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Oscillation criteria for second order Emden-Fowler functional differential equations of neutral type

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Abstract

In this article, some new oscillation criterion for the second order Emden-Fowler functional differential equation of neutral type

$$(r(t)|z'(t)|^{\alpha-1}z'(t))' + q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = 0,$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $\alpha > 0$ and $\beta > 0$ are established. Our results improve some well-known results which were published recently in the literature. Some illustrating examples are also provided to show the importance of our results.

MSC: 34C10; 34K11

Keywords: Emden-Fowler equation; oscillation criterion; Riccati method

1 Introduction

In this article we are concerned with the second order Emden-Fowler functional differential equation of neutral type of the form

$$(r(t)|z'(t)|^{\alpha-1}z'(t))' + q(t)|x(\sigma(t))|^{\beta-1}x(\sigma(t)) = 0, \quad t \geq t_0, \tag{1}$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $\alpha > 0$ and $\beta > 0$ are constants.

In the following we assume that

(A_1) $r(t) \in C^1([t_0,\infty),R)$, $r(t) > 0$, $r'(t) \geq 0$;
(A_2) $p(t), q(t) \in C([t_0,\infty),R)$, $0 \leq p(t) \leq 1$, $q(t) \geq 0$;
(A_3) $\tau(t) \in C([t_0,\infty),R)$, $\tau(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \infty$;
(A_4) $\sigma(t) \in C^1([t_0,\infty),R)$, $\sigma(t) > 0$, $\sigma'(t) > 0$, $\sigma(t) \leq t$, $\lim_{t\to\infty}\sigma(t) = \infty$.

A function $x(t) \in C^1([t_0,\infty),R)$, $T_x \geq t_0$, is called a solution of equation (1) if it satisfies the property $r(t)|z'(t)|^{\alpha-1}z'(t) \in C^1([T_x,\infty),R)$ and equation (1) on $[T_x,\infty)$. In this article we only consider the nontrivial solutions of equation (1), which ensure $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of equation (1) is said to be oscillatory if it has arbitrarily large zero point on $[T_0,\infty)$; otherwise, it is called nonoscillatory. Moreover, equation (1) is said to be oscillatory if all its solutions are oscillatory.

Recently, there were a large number of papers devoted to the oscillation of the delay and neutral differential equations. We refer the reader to [1–20].

Dzurina and Stavroulakis [1] studied the oscillation for the second order half-linear differential equations

$$(E_1):\quad \bigl(r(t)\bigl|u'(t)\bigr|^{\alpha-1}u'(t)\bigr)' + p(t)\bigl|u\bigl(\tau(t)\bigr)\bigr|^{\alpha-1}u\bigl(\tau(t)\bigr) = 0, \tag{2}$$

and established some sufficient conditions for oscillation of (2).

Sun and Meng [2] examined further the oscillation of (2). Their results hold for the condition

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,dt = \infty \tag{3}$$

or

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,dt < \infty, \tag{4}$$

which improves the results of Dzurina and Stavroulakis [1].

In 2008, Erbe *et al.* [3] studied the oscillatory behavior of the following second order neutral Emden-Fowler differential equation:

$$(E_2):\quad \bigl(a(t)\bigl[x(t) + p(t)x(t-\tau)\bigr]'\bigr)' + q(t)\bigl|x\bigl(\sigma(t)\bigr)\bigr|^{\alpha-1}x\bigl(\sigma(t)\bigr) = 0, \tag{5}$$

where $\int_{t_0}^{\infty}\frac{1}{a(t)}\,dt = \infty$ and $\alpha > 1$. Some new oscillation criteria of Philos type were established for equation (5).

In 2011, Li *et al.* [4] considered further the oscillation criteria for equation (5), where $\int_{t_0}^{\infty}\frac{1}{a(t)}\,dt < \infty$ and $\alpha \geq 1$. In fact, equations (2) and (5) cannot be contained in each other. So in 2012, Liu *et al.* [5] considered the oscillation criteria for second order generalized Emden-Fowler equation (1) for the condition $\alpha \geq \beta > 0$.

