq 2n - 2$ with equality if and only if $G \cong K_n$.*

Lemma 2.10 ([5]) *Let G be a connected graph of order n with diameter $d \geq 3$. Then*

$$\partial_1^Q > 2n - 4 + 2d.$$

Lemma 2.11 ([17]) *Let G be a connected graph of order n with Wiener index W . Then*

$$\partial_1^Q \geq \frac{4W}{n},$$

and the equality holds if and only if G is transmission regular.

Lemma 2.12 ([5]) *Let $G = K_{n_1, \dots, n_k}$ be a complete k -partite graph, where $\sum_{i=1}^k n_i = n$ and $2 \leq k \leq n - 1$. Then the characteristic polynomial of $D^Q(G)$ is*

$$P_{D^Q}(q) = \prod_{i=1}^k (q - n - n_i + 4)^{n_i-1} \left[\prod_{i=1}^k (q - n - 2n_i + 4) - \sum_{i=1}^k n_i \prod_{j=1, j \neq i}^k (q - n - 2n_j + 4) \right].$$

3 Remoteness and distance eigenvalues of a graph

Aouchiche et al. [11] and Lin et al. [12] obtained the following two results.

Theorem 3.1 ([11]) *Let G be a connected graph on $n \geq 4$ vertices with the distance spectral radius ∂_1 and remoteness ρ . Then*

$$\partial_1 - \rho \geq n - 2$$

with equality if and only if $G \cong K_n$.

Theorem 3.2 ([12]) *Let $G (\not\cong K_n)$ be a connected graph on $n \geq 4$ vertices with remoteness ρ . Then*

$$\partial_1 - \rho \geq \frac{n - 1 + \sqrt{(n - 1)^2 + 8}}{2} - \frac{n}{n - 1}$$

with equality holding if and only if $G \cong K_n - e$, where e is an edge of G .

Naturally, we consider the bounds on $\rho + \partial_1$ in this paper.

Theorem 3.3 *Let G be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$n \leq \rho + \partial_1 \leq \rho(P_n) + \partial_1(P_n)$$

with the left equality holding if and only if $G \cong K_n$ and the right equality holding if and only if $G \cong P_n$.

Proof By Lemma 2.1, $\rho \leq d - \frac{d^2-d}{2(n-1)} = f(d)$. Note that $f(d)$ is a strictly increasing function on d , then $\rho \leq f(n - 1) = \frac{n}{2}$, and the equality holds if and only if $G \cong H_{n,n-1} = P_n$. Meanwhile, it was shown in [18] that $\partial_1(G) \leq \partial_1(P_n)$ with equality holding if and only if $G \cong P_n$. Hence the right side of the theorem holds.

By the definition of ρ , we have $\rho \geq 1$ with the equality if and only if $G \cong K_n$. As is well known [18] $\partial_1(G) \geq n - 1$ with equality if and only if $G \cong K_n$. So the lower bound is completed. □

Theorem 3.4 *Let $G (\not\cong K_n)$ be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$\rho + \partial_1 \geq \frac{n}{n - 1} + \frac{n - 1 + \sqrt{(n - 1)^2 + 8}}{2}$$

with equality holding if and only if $G \cong K_n - e$.

Proof For $G \cong K_n - e$, note that $K_n - e$ is a complete multipartite graph, by Lemma 2.5, then

$$\rho + \partial_1 = \frac{n}{n - 1} + \frac{n - 1 + \sqrt{(n - 1)^2 + 8}}{2} < 1 + \frac{1}{n - 1} + \frac{2(n - 1) + \frac{4}{n - 1}}{2} = n + \frac{3}{n - 1}.$$

Let G be a connected graph with diameter d . If $d \geq 3$, by Lemmas 2.1 and 2.3, then

$$\rho + \partial_1 > \frac{d}{2} + n - 2 + d = \frac{3}{2}d + n - 2 > n + \frac{3}{n-1} > \rho(K_n - e) + \partial_1(K_n - e).$$

If $d = 2$, we have

$$\rho = \frac{\delta + 2(n-1-\delta)}{n-1} = \frac{2(n-1)-\delta}{n-1} = 2 - \frac{\delta}{n-1}.$$

