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Dynamical behaviors of stochastic local Swift-Hohenberg equation on unbounded domain

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Abstract

In this paper, we first study the deterministic Swift-Hohenberg equation on a bounded domain. After obtaining some a priori estimates by the uniform Gronwall inequality, we prove the existence of an attractor by the Sobolev compact embeddings. Then, we consider the stochastic Swift-Hohenberg equation driven by additive noise on an unbounded domain and prove that the random dynamical system is asymptotically compact by uniform a priori estimates for the far-field values of the solution, which implies the existence of a random attractor for the random dynamical system associated with the stochastic Swift-Hohenberg equation.

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Keywords: stochastic Swift-Hohenberg equation; random attractor; additive noise; asymptotically compact

1 Introduction

The Swift-Hohenberg (SH) equation describes the pattern formation in fluid layers confined between horizontal well-conducting boundaries, which was proposed by Swift and Hohenberg [1] as a model for the convective instability in the Rayleigh-Bénard convection. The localized one-dimensional version of the model is as follows:

$$u_t = \mu u - (1 + \partial_{xx})^2 u - u^3. \tag{1.1}$$

There have been some results for the local one-dimensional SH equation [2–5]. The Swift-Hohenberg equation has featured in different branches of physics, ranging from hydrodynamics to nonlinear optics, such as the Taylor-Couette flow [6, 7], study of lasers [8], and so on. The dynamical properties of the Swift-Hohenberg equation, such as the existence of a global attractor, are important for the studies of pattern formation, which ensure the stability of pattern formation and provide a mathematical foundation for the study of pattern dynamics. The authors considered the asymptotic dynamical difference between the nonlocal and local Swift-Hohenberg models in [9]. Recently, the global attractor, the stability of stationary solution, and pattern selections of the modified local Swift-Hohenberg equation have been investigated; see the references [10, 11].

After consulting the literature, we have found that there are few results about the existence of a global attractor for the local Swift-Hohenberg equation. Therefore, the existence of a global attractor for the local Swift-Hohenberg equation on a bounded domain will be given in Section 4.

In fact, when the distance from the change of stability is sufficiently small, or Rayleigh number is near thermal equilibrium, the influence of small noise or molecular noise is detected in various convection experiments [12–14]. It is difficult to stabilize the control parameters (e.g. temperature in the Rayleigh-Bénard convection) to the precision of the noise strength, which is extremely small in the case of thermal fluctuations. When the effects of thermal fluctuations on the onset of convective motion into the Bénard system are considered, the local stochastic Swift-Hohenberg equation with additive noise [1] is proposed:

$$u_t = \mu u - (1+\partial_{xx})^2 u - u^3 + \sigma\xi. \tag{1.2}$$

Furthermore, it is also allowed to consider the effects of small possible noise from μ. So a local stochastic Swift-Hohenberg equation with multiplicative noise [15] arises:

$$u_t = \mu u - (1+\partial_{xx})^2 u - u^3 + \sigma u \circ \xi, \tag{1.3}$$

where $\sigma > 0$, and $\xi = \frac{dW}{dt}$ is the generalized derivative of a real-valued one-dimensional Brownian motion $W(t)$.

There are few results on the dynamical behavior of the stochastic Swift-Hohenberg equation. Recently, some authors [16] proved the dynamics and invariant manifolds for a nonlocal stochastic Swift-Hohenberg equation. Here, the existence of a global random attractor for the stochastic Swift-Hohenberg equation with additive noise on an unbounded domain is considered. This is the main motivation of this paper. The Sobolev embeddings are no longer compact on unbounded domains. In order to overcome this difficulty, we use the method developed in [17] to prove the existence of a random attractor in the entire space. Specifically, the stochastic equation is transformed into the corresponding deterministic equation with random parameter by making use of the Ornstein-Uhlenbeck transform, and the asymptotic compactness of the random dynamical system is proved by using uniform a priori estimates for the far-field values of the solution via a truncation function.

Remark 1.1 For bounded case, "*a*" can be an arbitrary constant. After obtaining some a priori estimates of the solution, we can prove the existence of a global attractor by applying the compact Sobolev embeddings. In the case of unbounded domain, since the Sobolev inequality $\|u\|_{L^2} \leq C\|u\|_{L^4}$ is invalid, we need the additional condition $a > 5$ to prove the existence of a random attractor using the current method.

In this paper, we consider the two-dimensional stochastic Swift-Hohenberg equation with additive noise

$$du + \left(\Delta^2 u + 2\Delta u + au + u^3\right) dt = \Phi(x)\, d\omega(t), \quad t \in R^+, \tag{1.4}$$

with initial condition

$$u(x,0) = u_0(x), \tag{1.5}$$

The paper is organized as follows. In Section 2, we recall some definitions and known results concerning global random attractors. In Section 3, we introduce the O-U transformer and transform (1.4)-(1.5) into a continuous stochastic dynamical system. In Section 4, we prove the existence of a global attractor for the corresponding deterministic dynamical system on an bounded domain. In Section 5, we obtain some uniform a priori estimates for the far-field values of the solution by the technique of a cut-off function. In Section 6, we prove the asymptotic compactness of the random dynamical system and thus deduce the existence of a global random attractor for the stochastic Swift-Hohenberg equation.

2 Preliminaries

We recall some basic concepts related to random attractors. Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with Borel σ-algebra $\mathscr{B}(X)$ endowed with the distance d, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We also consider the mappings $S(t,s;\omega) : X \to X$, $-\infty < s \leq t < \infty$, parameterized by ω. There exists a group θ_t, $t \in \mathbb{R}$, of measure-preserving transformations of $(\Omega, \mathcal{F}, \mathbb{P})$ such that, for all $s < t$ and $x \in X$,

$$S(t,s;\omega)x = S(t-s,0;\theta_s\omega)x, \quad \mathbb{P}\text{-a.e.},$$

where $\omega(t)$ is from the two-sided Wiener space $C_0(\mathbb{R};X)$ of continuous functions with values in a Banach space X, equal to 0 at $t = 0$. In this case, θ_t is defined as

$$(\theta_t\omega)(s) = \omega(t+s) - \omega(t), \quad s,t \in \mathbb{R}.$$

Definition 2.1 Let $t \in \mathbb{R}$ and $\omega \in \Omega$. A stochastic dynamical system with time t on a complete separable metric space (X,d) with Borel σ-algebra $\mathscr{B}$ over $\{\theta_t\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable map

$$S(t,s;\omega) : X \to X, \quad -\infty < s \leq t < \infty,$$

such that $S(0,0;\omega) = id$ and $S(t,0;\omega) = S(t,s;\omega)S(s,0;\omega)$ for all $t,s \in \mathbb{R}$ and all $\omega \in \Omega$.

Definition 2.2 Given $t \in \mathbb{R}$ and $\omega \in \Omega$, $K(t,\omega) \subset X$ is called an attracting set if for all bounded sets $B \subset X$,

$$d\big(S(t,s;\omega)B, K(t,\omega)\big) \to 0, \quad s \to -\infty,$$

where $d(A,B)$ is the semidistance defined by

$$d(A,B) = \sup_{x\in A} \inf_{y\in B} d(x,y).$$

Definition 2.3 A family $A(\omega)$ $(\omega \in \Omega)$ of the closed subsets of X is measurable if for all $x \in X$, the mapping $\omega \mapsto d(A(\omega),x)$ is measurable.

