Wen et al. *Journal of Inequalities and Applications* (2015) 2015:261
DOI 10.1186/s13660-015-0791-4

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Oscillations of even order half-linear impulsive delay differential equations with damping

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Abstract

In this paper, a kind of half-linear impulsive delay differential equations with damping is studied. By employing a generalized Riccati technique and the impulsive differential inequality, we derive several oscillation criteria which are either new or improve several recent results in the literature. In addition, we provide several examples to illustrate the use of our results.

MSC: 34K06; 34K11

Keywords: even order; impulsive delay differential equation; half-linear; damping; oscillation

1 Introduction

Impulsive differential equations are used to simulate processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, *etc.*, and therefore their qualitative properties are important. The phenomenon of oscillations is observed in ecology, physics, economic, *etc.* In [1], Chen and Feng showed a few examples and indicated that some of the oscillations did favor the stability of system, but some might destroy the balance of the system. Oscillatory properties are so important for the balance of the system that there are now quite a few results on oscillatory properties of their solutions since recent years [1–15]. In particular, Agarwal *et al.* in [14, 15] discussed oscillation theory of differential equations and nonoscillation theory of functional differential equations with applications. Chen and Feng in [1] investigated oscillations of second order nonlinear impulsive differential equation by impulsive differential inequality. From then on, the authors in [3–7] generalized and improved the results of [1]. Furthermore, in [8–13], the delay effect to impulsive equations is considered and some interesting results of oscillations are obtained. Those papers have only considered first or second order differential equations (delay differential equations) with impulses. Recently, some scholars have been attracted by the problems of the oscillations of higher order differential equations and higher order impulsive differential equations and made relative advances therein in [16–26]. For example, Grace *et al.* in [22, 23] first studied oscillations of higher order nonlinear dynamic equations on time scales and got some interesting and exciting results. Pan *et al.* in [18] considered even order nonlinear differential equations with impulses of the form

$$\begin{cases} x^{(2n)}(t) + f(t,x) = 0, & t \geq t_0, t \neq t_k, \\ x^{(i)}(t_k^+) = g_k^{[i]}(x^{(i)}(t_k)), & i = 0,1,\ldots,2n-1, k = 1,2,\ldots, \\ x^{(i)}(t_0^+) = x_0^{(i)}, & i = 0,1,\ldots,2n-1, \end{cases} \tag{1}$$

where n is positive integer and $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$ such that $\lim_{k\to\infty} t_k = \infty$. They obtained sufficient conditions which guaranteed oscillation of every solution of (1). Wen *et al.* in [19] considered even order nonlinear differential equations with impulses of the form

$$\begin{cases} (r(t)x^{(2n-1)}(t))' + f(t,x) = 0, & t \geq t_0, t \neq t_k, \\ x^{(i)}(t_k^+) = g_k^{[i]}(x^{(i)}(t_k)), & i = 0,1,\ldots,2n-1, k = 1,2,\ldots, \\ x^{(i)}(t_0^+) = x_0^{(i)}, & i = 0,1,\ldots,2n-1, \end{cases} \tag{2}$$

where n is a positive integer and $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$ such that $\lim_{k\to\infty} t_k = \infty$, $p(t) > 0$. They generalized and improved the results in [16–18]. Pan in [20] considered nonlinear impulsive differential equations with damping of the form

$$\begin{cases} (r(t)x^{(2n-1)}(t))' + q(t)x^{(2n-1)}(t) + f(t,x(t)) = 0, & t \geq t_0, t \neq t_k, \\ x^{(i)}(t_k^+) = g_k^{[i]}(x^{(i)}(t_k)), & i = 0,1,\ldots,2n-1, k = 1,2,\ldots, \\ x^{(i)}(t_0^+) = x_0^{(i)}, & i = 0,1,\ldots,2n-1, \end{cases} \tag{3}$$

where n is a positive integer and $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$ such that $\lim_{k\to\infty} t_k = \infty$. He obtained sufficient conditions which guaranteed the oscillation of every solution of (3).

References devoted to the study of the oscillations of higher order impulsive differential equations are [18–20]. Impulsive delay differential equations may be used for the mathematical simulation of processes which are characterized by the fact that their state changes by jumps and by the dependence of the process on its history at each moment of time. Those equations can more precisely describe the real processes of a system than impulsive differential equations. Therefore, it is necessary to consider both impulsive effect and delay effect on the oscillation of a differential equation. Many useful results on oscillation and nonoscillation of first order or second order impulsive delay differential equations have been obtained in [8–13], but references devoted to the study of the oscillations of higher order impulsive delay differential equations are relatively scarce.

This paper is motivated by several recent studies [14–26] of such higher order equations. Using impulsive differential inequality and the Riccati transformation, we study the oscillatory properties of even order half-linear impulsive delay differential equation with damping of the form

$$\begin{cases} (r(t)|x^{(2n-1)}(t)|^{\alpha-1}x^{(2n-1)}(t))' + q(t)|x^{(2n-1)}(t)|^{\alpha-1}x^{(2n-1)}(t) \\ \quad + f(t,x(t),x(t-\tau)) = 0, \quad t \geq t_0, t \neq t_k, \\ x^{(i)}(t_k^+) = g_k^{[i]}(x^{(i)}(t_k)), \quad i = 0,1,\ldots,2n-1, k = 1,2,\ldots, \\ x^{(i)}(t_0^+) = x_0^{(i)}, \quad i = 0,1,\ldots,2n-1, \\ x(t) = \phi(t), \quad t_0 - \tau \leq t \leq t_0, \end{cases} \tag{4}$$

where

$$x^{(i)}(t_k) = \lim_{h\to 0^-} \frac{x^{(i-1)}(t_k+h) - x^{(i-1)}(t_k)}{h},$$
$$x^{(i)}(t_k^+) = \lim_{h\to 0^+} \frac{x^{(i-1)}(t_k+h) - x^{(i-1)}(t_k^+)}{h}.$$