By Lemma 2.4, then

$$\partial_1 \geq \frac{2W}{n} \geq \frac{n(n-1) + 2(n-1-\delta)}{n} = n + 1 - \frac{2(1+\delta)}{n}.$$

Hence

$$\rho + \partial_1 \geq n + 3 - \frac{2(1+\delta)}{n} - \frac{\delta}{n-1}.$$

If $\delta \leq n - 3$, then

$$\rho + \partial_1 \geq n + 3 - \frac{2(n-2)}{n} - \frac{n-3}{n-1} = n + \frac{4}{n} + \frac{2}{n-1} > n + \frac{3}{n-1} > \rho(K_n - e) + \partial_1(K_n - e).$$

If $\delta = n - 2$, we have $\rho(G) = \rho(K_n - e)$ and $\partial_1(G) > \partial_1(G + e)$, and thus $\rho(G) + \partial_1(G) \geq \rho(K_n - e) + \partial_1(K_n - e)$ with equality if and only if $G \cong K_n - e$. The result follows. \square

Theorem 3.5 *Let $G (\not\cong K_n, K_n - e)$ be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$\rho + \partial_1 \geq \frac{n}{n-1} + \frac{n-1 + \sqrt{(n-1)^2 + 16}}{2}$$

with equality holding if and only if $G \cong K_n - 2e$, where $2e$ are two matching edges.

Proof For $G \cong K_n - 2e$,

$$\rho + \partial_1 = \frac{n}{n-1} + \frac{n-1 + \sqrt{(n-1)^2 + 16}}{2} < \frac{n}{n-1} + \frac{2(n-1) + \frac{8}{n-1}}{2} = n + \frac{5}{n-1}.$$

Let G be a connected graph with d . If $d \geq 3$, then

$$\rho + \partial_1 > \frac{d}{2} + n - 2 + d = \frac{3}{2}d + n - 2 > n + \frac{5}{n-1} > \rho(K_n - 2e) + \partial_1(K_n - 2e).$$

If $d = 2$, similarly we have

$$\rho + \partial_1 \geq n + 3 - \frac{2(1+\delta)}{n} - \frac{\delta}{n-1}.$$

If $\delta \leq n - 3$, then

$$\rho + \partial_1 \geq n + 3 - \frac{2(n-2)}{n} - \frac{n-3}{n-1} = n + \frac{4}{n} + \frac{2}{n-1} \geq n + \frac{5}{n-1} > \rho(K_n - 2e) + \partial_1(K_n - 2e).$$

If $\delta = n - 2$, then $\partial_1(G) > \partial_1(G + e)$ and $\rho(G) = \rho(K_n - 2e)$, and hence $\partial_1(G) + \rho(G) \geq \partial_1(K_n - 2e) + \rho(K_n - 2e)$ with equality holding if and only if $G \cong K_n - 2e$. \square

In [11], Aouchiche and Hansen showed a lower bound on the sum of the remoteness and the second largest distance eigenvalue, $\rho + \partial_2$, of a graph with given number of vertices n .

Theorem 3.6 ([11]) *Let G be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$\rho + \partial_2 \geq 0,$$

with equality holding if and only if $G \cong K_n$.

In fact, the bound in the above corollary is best possible among the bounds of the form $\rho + \partial_k \geq 0$, with a fixed integer k , over the class of all connected graphs. First, we prove a lower bound on $\rho + \partial_2$ among all the complete bipartite graphs $K_{a,b}$.

Theorem 3.7 *Let G be a complete bipartite graph of order $n \geq 4$ with remoteness ρ . Then*

$$\rho(K_{a,b}) + \partial_2(K_{a,b}) \geq n - \frac{1}{n-1} - \sqrt{n^2 - 3n + 3}$$

with equality holding if and only if $G \cong K_{1,n-1}$.