Definition 2.4 Define the random omega limit set of a bounded set $B \subset X$ at time t as

$$A(B,t;\omega) = \bigcap_{T<t} \overline{\bigcup_{s<T} S(t,s;\omega)B}.$$

Definition 2.5 Let $S(t,s;\omega)_{t\geq s,\omega\in\Omega}$ be a stochastic dynamical system, and let $A(\omega)$ be a stochastic set satisfying the following conditions:

(1) It is the minimal closed set such that, for $t \in \mathbb{R}$ and $B \subset X$,

$$d\big(S(t,s;\omega)B, A(\omega)\big) \to 0, \quad s \to -\infty.$$

Then $A(\omega)$ is said to attract B (B is a deterministic set).

(2) $A(\omega)$ is the largest compact measurable set that is invariant in sense that

$$S(t,s;\omega)A(\theta_s\omega) = A(\theta_t\omega), \quad s \leq t.$$

Then $A(\omega)$ is said to be the random attractor.

Theorem 2.6 (see [18, 19]) *Let $S(t,s;\omega)_{t\geq s,\omega\in\Omega}$ be a stochastic dynamical system satisfying the following conditions*:

(i) *$S(t,r;\omega)S(r,s;\omega)x = S(t,s;\omega)x$ for all $s \leq r \leq t$ and $x \in X$,*

(ii) *$S(t,s;\omega)$ is continuous in X for all $s \leq t$,*

(iii) *for all $s < t$ and $x \in X$, the mapping $\omega \mapsto S(t,s;\omega)x$ is measurable from $(\Omega,\mathcal{F})$ to $(X,\mathscr{B}(X))$,*

(iv) *for all $t \in \mathbb{R}, x \in X$, and $\mathbb{P}$-a.e. ω, the mapping $s \mapsto S(t,s;\omega)x$ is right continuous at any point.*

Assume that there exists a group θ_t, $t \in \mathbb{R}$, of measure-preserving mappings such that

$$S(t,s;\omega)x = S(t-s,0;\theta_s\omega)x, \quad \mathbb{P}\text{-a.e.}$$

and for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists a compact attracting set $K(\omega)$ at time 0. We set $\Lambda(\omega) = \overline{\bigcup_{B\subset X} A(B,\omega)}$, where the union is taken over all bounded subsets of X, and $A(B,\omega)$ is given by

$$A(B,0;\omega) = \bigcap_{T<0} \overline{\bigcup_{s<T} S(0,s;\omega)B}.$$

Then $\Lambda(\omega)$ is a random attractor.

Theorem 2.7 (Uniform Gronwall lemma; see [20]) *Let g, h, y be three positive locally integrable functions on (t_0,∞) satisfying*

$$y'(t) \leq gy + h \quad \text{for } t \geq t_0,$$

$$\int_t^{t+r} g(s)\,ds \leq a_1, \qquad \int_t^{t+r} h(s)\,ds \leq a_2, \qquad \int_t^{t+r} y(s)\,ds \leq a_3 \quad \text{for all } t \geq t_0,$$

where r, a_1, a_2, a_3 are positive constants. Then

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right)\exp(a_1) \quad \textit{for all } t \geq t_0.$$

For the convenience of the following contents, we introduce some functional spaces and some notations.

$L^q(D)$ is the Lebesgue space with norm $\|\cdot\|_{L^q}$. The inner product on $L^2(D)$ is denoted by

$$(f,g) = \int_D fg\,dx.$$

Particularly, $\|u\|_{L^\infty} = \operatorname{ess\,sup}_{x\in D}|u(x)|$ for $q = \infty$.

$H^\sigma(D)$ is the Sobolev space $\{u \in L^2(D), D^k u \in L^2(D), k \leq \sigma\}$ with norm $\|\cdot\|_{H^\sigma}$. If $D = R^2$, then we use the same notations. In particular, $H_0^2(D)$ is the Sobolev space $\{u \in L^2(D), D^k u \in L^2(D), k \leq 2, \Delta u|_{\partial D} = 0\}$.

$\mathscr{C}(I, X)$ is the space of continuous functions from the interval I to X.

For notational simplicity, C is a generic constant and may assume various values from line to line.

3 The hydrodynamical equation with additive noise

Here we show that there is a continuous random dynamical system $(S(t,s;\omega); L^2(R^2))$ generated by the stochastic local Swift-Hohenberg equation on R^2

$$du + \left(\Delta^2 u + 2\Delta u + au + u^3\right)dt = \sum_{i=1}^{m}\Phi_i(x)\,d\omega_i(t) \tag{3.1}$$

with the initial condition

$$u(x,s) = u_s(x), \tag{3.2}$$

where $\Phi_i(x)$ is a given smooth enough function on R^2. We need to convert the stochastic equation with random additive term into a deterministic equation with random parameter.

Now, we introduce the Ornstein-Uhlenbeck process

$$z_i(t) = \int_{-\infty}^{t} e^{-A(t-s)}\,d\omega_i(s),$$

where $A = \Delta^2$ is a positive operator. It is well known and easy to check that $z_i(\cdot)$ is a stationary process with $\mathbb{P}$-a.e. continuous trajectories.

Putting $z(t) = \sum_{i=1}^{m}\Phi_i(x)z_i(t)$, we have

$$dz + \Delta^2 z\,dt = \sum_{i=1}^{m}\Phi_i(x)\,d\omega_i(t).$$

In addition, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have that

$$\sum_{i=1}^{m}\left(\left|z_i(t)\right|^2 + \left|z_i(t)\right|^p\right) \tag{3.3}$$

at most polynomically grows as $t \to -\infty$, where $p \geq 2$ (see [17]).

To study (3.1)-(3.2), it is usual to translate the known $v = u - z$ (z has the above form) and obtain the following equation:

$$\frac{dv}{dt} + \Delta^2 v + 2\Delta v + 2\Delta z + av + az + v^3 + z^3 + 3v^2 z + 3vz^2 = 0, \tag{3.4}$$

$$v(s,\omega) = v_s = u_s - z(s,\omega). \tag{3.5}$$

By the Galerkin method one can show that, for all $v_s \in L^2(R^2)$, system (3.4)-(3.5) has a unique solution $v \in \mathscr{C}(s,T;L^2(R^2)) \cap L^2(s,T;H^2(R^2))$ with $v(s) = v_s$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. It is obvious that there is a continuous stochastic dynamical system $(S(t,s;\omega);L^2(R^2))$ generated by the stochastic local Swift-Hohenberg equation with additive noise.

4 Global attractor on a bounded domain

For completeness, we first consider the following initial-boundary value problem for the deterministic local Swift-Hohenberg equation on a bounded domain:

$$u_t + \Delta^2 u + 2\Delta u + au + u^3 = 0, \quad x \in D, t \in R^+, \tag{4.1}$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in D, \tag{4.2}$$

and boundary conditions

$$u|_{\partial D} = \Delta u|_{\partial D} = 0, \quad x \in \partial D, \tag{4.3}$$

where D is an open connected bounded domain in R^2, and a is an arbitrary constant.