$\phi : [t_0 - \tau, t_0] \to R$ has at most finite discontinuous points of the first kind and is left-continuous at these points. $\alpha > 0$, $\tau > 0$, $0 \le t_0 < t_1 < \cdots < t_k < \cdots$ such that $\lim_{k\to\infty} t_k = \infty$, $x^{(0)}(t) = x(t)$, n is a positive integer.

Definition 1 A function $x : [t_0 - \tau, t_0 + \gamma) \to R$ ($\gamma > 0$) is said to be a solution of (4) on $[t_0 - \tau, t_0 + \gamma)$ starting from $(t, \phi, x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(2n-1)})$ if

(i) $x^{(i)}(t)$ is continuous on $[t_0, t_0 + \gamma) \setminus \{t_k, k \in N\}$, $i = 0, 1, \dots, 2n-1$,
(ii) $x(t) = \phi(t)$, $t \in [t_0 - \tau, t_0]$, $x^{(i)}(t_0^+) = x_0^{(i)}$, $i = 0, 1, \dots, 2n-1$,
(iii) $x(t)$ satisfies the first equality of (4) on $[t_0, t_0 + \gamma) \setminus \{t_k, k \in N\}$,
(iv) $x^{(i)}(t)$ has two-side limits and left-continuous at points t_k, $x^{(i)}(t_k)$ satisfies the second equality of (4), $i = 0, 1, \dots, 2n-1$, $k = 1, 2, \dots$.

Remark 1 Let $x_0(t) = x(t), x_1(t) = x'(t), \dots, x_{2n-1}(t) = x^{(2n-1)}(t)$. Then (4) can be changed into a differential system with impulses. By the same method in [21], one can get sufficient conditions that can guarantee the solution of (4) exists on $[t_0, \infty)$. In the following, we always assume the solution of (4) exists on $[t_0, \infty)$.

Definition 2 A solution of (4) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

In this paper, we investigate the oscillatory properties of (4). We first obtain two theorems to ensure every solution of (4) is oscillatory. The results extend and improve the earlier publications. Next, we obtain three corollaries by Theorem 1 and Theorem 2, and provide examples to show that although even order nonlinear delay differential equations without impulses may have nonoscillatory solutions, adding impulses may lead to oscillatory solutions. That is, impulses may change the oscillatory behavior of an equation.

2 Main results

We will establish oscillatory results based on combinations of the following conditions:

(A) $r(t) > 0$ and $r(t)$, $q(t)$ are both continuous on $[t_0 - \tau, \infty)$, $f(t, u, v)$ is continuous on $[t_0 - \tau, \infty) \times (-\infty, \infty) \times (-\infty, \infty)$, $uf(t, u, v) > 0$ ($uv > 0$), and $f(t, u, v)/\varphi(v) \ge p(t)$ ($v \ne 0$), where $p(t)$ is positive and continuous on $[t_0 - \tau, \infty)$ and for any $t \ge t_0$, $p(t)$ is not always equal to 0 on $[t, \infty)$, φ is differentiable on $(-\infty, \infty)$ such that $x\varphi(x) > 0$ ($x \ne 0$), $\varphi'(x) \ge 0$.
(B) For $k = 1, 2, \dots$, $g_k^{[i]}(x)$ are continuous on $(-\infty, \infty)$ and there exist positive numbers $a_k^{[i]}$, $b_k^{[i]}$ such that

$$a_k^{[i]} \le g_k^{[i]}(x)/x \le b_k^{[i]}, \quad i = 0, 1, 2, \dots, 2n-1.$$

(C) For $i = 1, 2, \dots, 2n-2$,

$$\int_{t_0}^{\infty} \prod_{t_0<t_k<s} \frac{a_k^{[i]}}{b_k^{[i-1]}}\, ds = \infty$$

and

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(s)} \prod_{t_0<t_k<s} \frac{a_k^{[2n-1]}}{b_k^{[2n-2]}}\, ds = \infty.$$

(D) $\int_{t_0}^{\infty} \prod_{t_0<t_k<s} \frac{a_k^{[2n-1]}}{b_k^{[2n-2]}} \exp(-\int_{t_0}^{s} \frac{r'(v)+q(v)}{\alpha r(v)}\, dv)\, ds = \infty$.

The main results of the paper are as follows.