Proof For $G \cong K_{a,b}$, where $1 \leq a \leq \lfloor \frac{n}{2} \rfloor$, we have

$$\rho(K_{a,b}) + \partial_2(K_{a,b}) = n - \frac{a}{n-1} - \sqrt{3a^2 - 3na + n^2}.$$

Let $f(a) = n - \frac{a}{n-1} - \sqrt{3a^2 - 3na + n^2}$. By a direct calculation, $f''(a) < 0$, $f'(0) > 0$ and $f'(\frac{n}{2}) < 0$. Then there exists a zero root a_0 such that $f'(a_0) = 0$. So $f(a)$ is an increasing function of a on $[0, a_0]$ and is a decreasing function on $[a_0, \lfloor \frac{n}{2} \rfloor]$. Note that $1 < a_0$ and $f(1) < f(\lfloor \frac{n}{2} \rfloor)$. Hence $\rho(K_{a,b}) + \partial_2(K_{a,b}) \geq f(1) = n - \frac{1}{n-1} - \sqrt{n^2 - 3n + 3}$, with equality holding if and only if $G \cong K_{1,n-1}$. \square

Naturally, for a connected graph, we propose the following conjecture.

Conjecture 3.8 *Let $G (\not\cong K_n, K_n - e)$ be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$\rho + \partial_2 \geq \frac{n}{n-1} + \frac{n-1 - \sqrt{(n-1)^2 + 8}}{2},$$

with equality holding if and only if $G \cong K_n - 2e$, where $2e$ are two matching edges.

Furthermore, Aouchiche [11] et al. proved the following result.

Theorem 3.9 ([11]) *Let G be a connected graph on $n \geq 4$ vertices with the least distance eigenvalues ∂_n and remoteness ρ . Then*

$$\rho + \partial_n \leq 0$$

with equality if and only if $G \cong K_n$.

Lin [19] showed that $\partial_n \leq -d$ with equality if and only if G is a complete multipartite graph. Using this, we obtain the following result.

Theorem 3.10 *Let $G (\not\cong K_n)$ be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$\rho + \partial_n \leq -\frac{1}{n-1}$$

with equality holding if and only if $G \cong K_{1,n-1}$.

Proof Let G be a connected graph with diameter d . Then $2 \leq d \leq n - 1$. Clearly, by Lemma 2.2, we know that $\rho + \partial_n \leq -\frac{d^2-d}{2(n-1)} = f(d)$. Note that $f(d)$ is a strictly decreasing function on d . Hence $\rho + \partial_n \leq -\frac{d^2-d}{2(n-1)} \leq f(2) = -\frac{1}{n-1}$, with the equality holding if and only if $G \cong H_{n,2}$ and G is a complete multipartite graph. Thus $G \cong K_{1,n-1}$. \square

Next, we start to consider the lower bound on $\rho - \partial_n$.

Theorem 3.11 *Let G be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$\rho - \partial_n \geq 2$$

with equality holding if and only if $G \cong K_n$.

Proof Let G be a connected graph with d . If $d \geq 2$, then $\rho - \partial_n > \frac{d}{2} + d = \frac{3}{2}d \geq 3$. Note that $\rho(K_n) - \partial_n(K_n) = 2$, the result follows. \square

Theorem 3.12 *Let $G (\not\cong K_n)$ be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$\rho - \partial_n \geq 3 + \frac{1}{n-1}$$

with equality holding if and only if $G \cong K_n - me$, where me denotes m matching edges.

Proof Note that $\rho(K_n - me) - \partial_n(K_n - me) = \frac{n}{n-1} + 2 = 3 + \frac{1}{n-1}$. Let G be a connected graph with d . If $d \geq 3$, then

$$\rho - \partial_n > \frac{d}{2} + d = \frac{3}{2}d \geq \frac{9}{2} > 3 + \frac{1}{n-1}.$$

If $d = 2$, then $\rho = 2 - \frac{\delta}{n-1}$, and hence $\rho - \partial_n \geq 2 - \frac{\delta}{n-1} + d = 4 - \frac{\delta}{n-1}$. If $\delta \leq n - 3$, we have $\rho - \partial_n > 3 + \frac{1}{n-1}$. If $\delta = n - 2$, then $G \cong K_n - me$, where me are m matching edges. This completes the proof. \square

4 Remoteness and distance Laplacian eigenvalues of a graph

In this section, we mainly investigate the relations between remoteness and the distance Laplacian eigenvalues of a graph.

Theorem 4.1 *Let G be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$n + 1 \leq \rho + \partial_1^L \leq \rho(P_n) + \partial_1^L(P_n)$$

with the left equality holding if and only if $G \cong K_n$ and the right equality holding if and only if $G \cong P_n$.

Proof By Lemma 2.6, we have $\rho + \partial_1^L \geq \rho + n \geq n + 1$, with the left equality holding if and only if $G \cong K_n$.