Theorem 4.1 *For any $u_0(x) \in H_0^2(D)$, there exists a unique, globally defined solution $V(t)u_0 = \sigma(u_0,t)$ in $H_0^2(D)$ of system* (4.1)-(4.3), *and $V(t)$ is a semigroup on $H_0^2(D)$. Moreover, the semigroup is point dissipative in $H_0^2(D)$ and compact in $H_0^2(D)$ for $t > 0$. Hence, system* (4.1)-(4.3) *has a global attractor in $H_0^2(D)$.*

Proof (1) Taking the inner product of (4.1) with u in H, we can obtain that

$$\frac{1}{2}\frac{d}{dt}\left\|u(t)\right\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 - 2\|\nabla u\|_{L^2}^2 + a\|u\|_{L^2}^2 + \|u\|_{L^4}^4 = 0. \tag{4.4}$$

Adding the term $\|u(t)\|_{L^2}^2$ to the above equation, we have

$$\frac{1}{2}\frac{d}{dt}\left\|u(t)\right\|_{L^2}^2 + \left\|u(t)\right\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|u\|_{L^4}^4 = (-a+1)\|u\|_{L^2}^2 + 2\|\nabla u\|_{L^2}^2. \tag{4.5}$$

For the first term on the right-hand side of (4.5), using the inequality $\|u\|_{L^2} \le C\|u\|_{L^4}$ and ε-Young inequality, we can easily deduce that

$$(-a+1)\|u\|_{L^2}^2 \le C\big(|a|+1\big)\|u\|_{L^4}^2 \le \frac{1}{3}\|u\|_{L^4}^4 + C.$$

For the second term on the right-hand side of (4.5), we have the estimate

$$\begin{aligned}\|\nabla u\|_{L^2}^2 &\le C\|u\|_{L^4}^{\frac{4}{3}}\|\Delta u\|_{L^2}^{\frac{2}{3}} \\ &\le \frac{1}{3}\|u\|_{L^4}^4 + C\|\Delta u\|_{L^2} \\ &\le \frac{1}{3}\|u\|_{L^4}^4 + \frac{1}{4}\|\Delta u\|_{L^2}^2 + C,\end{aligned}$$

where we applied the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^2} \le C\|u\|_{L^4}^{\frac{2}{3}}\|\Delta u\|_{L^2}^{\frac{1}{3}}$$

and the ε-Young inequality.

Combining the above consequences, we get the inequality

$$\frac{d}{dt}\big\|u(t)\big\|_{L^2}^2 + 2\big\|u(t)\big\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \le C. \tag{4.6}$$

By the Gronwall inequality, we have

$$\big\|u(t)\big\|_{L^2}^2 \le e^{-2t}\big\|u(0)\big\|_{L^2}^2 + C. \tag{4.7}$$

Now we integrate with respect to s from t to $t+1$ on the both sides of (4.6) and deduce that

$$\big\|u(t+1)\big\|_{L^2}^2 + \int_t^{t+1}\big\|\Delta u(s)\big\|_{L^2}^2\,ds \le C + \big\|u(t)\big\|_{L^2}^2.$$

By (4.7) we have

$$\int_t^{t+1}\big\|\Delta u(s)\big\|_{L^2}^2\,ds \le C.$$

(2) Taking the inner product of (4.1) with $\Delta^2 u$ in H, we have

$$\frac{1}{2}\frac{d}{dt}\big\|\Delta u(t)\big\|_{L^2}^2 + \big\|\Delta^2 u\big\|_{L^2}^2 + 2\big(\Delta u, \Delta^2 u\big) + a\|\Delta u\|_{L^2}^2 + \big(u^3, \Delta^2 u\big) = 0. \tag{4.8}$$

Using the Hölder inequality and ε-Young inequality, we then easily obtain

$$2\big|\big(\Delta u, \Delta^2 u\big)\big| \le 2\|\Delta u\|_{L^2}\big\|\Delta^2 u\big\|_{L^2} \le \frac{\varepsilon}{2}\big\|\Delta^2 u\big\|_{L^2}^2 + \frac{2}{\varepsilon}\|\Delta u\|_{L^2}^2. \tag{4.9}$$

For the estimate of $(u^3, \Delta^2 u)$, by the Gagliardo-Nirenberg inequality

$$\|u\|_{L^6} \le C\|u\|_{L^2}^{\frac{5}{6}}\big\|\Delta^2 u\big\|_{L^2}^{\frac{1}{6}},$$

the Hölder inequality, and ε-Young inequality, we obtain the estimate

$$\left|\left(u^3,\Delta^2 u\right)\right| \le \|u\|_{L^6}^3 \left\|\Delta^2 u\right\|_{L^2} \le C\|u\|_{L^2}^{\frac{5}{2}} \left\|\Delta^2 u\right\|_{L^2}^{\frac{3}{2}} \le \epsilon \left\|\Delta^2 u\right\|_{L^2}^2 + C, \tag{4.10}$$

where the last inequality is owing to the boundedness of $\|u\|_{L^2}$.

Plugging (4.9)-(4.10) into (4.8) yields

$$\begin{aligned}\frac{d}{dt}\left\|\Delta u(t)\right\|_{L^2}^2 + 2\left\|\Delta^2 u\right\|_{L^2}^2 + 2a\|\Delta u\|_{L^2}^2 &\le 4\left|\left(\Delta u,\Delta^2 u\right)\right| + 2\left|\left(u^3,\Delta^2 u\right)\right| \\ &\le 3\epsilon\left\|\Delta^2 u\right\|_{L^2}^2 + \frac{4}{\epsilon}\|\Delta u\|_{L^2}^2 + C,\end{aligned}$$

Letting $\epsilon = \frac{2}{3}$, we can easily write it as

$$\frac{d}{dt}\left\|\Delta u(t)\right\|_{L^2}^2 + (2a-6)\|\Delta u\|_{L^2}^2 \le C. \tag{4.11}$$

By (4.10) and Theorem 2.7 we get that

$$\begin{aligned}&\int_t^{t+1} g(s)\,ds = \int_t^{t+1} (2a-6)\,ds = 2a-6, \qquad \int_t^{t+1} C\,ds = C, \\ &\int_t^{t+1} \left\|\Delta u(s)\right\|_{L^2}^2\,ds \le C.\end{aligned}$$

Then

$$\left\|\Delta u(t+1)\right\|_{L^2}^2 \le C. \tag{4.12}$$

Hence, (4.12) provides a uniform bound for $\|\Delta u(t+1)\|_{L^2}$. Thus, the existence of an absorbing ball in the $H_0^2(D)$ is proved. This implies that $V(t)$ is point dissipative in $H_0^2(D)$. Similarly to [10], we can obtain that $V(t)$ is compact for $t>0$; for details, we refer to Theorem 3.2 in [10]. Hence, Theorem 4.1 implies the existence of a global attractor for problem (4.1)-(4.3) in the space $H_0^2(D)$. □

5 Uniform estimates on an unbounded domain

In this section, we derive uniform estimates of a solution for system (3.4)-(3.5) on R^2 for the purpose of proving the existence of a bounded random absorbing set and the asymptotic compactness of the stochastic dynamical system associated with the equation. In particular, we show that the tails of the solution, that is, the norms of solutions evaluated at large values of $|x|$, are uniformly small as $t \to 0$.