Theorem 1 *Assume that the conditions* (A), (B), (C), *and* (D) *hold. Suppose further that* $a_k^{[0]} \geq 1$ *and*

$$\int_{t_0}^{\infty} \prod_{t_0<t_{0,w}<s} \frac{1}{\theta_{0,w}} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds = \infty, \tag{5}$$

where

$$\theta_{0,w} = \begin{cases} 1, & t_{0,w} = t_k + \tau \neq t_m\ (m > k), \\ (b_k^{[2n-1]})^{\alpha}, & t_{0,w} = t_k, \\ (b_m^{[2n-1]})^{\alpha}, & t_{0,w} = t_k + \tau = t_m, \end{cases} \tag{6}$$

and $t_{0,w} = t_k$ *or* $t_k + \tau$ $(t_1 = t_{0,1} < t_{0,2} < \cdots < t_{0,w} < t_{0,w+1} < \cdots)$, *then every solution of* (4) *is oscillatory.*

Theorem 2 *Assume that the conditions* (A), (B), (C), *and* (D) *hold and that* $\varphi(ab) \geq \varphi(a)\varphi(b)$ *for* $ab > 0$. *Furthermore suppose that*

$$\int_{t_0}^{\infty} \prod_{t_0<t_{0,w}<t} \frac{1}{\mu_{0,w}} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds = \infty, \tag{7}$$

where

$$\mu_{0,w} = \begin{cases} \frac{(b_m^{[2n-1]})^{\alpha}}{\varphi(a_k^{[0]})}, & t_{0,w} = t_k + \tau = t_m\ (m > k), \\ (b_k^{[2n-1]})^{\alpha}, & t_{0,w} = t_k \text{ and } t_k - \tau \neq t_m\ (0 < m < k), \\ \frac{1}{\varphi(a_k^{[0]})}, & t_{0,w} = t_k + \tau \neq t_m\ (m > k), \\ \frac{(b_k^{[2n-1]})^{\alpha}}{\varphi(a_m^{[0]})}, & t_{0,w} = t_k \text{ and } t_k - \tau = t_m\ (0 < m < k), \end{cases} \tag{8}$$

and $t_{0,w} = t_k$ *or* $t_k + \tau$ $(t_1 = t_{0,1} < t_{0,2} < \cdots < t_{0,w} < t_{0,w+1} < \cdots)$, *then every solution of* (4) *is oscillatory.*

Remark 2 When $\alpha = 1$ and not considering the delay effect, (4) reduces to (3). Our Theorem 1 and Theorem 2 generalize and contain results in [20]. When $\alpha = 1$, $q(t) = 0$ and not considering a delay effect, (4) reduces to (2). Our Theorem 1 and Theorem 2 are extensions of Theorem 1, Theorem 2 of [19], respectively.

3 Corollaries and examples

Corollary 1 *Assume that the conditions* (A), (B), (C), *and* (D) *hold. Furthermore suppose that* $a_k^{[0]} \geq 1, b_k^{[2n-1]} \leq 1$ *and*

$$\int_{t_0}^{\infty} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds = \infty, \tag{9}$$

then every solution of (4) *is oscillatory.*

Proof By $a_k^{[0]} \geq 1$, $b_k^{[2n-1]} \leq 1$, we know that $\frac{1}{\theta_{0,w}} \geq 1$. Therefore

$$\int_{t_0}^{t} \prod_{t_0<t_{0,w}<t} \frac{1}{\theta_{0,w}} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds \geq \int_{t_0}^{t} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds, \tag{10}$$

letting $t \to \infty$, it follows from (9), (10) that (5) holds. By Theorem 1, we see that all solutions of (4) are oscillatory. □

Corollary 2 *Assume that the conditions* (A), (B), (C), *and* (D) *hold and that there exists a constant* $\delta > 0$ *such that*

$$a_k^{[0]} \geq 1, \qquad \frac{1}{(b_k^{[2n-1]})^{\alpha}} \geq \left(\frac{t_{k+1}}{t_k}\right)^{\delta}. \tag{11}$$

If

$$\int_{t_0}^{\infty} s^{\delta} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds = \infty, \tag{12}$$

then every solution of (4) *is oscillatory.*

Proof By $a_k^{[0]} \geq 1$, $\frac{1}{(b_k^{[2n-1]})^{\alpha}} \geq (\frac{t_{k+1}}{t_k})^{\delta}$, then for $t \in (t_w, t_{w+1}]$, we have

$$\begin{aligned}
&\int_{t_0}^{t} \prod_{t_0<t_{0,w}<t} \frac{1}{\theta_{0,w}} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds \\
&\quad = \int_{t_0}^{t_1} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds + \frac{1}{(b_1^{[2n-1]})^{\alpha}} \int_{t1}^{t_2} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds \\
&\qquad + \frac{1}{(b_1^{[2n-1]} b_2^{[2n-1]})^{\alpha}} \int_{t_2}^{t_3} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds + \cdots \\
&\qquad + \frac{1}{(b_1^{[2n-1]} b_2^{[2n-1]} \cdots b_w^{[2n-1]})^{\alpha}} \int_{t_w}^{t} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds \\
&\quad \geq \frac{1}{(b_1^{[2n-1]})^{\alpha}} \int_{t1}^{t_2} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds \\
&\qquad + \frac{1}{(b_1^{[2n-1]} b_2^{[2n-1]})^{\alpha}} \int_{t_2}^{t_3} p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\, dv\right) ds + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{(b_1^{[2n-1]}b_2^{[2n-1]}\cdots b_w^{[2n-1]})^\alpha}\int_{t_w}^{t} p(s)\exp\left(\int_{t_0}^{s}\frac{q(v)}{r(v)}\,dv\right)ds \\
&\geq \frac{1}{t_1^\delta}\left[\int_{t1}^{t_2} t_2^\delta p(s)\exp\left(\int_{t_0}^{s}\frac{q(v)}{r(v)}\,dv\right)ds + \int_{t_2}^{t_3} t_3^\delta p(s)\exp\left(\int_{t_0}^{s}\frac{q(v)}{r(v)}\,dv\right)ds + \cdots\right. \\
&\left. + \int_{t_w}^{t} t_{w+1}^\delta p(s)\exp\left(\int_{t_0}^{s}\frac{q(v)}{r(v)}\,dv\right)ds\right] \\
&\geq \frac{1}{t_1^\delta}\left[\int_{t1}^{t_2} s^\delta p(s)\exp\left(\int_{t_0}^{s}\frac{q(v)}{r(v)}\,dv\right)ds + \int_{t_2}^{t_3} s^\delta p(s)\exp\left(\int_{t_0}^{s}\frac{q(v)}{r(v)}\,dv\right)ds + \cdots\right. \\
&\left. + \int_{t_w}^{t} s^\delta p(s)\exp\left(\int_{t_0}^{s}\frac{q(v)}{r(v)}\,dv\right)ds\right] \\
&= \frac{1}{t_1^\delta}\int_{t1}^{t} s^\delta p(s)\exp\left(\int_{t_0}^{s}\frac{q(v)}{r(v)}\,dv\right)ds, \qquad (13)
\end{aligned}
$$