In 2015, Zeng *et al.* [6] used the Riccati transformation technique to get some new oscillation criterion for equation (1) under the condition $\alpha \geq \beta > 0$ or $\beta \geq \alpha > 0$, which improves the related results reported in [5].

Now in this article we shall apply the generalized Riccati inequality to study of the oscillation criteria of equation (1) under a more general case, namely, for all $\alpha > 0$ and $\beta > 0$.

2 Results and proofs

Theorem 1 *Suppose that* (A_1)-(A_4) *and* (3) *hold. If there exists a function* $\rho(t) \in C^1([t_0,\infty), (0,\infty))$ *such that*

$$\int_{t_0}^{\infty}\left[\rho(t)Q_1(t) - \frac{(\rho'(t))^{\lambda+1}r(\lambda(t))}{(\lambda+1)^{\lambda+1}(m\rho(t)\sigma'(t))^{\lambda}}\right]dt = \infty, \tag{6}$$

where

$$Q_1(t) = q(t)\bigl(1 - p\bigl(\sigma(t)\bigr)\bigr)^{\beta}, \qquad \lambda = \min\{\alpha,\beta\}, \tag{7}$$

$$\lambda(t) = \begin{cases} \sigma(t), & \beta \geq \alpha, \\ t, & \alpha > \beta, \end{cases} \quad \textit{and} \quad m = \begin{cases} 1, & \alpha = \beta, \\ 0 < m \leq 1, & \alpha \neq \beta. \end{cases} \tag{8}$$

Then equation (1) *is oscillatory for all* $\alpha > 0$ *and* $\beta > 0$.

Proof Suppose that equation (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we assume that $x(t) > 0$ for all large t. The case of $x(t) < 0$ can be treated by the same method. In view of (A_3) and (A_4), there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ on $[t_1, \infty)$. It follows that $z(t) = x(t) + p(t)x(\tau(t)) \geq x(t) > 0$. It follows from (1) that

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' = -q(t)x^{\beta}\left(\sigma(t)\right) \leq 0, \quad t \geq t_1. \tag{9}$$

Hence, $r(t)|z'(t)|^{\alpha-1}z'(t)$ is nonincreasing on $[t_1, \infty)$.

We now claim that

$$z'(t) > 0, \quad t \geq t_2 \geq t_1. \tag{10}$$

If not, then there exists $t_3 \in [t_2, \infty)$ such that $z'(t_3) < 0$. Hence

$$r(t)\left|z'(t)\right|^{\alpha-1}z'(t) \leq \left(r(t_3)\left|z'(t_3)\right|^{\alpha-1}z'(t_3)\right)' = -c < 0, \quad t \geq t_3,$$

which implies that

$$z'(t) \leq -\left(\frac{c}{r(t)}\right)^{\frac{1}{\alpha}}. \tag{11}$$

Integrating (11) from t_3 to t, we find from (3) that

$$z(t) \leq z(t_3) - c^{\frac{1}{\alpha}} \int_{t_3}^{t} \frac{1}{r^{\frac{1}{\alpha}}(s)}\,ds \to -\infty, \quad \text{as } t \to \infty,$$

which implies that $z(t)$ is eventually negative. This contradicts $z(t) > 0$. Hence our claim is true.

Now we have

$$x(t) \geq \left(1 - p(t)\right)z(t), \quad t \geq T \geq t_3. \tag{12}$$

This inequality together with (1) and (10) suggest

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + Q_1(t)z^{\beta}\left(\sigma(t)\right) \leq 0, \quad t \geq T. \tag{13}$$

We set

$$w(t) = \frac{r(t)(z'(t))^{\alpha}}{z^{\beta}(\sigma(t))}, \quad t \geq T. \tag{14}$$

Then $w(t) > 0$. By (13) and (14) we have

$$w'(t) \leq -Q_1(t) - \frac{\beta\sigma'(t)z'(\sigma(t))r(t)(z'(t))^\alpha}{z^{\beta+1}(\sigma(t))}. \tag{15}$$