Lemma 5.1 *Let $\Phi_i(x) \in H^2(R^2)$, $a>5$, $v(t)$ be the solution of system* (3.4)-(3.5). *Then, for any given $\eta>0$ and $u_s \in H$ satisfying $\|u_s\| \le \eta$, there exist random radii $r_0(w)$, $r_1(w)$, $r_2(w)$ and $s_0(w) \le -1$ such that for all $s \le s_0(w)$, the following inequalities hold $\mathbb{P}$-a.e.:*

$$\begin{aligned}&\left\|v(t)\right\|^2 \le r_0(w), \qquad \left\|u(t)\right\|^2 \le r_1(w), \\ &\int_{-1}^0 \left\|\Delta v(\tau)\right\|_{L^2}^2\,d\tau \le r_2(w).\end{aligned}$$

Proof Taking the inner product of (3.4) with $v(t)$ in H, we can obtain that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + 2(\Delta v, v) + (2\Delta z, v) + a\|v\|_{L^2}^2 + a(z, v) + \|v\|_{L^4}^4 + (z^3, v) + (3v^2 z, v) + (3vz^2, v) = 0. \tag{5.1}$$

Applying the Hölder inequality and ε-Young inequality, we have

$$|2(\Delta v, v)| \le 2\|\Delta v\|_{L^2}\|v\|_{L^2} \le \frac{1}{4}\|\Delta v\|_{L^2}^2 + 4\|v\|_{L^2}^2.$$

Utilizing similar arguments, the following three estimates are also valid:

$$|(2\Delta z, v)| \le 2\|\Delta z\|_{L^2}\|v\|_{L^2} \le \frac{1}{3}\|v\|_{L^2}^2 + C\|\Delta z\|_{L^2}^2,$$
$$|a(z, v)| \le a\|z\|_{L^2}\|v\|_{L^2} \le \frac{1}{3}\|v\|_{L^2}^2 + C\|z\|_{L^2}^2,$$

and

$$|(z^3, v)| \le \|z\|_{L^6}^3\|v\|_{L^2} \le \frac{1}{3}\|v\|_{L^2}^2 + C\|z\|_{L^6}^6.$$

For the last term on the left-hand side of (5.1), applying the Gagliardo-Nirenberg inequality

$$\|v\|_{L^6} \le C\|v\|_{L^4}^{\frac{8}{9}}\|\Delta v\|_{L^2}^{\frac{1}{9}},$$

the Hölder inequality, and ε-Young inequality, we obtain that

$$\begin{aligned}|(3v^2 z, v)| &\le 3\|v\|_{L^6}^3\|z\|_{L^2} \\ &\le C\|v\|_{L^4}^{\frac{8}{3}}\|\Delta v\|_{L^2}^{\frac{1}{3}}\|z\|_{L^2} \\ &\le \frac{1}{2}\|v\|_{L^4}^4 + C\|\Delta v\|_{L^2}\|z\|_{L^2}^3 \\ &\le \frac{1}{2}\|v\|_{L^4}^4 + \frac{1}{4}\|\Delta v\|_{L^2}^2 + C\|z\|_{L^2}^6.\end{aligned}$$

Because $(3vz^2, v) = 3\int_\Omega v^2 z^2 \ge 0$, we drop $(3vz^2, v)$ on the left-hand side of (5.1).

Combining the above estimates, we obtain the inequality

$$\frac{d}{dt}\|v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 + 2(a-5)\|v\|_{L^2}^2 + \|v\|_{L^4}^4 \le C\|\Delta z\|_{L^2}^2 + C\|z\|_{L^2}^2 + C\|z\|_{L^6}^6 + C\|z\|_{L^2}^6.$$

Let $F(t) = C(\|\Delta z\|_{L^2}^2 + \|z\|_{L^2}^2 + \|z\|_{L^6}^6 + \|z\|_{L^2}^6)$. Then

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 + 2(a-5)\|v(t)\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 \le F(t). \tag{5.2}$$

By the Gronwall inequality for $s \le -1$ and $t \in [-1,0]$ we have

$$\begin{aligned}
\|v(t)\|_{L^2}^2 &\le e^{-2(a-5)(t-s)}\|v(s)\|_{L^2}^2 + \int_s^t F(\sigma)e^{-2(a-5)(t-\sigma)}\,d\sigma \\
&\le e^{2(a-5)(s+1)}\|v(s)\|_{L^2}^2 + \int_{-\infty}^0 F(\sigma)e^{2(a-5)(s+1)}\,d\sigma,
\end{aligned} \tag{5.3}$$

where $F(\sigma)$ grows at most polynomially as $\sigma \to -\infty$ $\mathbb{P}$-a.e. Because $F(\sigma)$ is multiplied by a function that decays exponentially, the integral in (5.3) converges. Then, given $\eta > 0$, we can choose $s_0(\omega) \le -1$ depending only on ω such that

$$e^{2(a-5)s}\eta^2 \le 1, \quad \forall s_0(\omega) \le -1.$$

We can deduce from (5.3) that, for all $s \le s_0(\omega)$,

$$\|v(t)\|_{L^2}^2 \le r_0(\omega) = e^{2(a-5)}\Bigl(1 + \sup_{s\le -1} e^{2(a-5)s}\|z(s)\|^2\Bigr) + \int_{-\infty}^0 F(\sigma)e^{2(a-5)(s+1)}\,d\sigma.$$

Similarly, because $z(s)$ grows at polynomially as $s \to -\infty$ and $z(s)$ is multiplied by a function that decays exponentially, the term

$$\sup_{s\le -1} e^{2(a-5)s}\|z(s)\|^2$$

is bounded. Now we integrate with respect to t from -1 to 0 on the both sides of (5.2) and deduce that

$$\int_{-1}^0 \|\Delta v(t)\|_{L^2}^2\,dt \le \int_{-1}^0 F(t)\,dt + r_0(\omega) = r_2(\omega).$$

On the other hand, we can obtain

$$\|u(t)\|_{L^2}^2 \le 2\|v(t)\|_{L^2}^2 + 2\|z(t)\|_{L^2}^2 \le 2r_0(\omega) + 2\sup_{-1\le t\le 0}\|z(t)\|_{L^2}^2.$$

The proof is complete. □

Lemma 5.2 *Let* $\Phi_i(x) \in H^2(R^2)$, $a > 5$, $v(t)$ *be the solution of system* (3.4)-(3.5). *Then, for any given* $\eta > 0$ *and* $u_s \in H$ *satisfying* $\|u_s\| \le \eta$, *there exist random radii* $r_3(\omega)$, $r_4(\omega)$, $r_5(\omega)$ *such that the following inequalities hold* $\mathbb{P}$-*a.e.*:

$$\begin{aligned}
&\int_{-\infty}^0 e^{(a-5)\tau}\|v(\tau)\|_{L^2}^2\,d\tau \le r_3(\omega), \\
&\int_{-\infty}^0 e^{(a-5)\tau}\|\Delta v(\tau)\|_{L^2}^2\,d\tau \le r_4(\omega), \\
&\int_{-\infty}^0 e^{(a-5)\tau}\|v(\tau)\|_{L^4}^4\,d\tau \le r_5(\omega).
\end{aligned}$$