letting $t \to \infty$, it follows from (12), (13) that (5) hold. By Theorem 1, we see that all solutions of (4) are oscillatory. □

Corollary 3 *Assume that the conditions* (A), (B), (C), *and* (D) *hold and that* $\varphi(ab) \geq \varphi(a)\varphi(b)$ *for* $ab > 0$. *If there exists a constant* $\delta > 0$ *such that*

$$
t_{k+1} - t_k > \tau, \qquad \frac{\varphi(a_k^{[0]})}{(b_k^{[2n-1]})^\alpha} \geq \left(\frac{t_{k+1}}{t_k}\right)^\delta \qquad (14)
$$

and

$$
\int_{t_0}^{\infty} s^\delta p(s)\exp\left(\int_{t_0}^{s}\frac{q(v)}{r(v)}\,dv\right)ds = \infty, \qquad (15)
$$

then every solution of (4) *is oscillatory.*

Corollary 3 can be deduced from Theorem 2. The proof is similar to that of Corollary 2 and it is omitted.

Example 1 Consider the equation

$$
\begin{cases}
(|x^{(2n-1)}(t)|x^{(2n-1)}(t))' - \frac{1}{t}|x^{(2n-1)}(t)|x^{(2n-1)}(t) + \frac{1}{t^2}x(t-\frac{1}{2}) = 0, & t \geq \frac{1}{2}, t \neq k, \\
x(k^+) = x(k), \quad x^{(i)}(k^+) = \frac{k}{k+1}x^{(i)}(k), & i = 1,2,\ldots,2n-1; k = 1,2,\ldots, \\
x(\frac{1}{2}) = x_0, \quad x^{(i)}(\frac{1}{2}) = x_0^{(i)}, & \\
x(t) = \phi(t), \quad t \in [0,\frac{1}{2}], &
\end{cases} \qquad (16)
$$

where $a_k^{[0]} = b_k^{[0]} = 1$, $a_k^{[i]} = b_k^{[i]} = \frac{k}{k+1}$, $i = 1,2,\ldots,2n-1$; $q(t) = -\frac{1}{t}$, $p(t) = \frac{1}{t^2}$, $r(t) = 1$, $t_0 = \frac{1}{2}$, $t_k = k$, $\tau = \frac{1}{2}$, $\alpha = 2$, $\varphi(x) = x$. It is easy to see that the conditions (A), (B), (C), and (D) hold. Since $\frac{1}{(b_k^{[2n-1]})^2} = (\frac{k+1}{k})^2 = (\frac{t_{k+1}}{t_k})^2$, we may let $\delta = 2$, furthermore,

$$
\begin{aligned}
\int_{t_0}^{\infty} s^\delta p(s)\exp\left(\int_{t_0}^{s}\frac{q(v)}{r(v)}\,dv\right)ds &= \int_{\frac{1}{2}}^{\infty} s^2\frac{1}{s^2}\exp\left(-\int_{\frac{1}{2}}^{s}\frac{1}{v}\,dv\right)ds \\
&= \int_{\frac{1}{2}}^{\infty}\exp\left(-\int_{\frac{1}{2}}^{s}\frac{1}{v}\,dv\right)ds = \int_{\frac{1}{2}}^{\infty}\frac{1}{2s}\,ds = \infty.
\end{aligned}
$$

By Corollary 2, every solution of (16) is oscillatory.

Example 2 Consider the equation

$$\begin{cases} (t|x^{(2n-1)}(t)|^{\alpha-1}x^{(2n-1)}(t))' - |x^{(2n-1)}(t)|^{\alpha-1}x^{(2n-1)}(t) \\ \quad + \frac{1}{t^2}x^3(t-\frac{1}{2}) = 0, \quad t \geq \frac{1}{2}, t \neq k, \\ x(k^+) = \frac{k+1}{k}x(k), \qquad x^{(i)}(k^+) = x^{(i)}(k), \quad i = 1, 2, \dots, 2n-1; k = 1, 2, \dots, \\ x(\frac{1}{2}) = x_0, \qquad x^{(i)}(\frac{1}{2}) = x_0^{(i)}, \\ x(t) = \phi(t), \quad t \in [0, \frac{1}{2}], \end{cases} \tag{17}$$