In the following we consider three cases for (15):

Case (i): $\alpha = \beta$. In view of the inequality $r^{\frac{1}{\alpha}}(t)z'(t) \leq r^{\frac{1}{\alpha}}(\sigma(t))z'(\sigma(t))$ and (15) we see that

$$w'(t) \leq -Q_1(t) - \frac{\alpha\sigma'(t)}{r^{\frac{1}{\alpha}}(\sigma(t))}w^{\frac{\alpha+1}{\alpha}}(t), \quad t \geq T. \tag{16}$$

Case (ii): $\alpha < \beta$. Noting that $z(\sigma(t))$ is increasing on $[T, \infty)$, then there exists a constant $m_1 > 0$ such that

$$\begin{aligned} w'(t) &\leq -Q_1(t) - \frac{\beta\sigma'(t)}{r^{\frac{1}{\alpha}}(\sigma(t))}\bigl[z(\sigma(t))\bigr]^{\frac{\beta-\alpha}{\alpha}} w^{\frac{\alpha+1}{\alpha}}(t) \\ &\leq -Q_1(t) - \frac{\alpha\sigma'(t)m_1}{r^{\frac{1}{\alpha}}(\sigma(t))}w^{\frac{\alpha+1}{\alpha}}(t). \end{aligned} \tag{17}$$

Case (iii): $\alpha > \beta$. From $(r(t)(z'(t))^\alpha)' \leq 0$ and $r'(t) \geq 0$, we get $z''(t) \leq 0$, then $z'(t)$ is non-increasing. Thus, there exists a positive constant m_2, such that

$$\begin{aligned} w'(t) &\leq -Q_1(t) - \frac{\beta\sigma'(t)}{r^{\frac{1}{\beta}}(t)}\bigl[z'(t)\bigr]^{\frac{\beta-\alpha}{\beta}} w^{\frac{\beta+1}{\beta}}(t) \\ &\leq -Q_1(t) - \frac{\beta\sigma'(t)m_2}{r^{\frac{1}{\beta}}(t)}w^{\frac{\beta+1}{\beta}}(t). \end{aligned} \tag{18}$$

Combining (16)-(18), we obtain for any $\alpha > 0$, $\beta > 0$,

$$w'(t) \leq -Q_1(t) - \frac{\lambda m\sigma'(t)}{r^{\frac{1}{\lambda}}(\lambda(t))}w^{\frac{\lambda+1}{\lambda}}(t), \quad t \geq T. \tag{19}$$

Multiplying (19) by $\rho(t)$ and integrating it from T to t, we obtain

$$\int_T^t \rho(s)Q_1(s)\,ds \leq \rho(T)w(T) + \int_T^t \left[\rho'(s)w(s) - \frac{\lambda m\rho(s)\sigma'(s)}{r^{\frac{1}{\lambda}}(\lambda(s))}w^{\frac{\lambda+1}{\lambda}}(s)\right] ds. \tag{20}$$

By the inequality

$$Aw - Bw^{1+\frac{1}{\lambda}} \leq \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}A^{\lambda+1}B^{-\lambda}, \tag{21}$$

where $A \geq 0$, $B > 0$, $w \geq 0$, and $\lambda > 0$, we now can rewrite inequality (20) as

$$\int_T^t \left[\rho(s)Q_1(s) - \frac{(\rho'(s))^{\lambda+1}r(\lambda(s))}{(\lambda+1)^{\lambda+1}(m\rho(s)\sigma'(s))^\lambda}\right] ds \leq \rho(T)w(T). \tag{22}$$

Letting $t \to \infty$ in the above inequality, we get a contradiction with (6). Hence the theorem is proved. □

Remark 1 Theorems 1-5 of [1], Theorem 1 of [2] and [7] hold only for equation (1) with $p(t) = 0$ and $\alpha = \beta$. Theorem 2.1 of [5] (or [6]) holds only for equation (1) with $\alpha \geq \beta$, and Theorem 3.1 of [6] holds only for equation (1) with $\beta \geq \alpha$. Hence our theorem improves and unifies the above results.