Proof Integrating from s ($s \leq -1$) to 0 on the both sides of (5.2), we have

$$\begin{aligned} &\|v(0)\|_{L^2}^2 + (a-5)\int_s^0 e^{(a-5)\tau}\|v(\tau)\|_{L^2}^2\,d\tau + \int_s^0 e^{(a-5)\tau}\big(\|\Delta v(\tau)\|_{L^2}^2 + \|v(\tau)\|_{L^4}^4\big)\,d\tau \\ &\quad\leq \int_s^0 e^{(a-5)\tau}F(\tau)\,d\tau + e^{(a-5)s}\|v(s)\|_{L^2}^2. \end{aligned} \tag{5.4}$$

For the first term of the right-hand side of (5.4), since $F(\tau)$ grows at most polynomially as $\tau \to -\infty$ $\mathbb{P}$-a.e., and it is multiplied by a function that decays exponentially, the integral in (5.4) converges. Then, for the second term of the right-hand side of (5.4), there exists a $s_2(\omega) \leq s_1(\omega)$ satisfying

$$e^{(a-5)s}\|v(s)\|_{L^2}^2 \leq e^{(a-5)s}\eta^2 \leq 1.$$

The proof is complete. □

Lemma 5.3 *Let $\Phi_i(x) \in H^2(R^2)$, $a > 5$, $v(t)$ be the solution of system* (3.4)-(3.5).*Then, for any given $\eta > 0$ and $u_s \in H$ satisfying $\|u_s\| \leq \eta$, there exist random radii $r_6(\omega), r_7(\omega)$, and $s_0(\omega) \leq -1$ such that for all $s \leq s_0(\omega)$, the following inequality holds $\mathbb{P}$-a.e.*:

$$\|\Delta v(t)\|^2 \leq r_6(\omega), \qquad \|\Delta u(t)\|^2 \leq r_7(\omega), \quad t \in [-1, 0].$$

Proof Taking the inner product of (3.4) with $\Delta^2 v$ in H, we can obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\Delta v(t)\|_{L^2}^2 + \|\Delta^2 v\|_{L^2}^2 + (2\Delta v, \Delta^2 v) + (2\Delta z, \Delta^2 v) + a\|\Delta v\|_{L^2}^2 + a(z, \Delta^2 v) \\ &\quad + (v^3, \Delta^2 v) + (z^3, \Delta^2 v) + (3v^2 z, \Delta^2 v) + (3vz^2, \Delta^2 v) = 0. \end{aligned} \tag{5.5}$$

Applying the Hölder inequality and ε-Young equality, we have

$$\big|(2\Delta v, \Delta^2 v)\big| \leq 2\|\Delta v\|_{L^2}\|\Delta^2 v\|_{L^2} \leq \frac{1}{4}\|\Delta^2 v\|_{L^2}^2 + 4\|\Delta v\|_{L^2}^2.$$

It can easily be shown that

$$\big|(2\Delta z, \Delta^2 v)\big| \leq 2\|\Delta z\|_{L^2}\|\Delta^2 v\|_{L^2} \leq \frac{1}{8}\|\Delta^2 v\|_{L^2}^2 + C\|\Delta z\|_{L^2}^2.$$

For the estimate of $a(z, \Delta^2 v)$, it is evident that

$$\big|a(z, \Delta^2 v)\big| \leq a\|z\|_{L^2}\|\Delta^2 v\|_{L^2} \leq \frac{1}{8}\|\Delta^2 v\|_{L^2}^2 + C\|z\|_{L^2}^2.$$

According to the Gagliardo-Nirenberg inequality

$$\|v\|_{L^6} \leq C\|\Delta^2 v\|_{L^2}^{\frac{1}{6}}\|v\|_{L^2}^{\frac{5}{6}},$$

the Hölder inequality, and ε-Young inequality, we deduce that

$$\big|(v^3, \Delta^2 v)\big| \leq \|v\|_{L^6}^3\|\Delta^2 v\|_{L^2} \leq C\|\Delta^2 v\|_{L^2}^{\frac{3}{2}}\|v\|_{L^2}^{\frac{5}{2}} \leq \frac{1}{8}\|\Delta^2 v\|_{L^2}^2 + C,$$

where the last inequality is owing to the boundedness of $\|v\|_{L^2}$. It is obvious that

$$\left|\left(z^3, \Delta^2 v\right)\right| \le \|z\|_{L^6}^3 \left\|\Delta^2 v\right\|_{L^2} \le \frac{1}{8}\left\|\Delta^2 v\right\|_{L^2}^2 + C\|z\|_{L^6}^6.$$

By the Gagliardo-Nirenberg inequality

$$\|v\|_{L^4} \le C\|v\|_{L^2}^{\frac{7}{8}} \left\|\Delta^2 v\right\|_{L^2}^{\frac{1}{8}},$$

Hölder inequality, and ε-Young inequality, we have the estimate

$$\begin{aligned}\left|\left(3v^2 z, \Delta^2 v\right)\right| &\le 3\|v\|_{L^4}^2 \|z\|_{L^\infty} \left\|\Delta^2 v\right\|_{L^2} \\ &\le C\left\|\Delta^2 v\right\|_{L^2}^{\frac{5}{4}} \|v\|_{L^2}^{\frac{7}{4}} \|z\|_{L^\infty} \\ &\le \frac{1}{8}\left\|\Delta^2 v\right\|_{L^2}^2 + C\|z\|_{L^\infty}^{\frac{8}{3}},\end{aligned}$$

where the last inequality is owing to the boundedness of $\|v\|_{L^2}$.

On account of the boundedness of $\|v(t)\|_{L^2}$, it is easy to check that

$$\left|\left(3vz^2, \Delta^2 v\right)\right| \le 3\|v\|_{L^2} \|z\|_{L^\infty}^2 \left\|\Delta^2 v\right\|_{L^2} \le \frac{1}{8}\left\|\Delta^2 v\right\|_{L^2}^2 + C\|z\|_{L^\infty}^4.$$

Combining the above estimates, we obtain the inequality

$$\frac{d}{dt}\left\|\Delta v(t)\right\|_{L^2}^2 + (2a-8)\|\Delta v\|_{L^2}^2 \le C\|\Delta z\|_{L^2}^2 + C\|z\|_{L^6}^6 + C\|z\|_{L^\infty}^{\frac{8}{3}} + C\|z\|_{L^\infty}^4 + C\|z\|_{L^2}^2 + C.$$

Let $G(t) = C(\|\Delta z\|_{L^2}^2 + \|z\|_{L^6}^6 + \|z\|_{L^\infty}^{\frac{8}{3}} + \|z\|_{L^\infty}^4 + \|z\|_{L^2}^2)$, which grows at most polynomially as $t \to -\infty$ $\mathbb{P}$-a.e. Then

$$\frac{d}{dt}\left\|\Delta v(t)\right\|_{L^2}^2 \le 8\|\Delta v\|_{L^2}^2 + G(t) + C.$$