where $a_k^{[0]} = b_k^{[0]} = \frac{k+1}{k}$, $a_k^{[i]} = b_k^{[i]} = 1$, $i = 1, 2, \dots, 2n-1$; $q(t) = -1$, $p(t) = \frac{1}{t^2}$, $r(t) = t$, $t_0 = \frac{1}{2}$, $t_k = k$, $\tau = \frac{1}{2}$, $t_{k+1} - t_k = 1 > \frac{1}{2} = \tau$, $\varphi(x) = x^3$. It is easy to see that the conditions (A), (B), (C), and (D) hold. Since $\frac{\varphi(a_k^{[0]})}{(b_k^{[2n-1]})^\alpha} = (\frac{k+1}{k})^3 = (\frac{t_{k+1}}{t_k})^3$, we may let $\delta = 3$, furthermore

$$\begin{aligned} \int_{t_0}^{\infty} s^\delta p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\,dv\right) ds &= \int_{\frac{1}{2}}^{\infty} s^3 \frac{1}{s^2} \exp\left(-\int_{\frac{1}{2}}^{s} \frac{1}{v}\,dv\right) ds \\ &= \int_{\frac{1}{2}}^{\infty} s \exp\left(-\int_{\frac{1}{2}}^{s} \frac{1}{v}\,dv\right) ds \\ &= \int_{\frac{1}{2}}^{\infty} s \exp\left(In\frac{1}{2} - Ins\right) ds \\ &= \int_{\frac{1}{2}}^{\infty} s \frac{1}{2s}\,ds \\ &= \frac{1}{2} \int_{\frac{1}{2}}^{\infty} ds = \infty. \end{aligned}$$

By Corollary 3, every solution of (17) is oscillatory.

Example 3 Consider the equation

$$\begin{cases} (t^2|x'''(t)|x'''(t))' + \frac{27}{64}t^{-4}(t-\frac{1}{2})^{-\frac{7}{2}}x^7(t-\frac{1}{2}) = 0, \quad t \geq \frac{1}{2}, t \neq k, \\ x(k^+) = \frac{k+1}{k}x(k), \qquad x^{(i)}(k^+) = x^{(i)}(k), \quad i = 1, 2, \dots, 2n-1; k = 1, 2, \dots, \\ x(\frac{1}{2}) = x_0, \qquad x^{(i)}(\frac{1}{2}) = x_0^{(i)}, \\ x(t) = \phi(t), \quad t \in [0, \frac{1}{2}], \end{cases} \tag{18}$$

where $a_k^{[0]} = b_k^{[0]} = \frac{k+1}{k}$, $a_k^{[i]} = b_k^{[i]} = 1$, $i = 1, 2, 3$; $r(t) = t^2$, $q(t) = 0$, $p(t) = \frac{27}{64}t^{-4}(t-\frac{1}{2})^{-\frac{7}{2}}$, $t_0 = \frac{1}{2}$, $t_k = k$, $\tau = \frac{1}{2}$, $t_{k+1} - t_k = 1 > \frac{1}{2} = \tau$, $\alpha = 2$, $\varphi(x) = x^7$. It is easy to see that the conditions (A), (B), (C), and (D) hold. Since $\frac{\varphi(a_k^{[0]})}{(b_k^{[2n-1]})^\alpha} = (\frac{k+1}{k})^7 = (\frac{t_{k+1}}{t_k})^7$, we may let $\delta = 7$, furthermore

$$\begin{aligned} \int_{t_0}^{\infty} s^\delta p(s) \exp\left(\int_{t_0}^{s} \frac{q(v)}{r(v)}\,dv\right) ds &= \int_{\frac{1}{2}}^{\infty} s^7 \frac{27}{64} s^{-4} \left(s - \frac{1}{2}\right)^{-\frac{7}{2}} ds \\ &= \frac{27}{64} \int_{\frac{1}{2}}^{\infty} s^3 \left(s - \frac{1}{2}\right)^{-\frac{7}{2}} ds \\ &\geq \frac{27}{64} \int_{\frac{1}{2}}^{\infty} \frac{1}{(s-\frac{1}{2})^{\frac{1}{2}}}\,ds \\ &= \infty. \end{aligned}$$

By Corollary 3, every solution of (18) is oscillatory. But the delay differential equation

$$\left(t^2\left|x'''(t)\right|x'''(t)\right)' + \frac{27}{64}t^{-4}\left(t-\frac{1}{2}\right)^{-\frac{7}{2}}x^7\left(t-\frac{1}{2}\right) = 0$$

has a nonnegative solution $x = \sqrt{t}$. This example shows that impulses play an important role in the oscillatory behavior of equations under perturbing impulses.

4 Preparatory lemmas

To prove Theorem 1 and Theorem 2, we need the following lemmas.

Lemma 1 (Lakshmikantham *et al.* [2]) *Assume that*

(H_0) *$m \in PC'(R^+, R)$ and $m(t)$ is left-continuous at t_k, $k = 1, 2, \dots$.*
(H_1) *For t_k, $k = 1, 2, \dots$ and $t \geq t_0$,*

$$m'(t) \leq p(t)m(t) + q(t), \quad t \neq t_k,$$
$$m\left(t_k^+\right) \leq d_k m(t_k) + b_k,$$

where $p, q \in PC(R^+, R)$, $d_k \geq 0$ and b_k are real constants. Then for $t \geq t_0$,

$$\begin{aligned} m(t) \leq m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^t p(s)\,ds\right) + \sum_{t_0<t_k<t}\left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^t p(s)\,ds\right)\right) b_k \\ + \int_{t_0}^t \prod_{s<t_k<t} d_k \exp\left(\int_s^t p(\sigma)\,d\sigma\right) q(s)\,ds. \end{aligned} \tag{19}$$