In the following, we shall use the generalized Riccati technique and the integral averaging technique to show a new Philos type oscillation criterion for equation (1).

For this purpose, we first define the sets $D_0 = (t,s)$: $t > s \geq t_0$ and $D = (t,s)$: $t \geq s \geq t_0$. We introduce a general class of parameter functions $H : D \to R$, which have continuous partial derivatives on D with respect to the second variable and satisfy

(H_1): $H(t,t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for all $(t,s) \in D_0$,
(H_2): $-\frac{\partial H(t,s)}{\partial s} \geq 0$ for all $(t,s) \in D$.

Suppose that $h : D_0 \to R$ is a continuous function and $\rho \in C^1([t_0,\infty), R^+)$, such that

(H_3): $\frac{\partial H(t,s)}{\partial s} + \frac{\rho'(s)}{\rho(s)} H(t,s) = -h(t,s) H^{\frac{\lambda}{\lambda+1}}(t,s)$ for all $(t,s) \in D_0$.

Theorem 2 *Suppose that* (A_1)-(A_4) *and* (3) *hold. Suppose there exist functions* H, h, *and* ρ, *such that* (H_1), (H_2), *and* (H_3) *hold. Further assume for all sufficiently large* T,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)Q_1(s) - \frac{\rho(s) r(\lambda(s)) |h(t,s)|^{\lambda+1}}{(\lambda+1)^{\lambda+1}(m\sigma'(s))^{\lambda}} \right] ds = \infty, \tag{23}$$

where λ, m, $Q_1(t)$, *and* $\lambda(t)$ *are given in* (7) *and* (8). *Then equation* (1) *is oscillatory for all* $\alpha > 0$ *and* $\beta > 0$.

Proof Similar to Theorem 1, we assume that there exists a solution x of equation (1) such that $x(t) > 0$ on $[t_1,\infty)$ for some $t_1 \geq t_0$. Multiplying both sides of (19) by $H(t,s)\rho(s)$ and integrating from T to t, we have, for all $t \geq T \geq t_1$,

$$\begin{aligned} &\int_T^t H(t,s)\rho(s)Q_1(s)\,ds \\ &\quad\leq -\int_T^t H(t,s)\rho(s)w'(s)\,ds - \int_T^t H(t,s)\rho(s)\xi(s) w^{\frac{\lambda+1}{\lambda}}(s)\,ds, \end{aligned} \tag{24}$$

where w is defined by (14) and

$$\xi(s) = \frac{\lambda m\sigma'(s)}{r^{\frac{1}{\lambda}}(\lambda(s))}. \tag{25}$$

Applying integration by parts, from (H_3) and (24) we have

$$\begin{aligned} &\int_T^t H(t,s)\rho(s)Q_1(s)\,ds \\ &\quad\leq H(t,T)\rho(T)w(T) \\ &\qquad + \int_T^t \left[\left|h(t,s)\right| H^{\frac{\lambda}{\lambda+1}}(t,s)\rho(s)w(s) - H(t,s)\rho(s)\xi(s) w^{\frac{\lambda+1}{\lambda}}(s) \right] ds. \end{aligned} \tag{26}$$

Using the inequality (21), combining (26) and (25), we get

$$\begin{aligned} &\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)Q_1(s)-\frac{\rho(s)r(\lambda(s))|h(t,s)|^{\lambda+1}}{(\lambda+1)^{\lambda+1}(m\sigma'(s))^{\lambda}}\right]ds \\ &\quad\le \rho(T)w(T). \end{aligned} \tag{27}$$

It follows that

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)Q_1(s)-\frac{\rho(s)r(\lambda(s))|h(t,s)|^{\lambda+1}}{(\lambda+1)^{\lambda+1}(m\sigma'(s))^{\lambda}}\right]ds<\infty,$$

which contradicts the assumption (23). Therefore, equation (1) is oscillatory. Now we finish the proof of this theorem. □

Corollary 1 *Theorem* 2 *remains true if the condition* (23) *is replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t H(t,s)\rho(s)Q_1(s)\,ds=\infty \tag{28}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \frac{\rho(s)r(\lambda(s))}{(\sigma'(s))^{\lambda}}\big|h(t,s)\big|^{\lambda+1}\,ds<\infty. \tag{29}$$

Notice that by choosing specific functions ρ and H, it is possible to derive several oscillation criteria for equation (1) and its special cases, the half-linear equation (2) and the Emden-Fowler equation (5).