Integrating from θ to t for any $-1 \le \theta \le t \le 0$, we have

$$\begin{aligned}\left\|\Delta v(t)\right\|_{L^2}^2 &\le e^{\int_\theta^t 8\,d\sigma}\left\|\Delta v(\theta)\right\|_{L^2}^2 + \int_\theta^t \left(G(\tau) + C\right)e^{\int_\theta^\tau 8\,d\sigma}\,d\tau \\ &\le \left(\left\|\Delta v(\theta)\right\|_{L^2}^2 + \int_{-1}^0 \left(G(\tau) + C\right)d\tau\right)e^{\int_{-1}^0 8\,d\sigma}.\end{aligned} \tag{5.6}$$

Now integrating with respect to θ on $[-1, 0]$ on both sides of (5.6), as Lemma 5.1, satisfies for all $s < s_0(\omega)$, there exists $s_0(\omega)$ such that, for all $s < s_0(\omega)$,

$$\begin{aligned}\left\|\Delta v(t)\right\|_{L^2}^2 &\le \left(\int_{-1}^0 \left\|\Delta v(\theta)\right\|_{L^2}^2 + \int_{-1}^0 \left(G(\tau) + C\right)d\tau\right)e^8 \\ &\le \left(r_2(\omega) + \int_{-1}^0 \left(G(\tau) + C\right)d\tau\right))e^8 = r_6(\omega).\end{aligned} \tag{5.7}$$

On the other hand, we can obtain

$$\|\Delta u(t)\|_{L^2}^2 \le 2\|\Delta v(t)\|_{L^2}^2 + 2\|\Delta z(t)\|_{L^2}^2 \le 2r_6(\omega) + 2\sup_{-1\le t\le 0}\|\Delta z(t)\|^2 = r_7(\omega).$$

The proof is complete. □

Lemma 5.4 *Let $\Phi_i(x) \in H^2(R^2)$, $a > 5$, $\eta > 0$ be given, and $u_s \in H$ satisfy $\|u_s\| \le \eta$. Then, for every $\epsilon > 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist $\bar{s}(\omega) \le -1$ and $\bar{k}(\epsilon) > 0$ such that, for all $s \le \bar{s}(\omega)$ and $k > \bar{k}(\epsilon)$, the solution $v(t)$ of system* (3.4)-(3.5) *with $v_s = u_s - z(s)$ satisfies*

$$\int_{|x|\ge k} |v(t)|^2\,dx \le \epsilon.$$

Proof Let $\theta(s)$ be a smooth function defined on R^+ such that $0 \le \theta(s) \le 1$ for all $s \in R^+$ and

$$\theta(s) = 0 \quad (0 \le s \le 1); \qquad \theta(s) = 1 \quad (s \ge 2).$$

Then there exists a positive constant C such that $|\theta'(s)| + |\theta''(s)| < C$ for all $s \in R^+$.

Multiplying (3.4) with $\theta(\frac{|x|^2}{k^2})v$ and then integrating the resulting identity, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx + \int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v\Delta^2 v\,dx \\
&\quad + 2\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v\Delta v\,dx + 2\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v\Delta z\,dx \\
&\quad + a\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v^2\,dx + a\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)vz\,dx + \int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v^4\,dx \\
&\quad + \int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)vz^3\,dx + 3\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v^3 z\,dx + 3\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v^2 z^2\,dx = 0. \qquad (5.8)
\end{aligned}$$

For the estimate of $\int_{R^2}\theta(\frac{|x|^2}{k^2})v\Delta^2 v\,dx$,

$$\begin{aligned}
\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v\Delta^2 v\,dx &= \int_{R^2}\Delta\left[\theta\left(\frac{|x|^2}{k^2}\right)v\right]\Delta v\,dx \\
&= \int_{R^2}\left(\Delta\left[\theta\left(\frac{|x|^2}{k^2}\right)\right]v + 2\nabla\left[\theta\left(\frac{|x|^2}{k^2}\right)\right]\nabla v + \theta\left(\frac{|x|^2}{k^2}\right)\Delta v\right)\Delta v\,dx \\
&= \int_{R^2}\theta''\left(\frac{|x|^2}{k^2}\right)\frac{4x^2}{k^4}v\Delta v\,dx + \int_{R^2}\theta'\left(\frac{|x|^2}{k^2}\right)\frac{2}{k^2}v\Delta v\,dx \\
&\quad + \int_{R^2}\theta'\left(\frac{|x|^2}{k^2}\right)\frac{2x}{k^2}\nabla v\Delta v\,dx + \int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|\Delta v|^2\,dx. \qquad (5.9)
\end{aligned}$$

For the first term on the right-hand side of (5.9), we have

$$\begin{aligned}
\left|\int_{R^2}\theta''\left(\frac{|x|^2}{k^2}\right)\frac{4x^2}{k^4}v\Delta v\,dx\right| &\le C\int_{k\le|x|\le\sqrt{2}k}\frac{4x^2}{k^4}|v||\Delta v|\,dx \\
&\le \frac{C}{k^2}\int_{k\le|x|\le\sqrt{2}k}|v||\Delta v|\,dx
\end{aligned}$$

$$\begin{aligned}
&\le \frac{C}{k^2}\int_{R^2}|v||\Delta v|\,dx \\
&\le \frac{C}{k^2}\left(\|v\|_{L^2}^2+\|\Delta v\|_{L^2}^2\right).
\end{aligned} \tag{5.10}$$

Similarly, for the second and third terms on the right-hand side of (5.9), we have

$$\left|\int_{R^2}\theta'\left(\frac{|x|^2}{k^2}\right)\frac{2}{k^2}v\Delta v\,dx\right| \le \frac{C}{k^2}\int_{R^2}|v||\Delta v|\,dx \le \frac{C}{k^2}\left(\|v\|_{L^2}^2+\|\Delta v\|_{L^2}^2\right) \tag{5.11}$$

and

$$\begin{aligned}
\left|\int_{R^2}\theta'\left(\frac{|x|^2}{k^2}\right)\frac{2x}{k^2}\nabla v\Delta v\,dx\right| &\le C\int_{k\le|x|\le\sqrt{2}k}\frac{2|x|}{k^2}|\nabla v||\Delta v|\,dx \\
&\le \frac{C}{k}\int_{k\le|x|\le\sqrt{2}k}|\nabla v||\Delta v|\,dx \\
&\le \frac{C}{k}\left(\|\nabla v\|_{L^2}^2+\|\Delta v\|_{L^2}^2\right).
\end{aligned} \tag{5.12}$$

Substituting estimates (5.10)-(5.12), it follows that

$$\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v\Delta^2 v\,dx \ge -\frac{C}{k}\left(\|v\|_{L^2}^2+\|\nabla v\|_{L^2}^2+\|\Delta v\|_{L^2}^2\right)+\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|\Delta v|^2\,dx.$$

By the Hölder inequality and ε-Young inequality we have

$$\begin{aligned}
\left|\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v\Delta v\,dx\right| &\le \left(\int_{|x|\ge k}\theta\left(\frac{|x|^2}{k^2}\right)|\Delta v|^2\,dx\right)^{\frac{1}{2}}\left(\int_{|x|\ge k}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx\right)^{\frac{1}{2}} \\
&\le \frac{1}{2}\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|\Delta v|^2\,dx+2\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\left|\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v\Delta z\,dx\right| &= \left|\int_{|x|\ge k}\theta\left(\frac{|x|^2}{k^2}\right)v\Delta z\,dx\right| \\
&\le \left(\int_{|x|\ge k}|\Delta z|^2\,dx\right)^{\frac{1}{2}}\left(\int_{|x|\ge k}\theta^2\left(\frac{|x|^2}{k^2}\right)v^2\,dx\right)^{\frac{1}{2}} \\
&\le 2\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx+C\|\Delta z\|_{L^2}^2 \\
&\le 2\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx+C\|z\|_{H^2}^2,
\end{aligned}$$

where the second inequality is owing to the boundedness of function θ.