Lemma 2 *Suppose that the conditions* (A), (B), *and* (C) *hold and $x(t)$ is a solution of* (4). *We have the following statements*:

(a) *If there exists some $T \geq t_0$ such that $x^{(2n-1)}(t) > 0$ and $(r(t)|x^{(2n-1)}(t)|^{\alpha-1}x^{(2n-1)}(t))' \geq 0$ for $t \geq T$, then there exists some $T_1 \geq T$ such that $x^{(2n-2)}(t) > 0$ for $t \geq T_1$.*
(b) *If there exist $i \in \{1, 2, \dots, 2n-2\}$ and some $T \geq t_0$ such that $x^{(i)}(t) > 0$ and $x^{(i+1)}(t) \geq 0$ for $t \geq T$, then there exists some $T_1 \geq T$ such that $x^{(i-1)}(t) > 0$ for $t \geq T_1$.*

Proof (a) Without loss of generality, we may assume that $T = t_0$, $x^{(2n-1)}(t) > 0$ and $(r(t)|x^{(2n-1)}(t)|^{\alpha-1}x^{(2n-1)}(t))' \geq 0$ for $t \geq t_0$. We first prove that there exists some j such that $x^{(2n-2)}(t_j) > 0$ for $t_j \geq t_0$. If this is not true, then for any $t_k > t_0$, we have $x^{(2n-2)}(t_k) \leq 0$. Since $x^{(2n-2)}(t)$ is increasing on intervals of the form $(t_k, t_{k+1}]$, we see that $x^{(2n-2)}(t) \leq 0$ for $t \geq t_0$. Since $r(t)|x^{(2n-1)}(t)|^{\alpha-1}x^{(2n-1)}(t)$ is increasing on intervals of the form $(t_k, t_{k+1}]$, we see that for $(t_1, t_2]$,

$$r(t)\left|x^{(2n-1)}(t)\right|^{\alpha-1}x^{(2n-1)}(t) \geq r(t_1)\left|x^{(2n-1)}\left(t_1^+\right)\right|^{\alpha-1}x^{(2n-1)}\left(t_1^+\right),$$

that is,

$$x^{(2n-1)}(t) \geq \frac{r^{\frac{1}{\alpha}}(t_1)}{r^{\frac{1}{\alpha}}(t)}x^{(2n-1)}\left(t_1^+\right).$$

In particular,

$$x^{(2n-1)}(t_2) \geq \frac{r^{\frac{1}{\alpha}}(t_1)}{r^{\frac{1}{\alpha}}(t_2)} x^{(2n-1)}\bigl(t_1^+\bigr).$$

Similarly, for $(t_2, t_3]$, we have

$$x^{(2n-1)}(t) \geq \frac{r^{\frac{1}{\alpha}}(t_2)}{r^{\frac{1}{\alpha}}(t)} x^{(2n-1)}\bigl(t_2^+\bigr) \geq \frac{r^{\frac{1}{\alpha}}(t_2)}{r^{\frac{1}{\alpha}}(t)} a_2^{[2n-1]} x^{(2n-1)}(t_2) \geq \frac{r^{\frac{1}{\alpha}}(t_1)}{r^{\frac{1}{\alpha}}(t)} a_2^{[2n-1]} x^{(2n-1)}\bigl(t_1^+\bigr).$$

By induction, we know that

$$x^{(2n-1)}(t) \geq \frac{r^{\frac{1}{\alpha}}(t_1)}{r^{\frac{1}{\alpha}}(t)} \prod_{t_1<t_k<t} a_k^{[2n-1]} x^{(2n-1)}\bigl(t_1^+\bigr), \quad t \neq t_k. \tag{20}$$

From the condition (B), we have

$$x^{(2n-2)}\bigl(t_k^+\bigr) \geq b_k^{[2n-2]} x^{(2n-2)}(t_k), \quad t > t_1, k = 2, 3, \ldots. \tag{21}$$

Set $m(t) = -x^{(2n-2)}(t)$. Then from (20) and (21), we see that

$$m'(t) \leq -\frac{r^{\frac{1}{\alpha}}(t_1)}{r^{\frac{1}{\alpha}}(t)} \prod_{t_1<t_k<t} a_k^{[2n-1]} x^{(2n-1)}\bigl(t_1^+\bigr), \quad t > t_1, t \neq t_k,$$

and

$$m\bigl(t_k^+\bigr) \leq b_k^{[2n-2]} m(t_k), \quad k = 2, 3, \ldots.$$

It follows from Lemma 1 that

$$\begin{aligned} m(t) &\leq m\bigl(t_1^+\bigr) \prod_{t_1<t_k<t} b_k^{[2n-2]} - x^{(2n-1)}\bigl(t_1^+\bigr) r^{\frac{1}{\alpha}}(t_1) \int_{t_1}^t \frac{1}{r^{\frac{1}{\alpha}}(s)} \prod_{s<t_k<t} b_k^{[2n-2]} \prod_{t_1<t_k<t} a_k^{[2n-1]}\, ds \\ &= \prod_{t_1<t_k<t} b_k^{[2n-2]} \left\{ m\bigl(t_1^+\bigr) - x^{(2n-1)}\bigl(t_1^+\bigr) r^{\frac{1}{\alpha}}(t_1) \int_{t_1}^t \frac{1}{r^{\frac{1}{\alpha}}(s)} \prod_{t_1<t_k<s} \frac{a_k^{[2n-1]}}{b_k^{[2n-2]}}\, ds \right\}. \end{aligned}$$