Remark 2 Theorem 2.1 of [3] holds only for equation (1) with $\alpha = 1$ and $\beta > 1$, Theorem 2.2 of [5] holds only for equation (1) with $\alpha \ge \beta$, Theorem 5 of [7] holds only for equation (1) with $\beta \ge \alpha$. Hence, Theorem 2 improves and unifies above oscillation criteria.

Note that the theorems above hold for the condition (3), now we consider the case for (4). In order to do this we first define

$$\pi(t)=\int_t^\infty \frac{1}{r^{\frac{1}{\alpha}}(s)}\,ds \tag{30}$$

and

$$Q_2(t)=q(t)\big(1-p(t)\big)^{\beta}. \tag{31}$$

Then we have the following.

Theorem 3 *Suppose that* (A_1)-(A_4) *and* (4) *hold. Suppose*

$$p'(t)\ge 0,\qquad \tau'(t)>0,\qquad \sigma(t)\le\tau(t), \tag{32}$$

and (6) *are satisfied. Further assume there exists a constant* $K > 0$ *such that*

$$\int_{t_0}^{\infty} \left[\pi^{\mu}(t) Q_2(t) - \frac{K(r(t))^{1-\frac{\mu+1}{\alpha}}}{\pi(t)} \right] dt = \infty, \tag{33}$$

where $\mu = \max\{\alpha, \beta\}$. *Then equation* (1) *is oscillatory for all* $\alpha > 0$ *and* $\beta > 0$.

Proof As in Theorem 1 we assume that there exists a solution x of equation (1) such that $x(t) > 0$ on $[t_1, \infty)$ for some $t_1 \geq t_0$. Then we have

$$\left(r(t) \left| z'(t) \right|^{\alpha-1} z'(t) \right)' \leq 0, \quad t \geq t_1, \tag{34}$$

from which we see that there exist two possible cases of the sign of $z'(t)$. If $z'(t) > 0$, then we come back to the proof of Theorem 1, and we can get a contradiction with (6). If $z'(t) < 0$, we have

$$z'(t) = x'(t) + p'(t) x\big(\tau(t)\big) + p(t) x'\big(\tau(t)\big) \tau'(t) < 0.$$

Therefore, $x'(t) < 0$ and

$$z(t) \leq x\big(\tau(t)\big) + p(t) x\big(\tau(t)\big) = \big(1 + p(t)\big) x\big(\tau(t)\big), \tag{35}$$

from which together with (32) we have

$$x\big(\sigma(t)\big) \geq x\big(\tau(t)\big) \geq \frac{z(t)}{1 + p(t)} \geq \big(1 - p(t)\big) z(t). \tag{36}$$

Then equation (1) becomes

$$\left(r(t) \big(-z'(t) \big)^{\alpha} \right)' - Q_2(t) z^{\beta}(t) \geq 0, \quad t \geq t_1. \tag{37}$$

Now we define a function ν by

$$\nu(t) = \frac{r(t)(-z'(t))^{\alpha}}{z^{\beta}(t)}, \quad t \geq t_1. \tag{38}$$

Obviously, $\nu(t) > 0$ for $t \geq t_1$. It follows from (34) that $r(t)|z'(t)|^{\alpha-1} z'(t)$ is nonincreasing. Hence we get

$$r^{\frac{1}{\alpha}}(s) z'(s) \leq r^{\frac{1}{\alpha}}(t) z'(t), \quad s \geq t \geq t_1.$$