For the estimate of $\int_{R^2}\theta(\frac{|x|^2}{k^2})vz\,dx$, we obtain

$$\begin{aligned}
\left|\int_{R^n}\theta\left(\frac{|x|^2}{k^2}\right)vz\,dx\right| &= \left|\int_{|x|\ge k}\theta\left(\frac{|x|^2}{k^2}\right)vz\,dx\right| \\
&\le \left(\int_{|x|\ge k}|z|^2\,dx\right)^{\frac{1}{2}}\left(\int_{|x|\ge k}\theta^2\left(\frac{|x|^2}{k^2}\right)v^2\,dx\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{1}{2}\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx + C\|z\|^2$$
$$\leq \frac{1}{2}\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx + C\|z\|_{H^2}^2.$$

For the estimate of $\int_{R^2}\theta(\frac{|x|^2}{k^2})vz^3\,dx$, we get

$$\begin{aligned}\left|\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)vz^3\,dx\right| &\leq \left(\int_{|x|\geq k}|z|^6\,dx\right)^{\frac{1}{2}}\left(\int_{|x|\geq k}\theta^2\left(\frac{|x|^2}{k^2}\right)v^2\,dx\right)^{\frac{1}{2}}\\ &\leq \frac{1}{2}\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx + C\|z\|^3\\ &\leq \frac{1}{2}\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx + C\|z\|_{H^2}^3.\end{aligned}$$

For the estimate of$\int_{R^2}\theta(\frac{|x|^2}{k^2})v^3z\,dx$, we have

$$\begin{aligned}\left|\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v^3z\,dx\right| &= \left(\int_{R^2}\left(\theta\left(\frac{|x|^2}{k^2}\right)v^3\right)^{\frac{4}{3}}dx\right)^{\frac{3}{4}}\left(\int_{R^2}z^4\,dx\right)^{\frac{1}{4}}\\ &\leq \left(\int_{R^2}\left(\theta\left(\frac{|x|^2}{k^2}\right)\right)^{\frac{4}{3}}v^4\,dx\right)^{\frac{3}{4}}\left(\int_{R^2}z^4\,dx\right)^{\frac{1}{4}}\\ &\leq \left(\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v^4\,dx\right)^{\frac{3}{4}}\left(\int_{R^2}z^4\,dx\right)^{\frac{1}{4}}\\ &\leq \frac{2}{3}\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v^4\,dx + C\|z\|_{L^4}^4\\ &\leq \frac{2}{3}\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)v^4\,dx + C\|z\|_{H^2}^4.\end{aligned}$$

Because $\int_{R^2}\theta(\frac{|x|^2}{k^2})v^2z^2\,dx \geq 0$, we drop it on the left-hand side of (5.8).

From the above estimates we can obtain the inequality

$$\begin{aligned}&\frac{d}{dt}\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx + \int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|\Delta v|^2\,dx + (a-5)\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)|v|^2\,dx\\ &\quad\leq \frac{C}{k}\left(\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2\right) + C\left(\|z\|_{H^2}^3 + \|z\|_{H^2}^2 + \|z\|_{H^2}^4\right).\end{aligned} \tag{5.13}$$

By the Gronwall inequality for $s \leq -1$ and $t \in [-1, 0]$ we have

$$\begin{aligned}&\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)\left|v(t)\right|^2\,dx\\ &\quad\leq e^{-\int_s^t(a-5)\,d\sigma}\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)\left|v(s)\right|^2\,dx\\ &\qquad+ \int_s^t e^{-\int_\tau^t(a-5)\,d\sigma}\left[\frac{C}{k}\left(\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2\right)\right.\\ &\qquad\left.+ C\left(\|z\|_{H^2}^2 + \|z\|_{H^2}^3 + \|z\|_{H^2}^4\right)\right]d\tau\end{aligned}$$

$$
\begin{aligned}
&\le e^{(a-5)(s+1)}\|v(s)\|^2 + e^{(a-5)}\frac{C}{k}\int_{-\infty}^{0} e^{(a-5)\tau}\left(\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2\right)d\tau \\
&\quad + e^{(a-5)}C\int_{-\infty}^{0} e^{(a-5)\tau}\left(\|z\|_{H^2}^2 + \|z\|_{H^2}^3 + \|z\|_{H^2}^4\right)d\tau. \qquad (5.14)
\end{aligned}
$$

For the first term of the right-hand side of (5.14), there exists $s_1(\omega)$ such that, for all $s \le s_1(\omega)$, $e^{(a-5)(s+1)}$ decays exponentially as $s \to -\infty$. Then there exists $\bar{s} < s_1(\omega)$ such that, for all $s \le \bar{s}$, we have

$$
e^{(a-5)(s+1)}\|v(s)\|^2 \le \frac{\epsilon}{3}. \qquad (5.15)
$$

For the second term of the right-hand side of (5.14), by the Gagliardo-Nirenberg inequality and ε-Young inequality we have

$$
\begin{aligned}
\|\nabla v\|_{L^2}^2 &\le C\|v\|_{L^4}^{\frac{4}{3}}\|\Delta v\|_{L^2}^{\frac{2}{3}} \\
&\le C\|v\|_{L^4}^4 + \|\Delta v\|_{L^2} \\
&\le C\left(\|v\|_{L^4}^4 + \|\Delta v\|_{L^2}^2 + 1\right).
\end{aligned}
$$

According to Lemma 5.2, there exists $k_1 > 0$ such that, for all $k > k_1$, we have

$$
e^{(a-5)}\frac{C}{k}\int_{-\infty}^{0} e^{(a-5)\tau}\left(\|v\|_{L^2}^2 + \|\nabla v\|_{L^4}^4 + \|\Delta v\|_{L^2}^2 + 1\right)d\tau \le \frac{\epsilon}{3}. \qquad (5.16)
$$

It follows from (5.15)-(5.16) that

$$
\int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)\left|v(t)\right|^2 dx \le \frac{2\epsilon}{3} + e^{(a-5)}C\int_{-\infty}^{0} e^{(a-5)\tau}\left(\|z\|_{H^2}^2 + \|z\|_{H^2}^3 + \|z\|_{H^2}^4\right)d\tau.
$$

Let $E(t) = C(\|z\|_{H^2}^2 + \|z\|_{H^2}^3 + \|z\|_{H^2}^4)$. When k is large enough, we have

$$
e^{(a-5)}C\int_{-\infty}^{0} e^{(a-5)\tau}\left(\|z\|_{H^2}^2 + \|z\|_{H^2}^3 + \|z\|_{H^2}^4\right)d\tau \le \frac{\epsilon}{3}
$$

because $E(t) = C(\|z\|_{H^2}^2 + \|z\|_{H^2}^3 + \|z\|_{H^2}^4) \le \epsilon\sum_{k=1}^{m}(|z_k|^2 + |z_k|^3 + |z_k|^4)$ when $|x| \ge k$ and $\sum_{k=1}^{m}(|z_k|^2 + |z_k|^3 + |z_k|^4)$ grows at most polynomially.