That is,

$$x^{(2n-2)}(t) \geq \prod_{t_1<t_k<t} b_k^{[2n-2]} \left\{ x^{(2n-2)}\bigl(t_1^+\bigr) + x^{(2n-1)}\bigl(t_1^+\bigr) r^{\frac{1}{\alpha}}(t_1) \int_{t_1}^t \frac{1}{r^{\frac{1}{\alpha}}(s)} \prod_{t_1<t_k<s} \frac{a_k^{[2n-1]}}{b_k^{[2n-2]}}\, ds \right\}. \tag{22}$$

Note that $a_k^{[i]} > 0$, $b_k^{[i]} > 0$, and the second equality of the condition (B) holds. Thus we get $x^{(2n-2)}(t) > 0$ for all sufficiently large t. The relation $x^{(2n-2)}(t) \leq 0$ leads to a contradiction. So there exists some j such that $t_j > T$ and $x^{(2n-2)}(t_j) > 0$. Since $x^{(2n-2)}(t)$ is increasing on $(t_{j+\lambda-1}, t_{j+\lambda}]$, $\lambda = 1, 2, \ldots$, for $(t_j, t_{j+1}]$, we have

$$x^{(2n-2)}(t) \geq x^{(2n-2)}\bigl(t_j^+\bigr) \geq a_j^{[2n-2]} x^{(2n-2)}(t_j) > 0.$$

Similarly, for $(t_{j+1}, t_{j+2}]$,

$$x^{(2n-2)}(t) \geq x^{(2n-2)}\big(t_{j+1}^+\big) \geq a_{j+1}^{[2n-2]} x^{(2n-2)}(t_{j+1}) \geq a_{j+1}^{[2n-2]} a_j^{[2n-2]} x^{(2n-2)}(t_j) > 0.$$

We can easily prove that, for any positive integer $\lambda \geq 2$ and $t \in (t_{j+\lambda}, t_{j+\lambda+1}]$,

$$x^{(2n-2)}(t) \geq a_j^{[2n-2]} a_{j+1}^{[2n-2]} \cdots a_{j+\lambda}^{[2n-2]} x^{(2n-2)}(t_j) > 0.$$

Thus $x^{(2n-2)}(t) > 0$ for $t \geq t_j$. So there exists $T_1 \geq T$ such that $x^{(2n-2)}(t) > 0$ for $t \geq T_1$. The proof of (a) is complete.

(b) Assume that for any $t_k > T$, we have $x^{(i-1)}(t_k) \leq 0$. By $x^{(i)}(t) > 0$, $x^{(i+1)}(t) \geq 0$, $t \in (t_k, t_{k+1}]$, we see that $x^{(i)}(t)$ is nondecreasing on $(t_k, t_{k+1}]$. For $t \in (t_1, t_2]$, we have

$$x^{(i)}(t) \geq x^{(i)}\big(t_1^+\big).$$

In particular,

$$x^{(i)}(t_2) \geq x^{(i)}\big(t_1^+\big).$$

Similarly, for $t \in (t_2, t_3]$, we have

$$x^{(i)}(t) \geq x^{(i)}\big(t_2^+\big) \geq a_2^{[i]} x^{(i)}(t_2) \geq a_2^{[i]} x^{(i)}\big(t_1^+\big).$$

By induction, we know that

$$x^{(i)}(t) \geq \prod_{t_1 < t_k < t} a_k^{[i]} x^{(i)}\big(t_1^+\big), \quad t > t_1, t \neq t_k. \tag{23}$$

From the condition (ii), we have

$$x^{(i-1)}\big(t_k^+\big) \geq b_k^{[i-1]} x^{(i-1)}(t_k), \quad k = 2, 3, \ldots. \tag{24}$$

Set $u(t) = -x^{(i-1)}(t)$. Then from (23) and (24), we see that

$$u'(t) \leq -\prod_{t_1 < t_k < t} a_k^{[i]} x^{(i)}\big(t_1^+\big), \quad t > t_1, t \neq t_k,$$

and

$$u\big(t_k^+\big) \leq b_k^{[i-1]} u(t_k), \quad k = 2, 3, \ldots.$$

It follows from Lemma 1 that

$$\begin{aligned} u(t) &\leq u\big(t_1^+\big) \prod_{t_1 < t_k < t} b_k^{[i-1]} - x^{(i)}\big(t_1^+\big) \int_{t_1}^t \prod_{s < t_k < t} b_k^{[i-1]} \prod_{t_1 < t_k < t} a_k^{[i]}\, ds \\ &= \prod_{t_1 < t_k < t} b_k^{[i-1]} \left\{ u\big(t_1^+\big) - x^{(i)}\big(t_1^+\big) \int_{t_1}^t \prod_{t_1 < t_k < s} \frac{a_k^{[i]}}{b_k^{[i-1]}}\, ds \right\}. \end{aligned}$$

That is,

$$x^{(i-1)}(t) \geq \prod_{t_1<t_k<t} b_k^{[i-1]} \left\{ x^{(i-1)}\left(t_1^+\right) + x^{(i)}\left(t_1^+\right) \int_{t_1}^{t} \prod_{t_1<t_k<s} \frac{a_k^{[i]}}{b_k^{[i-1]}}\, ds \right\}. \tag{25}$$