Similar to the proof of Theorem 3.3, we have $\rho(G) \leq \rho(P_n)$ with the equality if and only if $G \cong P_n$. Meanwhile, it was shown in [20] that $\partial_1^L(G) \leq \partial_1^L(P_n)$ with equality holding if and only if $G \cong P_n$. Hence the right side of the theorem holds. \square

Theorem 4.2 *Let $G (\not\cong K_n)$ be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$\rho + \partial_1^L \geq n + \frac{1}{n-1} + 3$$

with equality holding if and only if $G \cong K_n - me$, where me denotes m matching edges.

Proof For $G \cong K_n - me$, note that $K_n - me$ is a complete multipartite graph, by Lemma 2.8, then $\rho + \partial_1^L = \frac{n}{n-1} + n + 2 = n + \frac{1}{n-1} + 3$. Let G be a connected graph with d . If $d \geq 3$, by Lemma 2.7, we obtain

$$\rho + \partial_1^L \geq \rho + D_1 + \frac{D_1}{n-1} = (n+1)\rho > \frac{d}{2}(n+1) \geq \frac{3}{2}(n+1) > n + \frac{1}{n-1} + 3.$$

If $d = 2$, then $\rho + \partial_1^L \geq (n+1)\rho = (n+1)(2 - \frac{\delta}{n-1})$. If $\delta \leq n - 3$, then

$$\rho + \partial_1^L \geq (n+1)\left(2 - \frac{n-3}{n-1}\right) = n + \frac{4}{n-1} + 3 > n + \frac{1}{n-1} + 3.$$

If $\delta = n - 2$, then $G \cong K_n - me$. The result follows. \square

Theorem 4.3 *Let G be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$\partial_1^L - \rho \geq n - 1$$

with equality holding if and only if $G \cong K_n$.

Proof By Lemma 2.7, we have

$$\partial_1^L - \rho \geq D_1 + 1 - \rho = (n-2)\rho + 1 \geq n - 1,$$

with equality holding if and only if $G \cong K_n$. \square

Theorem 4.4 *Let $G (\not\cong K_n)$ be a connected graph of order $n \geq 5$ with remoteness ρ . Then*

$$\partial_1^L - \rho \geq n + 1 - \frac{1}{n-1}$$

with equality if and only if $G \cong K_n - me$, where $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$.

Proof If $G \cong K_n - me$, then $\partial_1^L - \rho = n + 2 - \frac{n}{n-1} = n + 1 - \frac{1}{n-1}$. Let G be a connected graph with d . If $d \geq 3$, then

$$\partial_1^L - \rho \geq D_1 + \frac{D_1}{n-1} - \rho = (n-1)\rho > \frac{3}{2}(n-1) > n + 1 - \frac{1}{n-1}.$$

If $d = 2$, then $\partial_1^L - \rho \geq (n-1)\rho = (n-1)(2 - \frac{\delta}{n-1})$. If $\delta \leq n-3$, then $\partial_1^L - \rho \geq (n-1)(2 - \frac{n-3}{n-1}) = n + 1 > n + 1 - \frac{1}{n-1}$. If $\delta = n-2$, then $G \cong K_n - me$. This completes the proof. \square

5 Remoteness and distance signless Laplacian eigenvalues of a graph

First, we consider the bounds on $2\rho + \partial_1^Q$.

Theorem 5.1 *Let G be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$2n \leq 2\rho + \partial_1^Q \leq 2\rho(P_n) + \partial_1^Q(P_n)$$

with the left equality holding if and only if $G \cong K_n$ and the right equality holding if and only if $G \cong P_n$.

Proof For $G \cong K_n$, then $2\rho + \partial_1^Q = 2n - 2 + 2 = 2n$.

If $d \geq 3$, we know that $\partial_1^Q(G) > 2n - 4 + 2d \geq 2n + 2$, thus $\partial_1^Q(G) + 2\rho > 2n$.

If $d = 2$, we know that $\rho = 2 - \frac{\delta}{n-1}$ and $\partial_1^Q \geq \frac{4W}{n} \geq 2(n+1) - \frac{4(1+\delta)}{n}$. Then $\partial_1^Q(G) + 2\rho = 2n + 2 - \frac{4(1+\delta)}{n} + 4 - \frac{2\delta}{n-1} = 2n + 6 - \frac{4(1+\delta)}{n} - \frac{2\delta}{n-1} > 2n$.