Then we can obtain that, for $\epsilon > 0$, there exist $\bar{s}, \bar{k} = \max\{k_1, k_2\}$ such that, for all $s \le \bar{s}$ and $k \ge \bar{k}$,

$$
\int_{|x|\ge k}\left|v(t)\right|^2 dx \le \int_{R^2}\theta\left(\frac{|x|^2}{k^2}\right)\left|v(t)\right|^2 dx \le \epsilon, \quad t \in [-1, 0]. \qquad \square
$$

Lemma 5.5 *Let $\Phi_i(x) \in H^2(R^2)$, $a > 5$, $\eta > 0$ be given, and $u_s \in H$ satisfy $\|u_s\| \le \eta$. Then, for every $\epsilon > 0$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist $s'(\omega) \le -1$ and $k'(\epsilon) > 0$ such that, for all $s \le s'(\omega)$ and $k > k'(\epsilon)$, the solution $u(t)$ satisfies the inequality*

$$
\int_{|x|\ge k}\left|u(t)\right|^2 dx \le \epsilon, \quad t \in [-1, 0].
$$

Proof Since $z(t) \in H^2$, we have

$$\int_{|x|\geq k} |z(t)|^2 \, dx \leq \frac{\epsilon}{4}, \quad t \in [-1, 0]. \tag{5.17}$$

By Lemma 5.4 we know that

$$\int_{|x|\geq k} |v(t)|^2 \, dx \leq \frac{\epsilon}{4}, \quad t \in [-1, 0]. \tag{5.18}$$

Let $\bar{s}(\omega)$ and $\bar{k}$ be the constants in Lemma 5.1. Then choosing $s' > \bar{s}$ and $k' > \bar{k}$, for all $s \leq s', k > k'$, by (5.17) and (5.18) we have

$$\begin{aligned} \int_{|x|\geq k} |u(t)|^2 \, dx &= \int_{|x|\geq k} |v(t) + z(t)|^2 \, dx \\ &\leq 2 \int_{|x|\geq k} |v(t)|^2 \, dx + 2 \int_{|x|\geq k} |z(t)|^2 \, dx \\ &\leq \epsilon, \quad t \in [-1, 0]. \end{aligned}$$

□

6 Random attractors

Motivated by these previous works, in this section, we are interested in the existence of a random attractor for the random dynamical system $S(t, s; \omega)$ associated with the stochastic Swift-Hohenberg equation on R^2.

Lemma 6.1 *Assume that* $\Phi_i(x) \in H^2(R^2)$. *Then the random dynamical system* $S(t, s; \omega)$ *is asymptotically compact in* $L^2(R^2)$; *that is, for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$, *the sequence* $u(0, s_n; \omega)$ *has a convergent subsequence in* $L^2(R^2)$, *provided that* $s_n \to -\infty$.

Proof Let $s_n \to -\infty$. Then by Lemma 5.1, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we obtain

$$\{u(0, s_n; \omega)\}_{n=1}^{\infty} \text{ is bounded in } L^2(R^2).$$

Hence, there is $\xi \in L^2(R^2)$ such that, up to a subsequence,

$$u(0, s_n; \omega) \to \xi \quad \text{weakly in } L^2(R^2) \text{ as } s_n \to -\infty. \tag{6.1}$$

Next, we prove that the weak convergence of (6.1) is in fact the strong convergence.

Given $\epsilon > 0$, by Lemma 5.5 there are $T_1(\eta, \omega, \epsilon)$ and $k(\omega, \epsilon)$ such that, for all $s < T_1$, we have

$$\int_{|x|\geq k} |u(0, s; \omega)|^2 \, dx \leq \frac{\epsilon}{3}. \tag{6.2}$$

Since $s_n \to -\infty$, there is $N_1(\eta, \omega, \epsilon,)$ such that $s_n < N_1$ for every $n > N_1$. Therefore, it follows from (6.2) that, for all $n > N_1$, we have

$$\int_{|x|\geq k} |u(0, s_n; \omega)|^2 \, dx \leq \frac{\epsilon}{3}. \tag{6.3}$$

On the other hand, by Lemma 5.3 there are $T_2(\eta,\omega)$ and $r(\omega)$ such that, for all $s < T_2$, we have

$$\|u(0,s_n;\omega)\|^2_{H^2(R^2)} \le r(\omega). \tag{6.4}$$

Denote $Q_R = \{x \in R^2 : |x| \le R\}$. By the compactness of embedding $H^2(Q_R) \hookrightarrow L^2(Q_R)$ it follows from (6.4) that there is a subsequence

$$u(0,s_n;\omega) \to \xi \quad \text{strongly in } L^2(Q_R) \text{ as } s_n \to -\infty, \tag{6.5}$$

which shows that, for given $\epsilon > 0$, there exists $N_2(\eta,\omega,\epsilon)$ such that, for all $n > N_2$,

$$\|u(0,s_n;\omega) - \xi\|^2_{L^2(Q_R)} \le \frac{\epsilon}{3}.$$

Note that $\xi \in L^2(R^2)$, so there exists $R'(\epsilon) > R$ such that

$$\int_{|x|\ge R'} |\xi|^2 \le \frac{\epsilon}{3} \tag{6.6}$$

and

$$\|u(0,s_n;\omega) - \xi\|^2_{L^2(Q_{R'})} \le \frac{\epsilon}{3}. \tag{6.7}$$

Let $N_3 = \max\{N_1, N_2\}$. By (6.3), (6.6), and (6.7) we find that, for all $n \ge N_3$, we have

$$\begin{aligned}\|u(0,s_n;\omega) - \xi\|^2_{L^2(R^2)} &\le \int_{|x|\ge R'} |u(0,s_n;\omega) - \xi|^2\,dx + \int_{|x|\le R'} |u(0,s_n;\omega) - \xi|^2\,dx,\\ &\le \epsilon,\end{aligned} \tag{6.8}$$

which shows that

$$u(0,s_n;\omega) \to \xi \quad \text{strongly in } L^2(R^2) \text{ as } s_n \to -\infty, \tag{6.9}$$

as desired. The proof is complete. □

We are now in a position to present our main result, the existence of a random attractor for $S(t,s;\omega)$ in $L^2(R^2)$.

Theorem 6.2 *Let $\Phi(x) \in H^2(R^2)$ and $a > 5$. Then the random dynamical system $S(t,s;\omega)$ has a unique random attractor in $L^2(R^2)$.*

Proof Notice that $S(t,s;\omega)$ has a closed random absorbing set in $H^1(R^2)$ by Lemmas 5.1 and 5.3 and is asymptotically compact in $L^2(R^2)$ by Lemma 6.1. Hence, the existence of a unique random attractor for $S(t,s;\omega)$ follows from Theorem 2.6 immediately. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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