Dividing the above inequality by $r^{\frac{1}{\alpha}}(t)$ and integrating it from t to l, we have

$$z(l) \leq z(t) + r^{\frac{1}{\alpha}}(t) z'(t) \int_t^l \frac{1}{r^{\frac{1}{\alpha}}(s)} \, ds, \quad l \geq t \geq t_1.$$

It follows that

$$z(t) \geq r^{\frac{1}{\alpha}}(t) \big(-z'(t) \big) \pi(t). \tag{39}$$

Moreover, we have

$$z^{\alpha}(t) \geq \left[r^{\frac{1}{\alpha}}(t)\left(-z'(t)\right)\right]^{\alpha}\pi^{\alpha}(t). \tag{40}$$

By (38) and the fact $z'(t) < 0$ we find that there exists a constant $c_1 > 0$ such that

$$c_1 \geq z^{\alpha-\beta}(t) \geq \pi^{\alpha}(t)v(t) > 0, \quad \text{for } \alpha > \beta. \tag{41}$$

On the other hand, from (39) we get

$$z^{\beta}(t) \geq r^{\frac{\beta}{\alpha}}(t)\left(-z'(t)\right)^{\beta}\pi^{\beta}(t).$$

Hence we have

$$1 \geq \frac{r^{\frac{\beta}{\alpha}}(t)(-z'(t))^{\beta}}{z^{\beta}(t)}\pi^{\beta}(t).$$

Since $r^{\frac{1}{\alpha}}(t)(-z'(t))$ is nondecreasing, then there exists a constant $c_2 > 0$ such that

$$c_2 \geq \left[r^{\frac{1}{\alpha}}(t)\left(-z'(t)\right)\right]^{\alpha-\beta} \geq \pi^{\beta}(t)v(t) > 0, \quad \text{for } \beta > \alpha. \tag{42}$$

Next, differentiating (38) yields

$$v'(t) \geq Q_2(t) + \frac{\beta r(t)(-z'(t))^{\alpha+1}}{z^{\beta+1}(t)}, \quad t \geq t_1. \tag{43}$$

We consider the following three cases:

Case (i): $\alpha > \beta$. In this case, since $z(t)$ is decreasing, it follows from (43) that

$$\begin{aligned} v'(t) &\geq Q_2(t) + \frac{\beta}{r^{\frac{1}{\alpha}}(t)}\left[z(t)\right]^{\frac{\beta-\alpha}{\alpha}}v^{\frac{\alpha+1}{\alpha}}(t) \\ &\geq Q_2(t) + \frac{c_1}{r^{\frac{1}{\alpha}}(t)}v^{\frac{\alpha+1}{\alpha}}(t), \quad t \geq t_1, \end{aligned} \tag{44}$$

where $c_1 = \beta[z(t_1)]^{\frac{\beta-\alpha}{\alpha}}$.

Case (ii): $\alpha = \beta$. In this case, we see that $[z(t)]^{\frac{\beta-\alpha}{\alpha}} = 1$, then (43) becomes

$$v'(t) \geq Q_2(t) + \frac{\alpha}{r^{\frac{1}{\alpha}(t)}}v^{\frac{\alpha+1}{\alpha}}(t), \quad t \geq t_1. \tag{45}$$

Case (iii): $\alpha < \beta$. By the inequality (37) we have $(r(t)(-z'(t))^{\alpha})' \geq 0$, from which together with $r'(t) \geq 0$ we find that $z''(t) \leq 0$. Hence we get $z'(t) \leq z'(t_2)$ for $t \geq t_2$. Now the inequality (43) suggests that

$$\begin{aligned} v'(t) &\geq Q_2(t) + \frac{\beta}{r^{\frac{1}{\beta}}(t)}\left[-z'(t)\right]^{\frac{\beta-\alpha}{\beta}}v^{\frac{\beta+1}{\beta}}(t) \\ &\geq Q_2(t) + \frac{c_2}{r^{\frac{1}{\beta}}(t)}v^{\frac{\beta+1}{\beta}}(t), \quad t \geq t_2 \geq t_1, \end{aligned} \tag{46}$$

where $c_2 = \beta[-z'(t_2)]^{\frac{\beta-\alpha}{\beta}}$.