It was shown in [20] that $\partial_1^Q(G) \leq \partial_1^Q(P_n)$ with equality holding if and only if $G \cong P_n$. Hence the right side of the theorem holds. \square

Theorem 5.2 *Let $G (\not\cong K_n)$ be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$2\rho + \partial_1^Q \geq \frac{3n-2 + \sqrt{(n-2)^2 + 16}}{2} + \frac{2n}{n-1}$$

with equality holding if and only if $G \cong K_n - e$.

Proof For $G \cong K_n - e$, by Lemma 2.12, then

$$\begin{aligned} 2\rho + \partial_1^Q &= \frac{3n-2 + \sqrt{(n-2)^2 + 16}}{2} + \frac{2n}{n-1} < \frac{3n-2 + n-2 + \frac{8}{n-2}}{2} + \frac{2n}{n-1} \\ &= 2n + \frac{4}{n-2} + \frac{2}{n-1}. \end{aligned}$$

Let G be a connected graph with d . If $d \geq 3$, by Lemma 2.10, then

$$2\rho + \partial_1^Q > 2n - 4 + 2d + d \geq 2n + 5 > 2n + \frac{4}{n-2} + \frac{2}{n-1}.$$

If $d = 2$, then $\rho = 2 - \frac{\delta}{n-1}$. By Lemma 2.11, $\partial_1^Q \geq \frac{4W}{n} \geq 2n + 2 - \frac{4(1+\delta)}{n}$. Thus

$$2\rho + \partial_1^Q \geq 2n + 2 - \frac{4(1+\delta)}{n} + 4 - \frac{2\delta}{n-1} = 2n + 6 - \frac{4(1+\delta)}{n} - \frac{2\delta}{n-1}.$$

If $\delta \leq n - 3$, then $2\rho + \partial_1^Q \geq 2n + 6 - \frac{4(n-2)}{n} - \frac{2(n-3)}{n-1} = 2n + \frac{8}{n} + \frac{4}{n-1} > 2n + \frac{4}{n-2} + \frac{2}{n-1}$. If $\delta = n - 2$, we know that $\rho(G) = \rho(K_n - e)$ and $\partial_1^Q(G) > \partial_1^Q(G + e)$, thus $2\rho(G) + \partial_1^Q(G) \geq 2\rho(K_n - e) + \partial_1^Q(K_n - e)$ with equality if and only if $G \cong K_n - e$. \square

Theorem 5.3 *Let $G (\not\cong K_n, K_n - e)$ be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$2\rho + \partial_1^Q \geq \frac{3n - 2 + \sqrt{(n - 2)^2 + 32}}{2} + \frac{2n}{n - 1}$$

with equality holding if and only if $G \cong K_n - 2e$, where $2e$ are two matching edges.

Proof For $G \cong K_n - 2e$, then

$$\begin{aligned} 2\rho + \partial_1^Q &= \frac{3n - 2 + \sqrt{(n - 2)^2 + 32}}{2} + \frac{2n}{n - 1} \\ &< \frac{3n - 2 + n - 2 + \frac{16}{n-2}}{2} + \frac{2n}{n - 1} \\ &= 2n + \frac{8}{n - 2} + \frac{2}{n - 1}. \end{aligned}$$

Let G be a connected graph with d . If $d \geq 3$, then

$$2\rho + \partial_1^Q(G) > 2n - 4 + 2d + d \geq 2n + 5 > 2n + \frac{8}{n - 2} + \frac{2}{n - 1}.$$

If $d = 2$, then $\rho = 2 - \frac{\delta}{n-1}$, $\partial_1^Q(G) \geq 2n + 2 - \frac{4(1+\delta)}{n}$, and thus

$$\partial_1^Q(G) + 2\rho \geq 2n + 2 - \frac{4(1 + \delta)}{n} + 4 - \frac{2\delta}{n - 1} = 2n + 6 - \frac{4(1 + \delta)}{n} - \frac{2\delta}{n - 1}.$$

If $\delta \leq n - 3$, then $\partial_1^Q(G) + 2\rho \geq 2n + 6 - \frac{4(n-2)}{n} - \frac{2(n-3)}{n-1} = 2n + \frac{8}{n} + \frac{4}{n-1} > 2n + \frac{8}{n-2} + \frac{2}{n-1}$. If $\delta = n - 2$, we have $\rho(G) = \rho(K_n - 2e)$ and $\partial_1^Q(G) > \partial_1^Q(G + e)$, thus $\partial_1^Q(G) + 2\rho(G) \geq \partial_1^Q(K_n - 2e) + 2\rho(K_n - 2e)$ with equality holding if and only if $G \cong K_n - 2e$. This completes the proof. \square

Next, we prove the lower bound on $\partial_1^Q - 2\rho$.