Combining (44)-(46), we obtain

$$v'(t) \geq Q_2(t) + \frac{c}{r^{\frac{1}{\mu}}(t)} v^{\frac{\mu+1}{\mu}}(t), \quad t \geq t_2, \tag{47}$$

where $\mu = \max\{\alpha, \beta\}$ and

$$c = \begin{cases} \alpha, & \alpha = \beta, \\ c = \min\{c_1, c_2\}, & \alpha \neq \beta. \end{cases}$$

Multiplying (47) by $\pi^{\mu}(t)$ and integrating it from t_2 to t, we have

$$\int_{t_2}^{t} \pi^{\mu}(s) Q_2(s)\,ds \leq \int_{t_2}^{t} \pi^{\mu}(s) v'(s)\,ds - c\int_{t_2}^{t} \frac{\pi^{\mu}(s)}{r^{\frac{1}{\mu}}(s)} v^{\frac{\mu+1}{\mu}}(s)\,ds. \tag{48}$$

Using integration by parts, the inequality (48) yields

$$\begin{aligned} \int_{t_2}^{t} \pi^{\mu}(s) Q_2(s)\,ds &\leq \pi^{\mu}(t)v(t) - \pi^{\mu}(t_2)v(t_2) \\ &\quad + \int_{t_2}^{t} \pi^{\mu}(s) \left[\frac{\mu v(s)}{\pi(s) r^{\frac{1}{\alpha}}(s)} - \frac{c v^{\frac{\mu+1}{\mu}}(s)}{r^{\frac{1}{\mu}}(s)} \right] ds. \end{aligned} \tag{49}$$

By the inequality (21), we get

$$\frac{\mu v(s)}{\pi(s) r^{\frac{1}{\alpha}}(s)} - \frac{c v^{\frac{\mu+1}{\mu}}(s)}{r^{\frac{1}{\mu}}(s)} \leq \frac{\mu^{2\mu+1}}{c^{\mu}(\mu+1)^{\mu+1}} \frac{(r(s))^{1-\frac{\mu+1}{\alpha}}}{\pi^{\mu+1}(s)}. \tag{50}$$

Substituting in (49), we obtain

$$\int_{t_2}^{t} \left[\pi^{\mu}(s) Q_2(s) - \frac{K(r(s))^{1-\frac{\mu+1}{\alpha}}}{\pi(s)} \right] ds \leq \pi^{\mu}(t)v(t) - \pi^{\mu}(t_2)v(t_2), \tag{51}$$

where $K = \frac{\mu^{2\mu+1}}{c^{\mu}(\mu+1)^{\mu+1}}$.

In view of (41) and (42), we have

$$\int_{t_2}^{t} \left[\pi^{\mu}(s) Q_2(s) - \frac{K(r(s))^{1-\frac{\mu+1}{\alpha}}}{\pi(s)} \right] ds \leq c_1 + c_2,$$

which contradicts condition (33). Then equation (1) is oscillatory for all $\alpha > 0$ and $\beta > 0$. Hence the theorem is proved. □

Remark 3 Theorem 2.2 of [2] holds only for equation (1) with $p(t) = 0$ and $\alpha = \beta$, Theorem 2.1-2.3 of [4] hold only for $\alpha = 1$ and $\beta \geq 1$, Theorem 2.5 of [5] and Theorem 2.3 of [6] hold only for $\alpha \geq \beta$. Our Theorem 3 holds for equation (1) with all $\alpha > 0$ and $\beta > 0$.

3 Examples

Now in this section we shall give two examples to illustrate our results.

Example 1 Consider the differential equation

$$(E_3):\quad \left(\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + \frac{1}{t^{1+\frac{\lambda}{2}}}\left|x(t-2)\right|^{\beta-1}x(t-2) = 0, \quad \text{for } t \in [2,\infty), \tag{52}$$

where $z(t) = x(t) + \frac{1}{2}x(t-1)$, $\alpha > 0$, $\beta > 0$, and $\lambda = \min\{\alpha,\beta\}$. Noticing that $r(t) = 1$, $p(t) = \frac{1}{2}$, $q(t) = \frac{1}{t^{1+\frac{\lambda}{2}}}$, $\tau(t) = t-1$, $\sigma(t) = t-2$ and

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,dt = \infty,$$

then (3) is satisfied. Since $Q_1(t) = (\frac{1}{2})^{\beta}\frac{1}{t^{1+\frac{\lambda}{2}}}$, $\lambda > 0$, then $\int_{t_0}^{\infty} Q_1(t)\,dt < \infty$. In order to apply Theorem 1, it remains to discuss condition (6). If we choose $\rho(t) = t^{\frac{\lambda}{2}}$, we have

$$\begin{aligned}
&\int_{t_0}^{\infty}\left[\rho(t)Q_1(t) - \frac{(\rho'(t))^{\lambda+1}r(\lambda(t))}{(\lambda+1)^{\lambda+1}(m\rho(t)\sigma'(t))^{\lambda}}\right]dt \\
&\quad = \int_{2}^{\infty}\left[\frac{(\frac{1}{2})^{\beta}}{t} - \left(\frac{\lambda}{2(\lambda+1)}\right)^{\lambda+1}\frac{1}{m^{\lambda}}\frac{1}{t^{1+\frac{\lambda}{2}}}\right]dt = \infty.
\end{aligned}$$

Then by Theorem 1, every solution of (52) is oscillatory for all $\alpha > 0$ and $\beta > 0$.

Example 2 Consider the differential equation

$$(E_4):\quad \left(t^{2\alpha}\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + t^{2\alpha+\beta}\left|x(t-3)\right|^{\beta-1}x(t-3) = 0, \quad \text{for } t \in [3,\infty), \tag{53}$$

where $z(t) = x(t) + \frac{1}{3}x(t-2)$, $\alpha > 0$, $\beta > 0$. Observe $r(t) = t^{2\alpha}$, $p(t) = \frac{1}{3}$, $q(t) = t^{2\alpha+\beta}$, $\tau(t) = t-2$, $\sigma(t) = t-3$ and

$$\int_{3}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,dt = \int_{3}^{\infty}\frac{1}{t^2}\,dt < \infty.$$

Then (4) is satisfied. It is clear that (32) is satisfied. Since $p(t) = \frac{1}{3}$, we have

$$Q_1(t) = Q_2(t) = \left(\frac{2}{3}\right)^{\beta} t^{2\alpha+\beta}.$$

If we choose $\rho(t) = 1$, then condition (6) is satisfied. To apply Theorem 3, it remains to discuss the condition (33); in view of $\pi(t) = \frac{1}{t}$, we have

$$\begin{aligned}
&\int_{t_0}^{\infty}\left[\pi^{\mu}(t)Q_2(t) - \frac{K(r(t))^{\frac{\alpha-\mu-1}{\alpha}}}{\pi(t)}\right]dt \\
&\quad = \int_{t_0}^{\infty}\left[\left(\frac{2}{3}\right)^{\beta} t^{2\alpha+\beta-\mu} - Kt^{2\alpha-2\mu-1}\right]dt
\end{aligned}$$

$$= \begin{cases} \int_3^\infty [(\frac{2}{3})^\beta t^{\alpha+\beta} - \frac{K}{t}]\,dt = \infty, & \mu = \alpha; \\ \int_3^\infty t^{2\alpha} [(\frac{2}{3})^\beta - \frac{K}{t^{1+2\beta}}]\,dt = \infty, & \mu = \beta. \end{cases}$$

Then by Theorem 3, (53) is oscillatory for all $\alpha > 0$ and $\beta > 0$.

Remark 4 We note that the results obtained for those equations in [1–20] cannot deal with (52) and (53).

Competing interests
The authors declare they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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