Theorem 5.4 *Let G be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$\partial_1^Q - 2\rho \geq 2n - 4$$

with equality holding if and only if $G \cong K_n$.

Proof By Lemma 2.11, we have

$$\begin{aligned} \partial_1^Q &\geq \frac{4W(G)}{n} = \frac{2 \sum_{v \in V(G)} \text{Tr}(v)}{n} = \frac{2(2(n - 1)\rho + \sum_{2 \leq i < j \leq n} d_{ij})}{n} \\ &\geq \frac{4(n - 1)\rho + 2(n - 1)(n - 2)}{n} \end{aligned}$$

$$\begin{aligned}
 &= \frac{2n\rho + 2(n-2)\rho + 2n(n-2) - 2(n-2)}{n} \\
 &= 2\rho + 2(n-2) + \frac{2(n-2)(\rho-1)}{n} \geq 2\rho + 2n - 4.
 \end{aligned}$$

Thus $\partial_1^Q - 2\rho \geq 2n - 4$ with equality if and only if $G \cong K_n$. □

Theorem 5.5 *Let $G (\not\cong K_n)$ be a connected graph of order $n \geq 4$ with remoteness ρ . Then*

$$\partial_1^Q - 2\rho \geq \frac{3n - 2 + \sqrt{(n-2)^2 + 16}}{2} - \frac{2n}{n-1}$$

with equality holding if and only if $G \cong K_n - e$.

Proof For $G \cong K_n - e$, then we have

$$\partial_1^Q - 2\rho = \frac{3n - 2 + \sqrt{(n-2)^2 + 16}}{2} - \frac{2n}{n-1} < 2n - 4 + \frac{4}{n-2} - \frac{2}{n-1}.$$

Let G be a connected graph with d . If $d \geq 3$, by Lemmas 2.2 and 2.10, then

$$\partial_1^Q - 2\rho > 2n - 4 + 2d - 2d + \frac{d^2 - d}{n-1} \geq 2n - 4 + \frac{6}{n-1} > 2n - 4 + \frac{4}{n-2} - \frac{2}{n-1}.$$

If $d = 2$, then

$$\partial_1^Q - 2\rho \geq 2n - 2 - \frac{4(1 + \delta)}{n} + \frac{2\delta}{n-1} = 2n - 2 - \frac{4}{n} - \left(\frac{4}{n} - \frac{2}{n-1}\right)\delta.$$

If $\delta \leq n - 3$, then

$$\partial_1^Q - 2\rho \geq 2n - 2 - \frac{4}{n} - \left(\frac{4}{n} - \frac{2}{n-1}\right)(n-3) > 2n - 4 + \frac{4}{n-2} - \frac{2}{n-1}.$$

If $\delta = n - 2$, we know that $\rho(G) = \rho(K_n - e)$ and $\partial_1^Q(G) > \partial_1^Q(G + e)$, thus $\partial_1^Q(G) - 2\rho(G) \geq \partial_1^Q(K_n - e) - 2\rho(K_n - e)$ with equality holding if and only if $G \cong K_n - e$. This completes the proof. □

6 Conclusions

We give lower bounds on $\rho + \partial_1, \rho - \partial_n, \rho + \partial_1^L, \partial_1^L - \rho, 2\rho + \partial_1^Q$ and $\partial_1^Q - 2\rho$ and the corresponding extremal graphs are characterized. Considering the distance, distance Laplacian and distance signless Laplacian eigenvalues of a graph is still an interesting and important problem.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

H CJ carried out the proofs of main results in the manuscript. H CJ and HYS participated in the design of the study and drafted the manuscripts. All the authors read and approved the final manuscripts.

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