Note that $a_k^{[i]} > 0$, $b_k^{[i]} > 0$, and the first equality of the condition (B) holds. Thus we get $x^{(i-1)}(t) > 0$ for all sufficiently large t. The relation $x^{(i-1)}(t) \leq 0$ leads to a contradiction. So there exists some j such that $t_j > T$ and $x^{(i-1)}(t_j) > 0$. Then

$$x^{(i-1)}\left(t_j^+\right) \geq a_j^{[i-1]} x^{(i-1)}(t_j) > 0.$$

Since $x^{(i)}(t) > 0$, we see that $x^{(i-1)}(t)$ is increasing on $(t_{j+m-1}, t_{j+m}]$, $m = 1, 2, \ldots$. For $(t_j, t_{j+1}]$, we have

$$x^{(i-1)}(t) \geq x^{(i-1)}\left(t_j^+\right) > 0.$$

In particular,

$$x^{(i-1)}(t_{j+1}) \geq x^{(i-1)}\left(t_j^+\right) > 0.$$

Similarly, for $(t_{j+1}, t_{j+2}]$, we have

$$x^{(i-1)}(t) \geq x^{(i-1)}\left(t_{j+1}^+\right) \geq a_{j+1}^{(i-1)} x^{(i-1)}(t_{j+1}) > 0.$$

By induction, for $(t_{j+m-1}, t_{j+m}]$, we have $x^{(i-1)}(t) > 0$. So when $t \geq t_{j+1}$, we have

$$x^{(i-1)}(t) > 0.$$

Summing up the above discussion, we know that there exists some $T_1 \geq T$ such that

$$x^{(i-1)}(t) > 0, \quad t \geq T_1.$$

The proof of Lemma 2 is complete. □

Remark 3 We may prove in a similar manner the following statements:

(a′) If we replace the condition (a) in Lemma 2 ‘$x^{(2n-1)}(t) > 0$ and $(r(t)|x^{(2n-1)}(t)|^{\alpha-1} \times x^{(2n-1)}(t))' \geq 0$ for $t \geq T$’ with ‘$x^{(2n-1)}(t) < 0$ and $(r(t)|x^{(2n-1)}(t)|^{\alpha-1} x^{(2n-1)}(t))' \leq 0$ for $t \geq T$’, under the conditions (A), (B), and (C), then there exists some $T_1 \geq T$ such that $x^{(2n-2)}(t) < 0$ for $t \geq T_1$.

(b′) If we replace the condition (b) in Lemma 2 ‘$x^{(i)}(t) > 0$ and $x^{(i+1)}(t) \geq 0$ for $t \geq T$’ with ‘$x^{(i)}(t) < 0$ and $x^{(i+1)}(t) \leq 0$ for $t \geq T$’ under the conditions (A), (B), and (C), then there exists some $T_1 \geq T$ such that $x^{(i-1)}(t) < 0$ for $t \geq T_1$.

Lemma 3 *Let* $x = x(t)$ *be a solution of* (4) *and suppose that the conditions* (A), (B), *and* (C) *hold*.

(a) *If there exists some* $T \geq t_0$ *such that* $x(t) > 0$ *and* $(r(t)|x^{(2n-1)}(t)|^{\alpha-1} x^{(2n-1)}(t))' < 0$ *for* $t \geq T$, *then* $x^{(2n-1)}(t) > 0$ *for all sufficiently large* t.

(b) *If there exist* $i \in \{1, 2, \dots, 2n-1\}$ *and some* $T \geq t_0$ *such that* $x(t) > 0$ *and* $x^{(i)}(t) < 0$ *for* $t \geq T$, *then* $x^{(i-1)}(t) > 0$ *for all sufficiently large* t.

Proof (a) We first prove that $x^{(2n-1)}(t) > 0$ for any $t_k \geq T$. If this is not true, then there exists some $t_j \geq T$ such that $x^{(2n-1)}(t_j) \leq 0$. Since $r(t) > 0$ and $r(t)|x^{(2n-1)}(t)|^{\alpha-1}x^{(2n-1)}(t)$ is strictly decreasing on $(t_{j+m-1}, t_{j+m}]$ for $m = 1, 2, \dots$ and for $t \in (t_j, t_{j+1}]$, we have

$$\begin{aligned} r(t)\left|x^{(2n-1)}(t)\right|^{\alpha-1}x^{(2n-1)}(t) &< r(t_j)\left|x^{(2n-1)}\left(t_j^+\right)\right|^{\alpha-1}x^{(2n-1)}\left(t_j^+\right) \\ &\leq \left(a_j^{[2n-1]}\right)^{\alpha} r(t_j)\left|x^{(2n-1)}(t_j)\right|^{\alpha-1}x^{(2n-1)}(t_j) \leq 0. \end{aligned}$$

Let $\beta = r(t_j)|x^{(2n-1)}(t_j)|^{\alpha-1}x^{(2n-1)}(t_j) < 0$, we have

$$r(t)\left|x^{(2n-1)}(t)\right|^{\alpha-1}x^{(2n-1)}(t) < \left(a_j^{[2n-1]}\right)^{\alpha}\beta < 0.$$

Similarly, for $t \in (t_{j+1}, t_{j+2}]$, we have

$$\begin{aligned} r(t)\left|x^{(2n-1)}(t)\right|^{\alpha-1}x^{(2n-1)}(t) &< r(t_{j+1})\left|x^{(2n-1)}\left(t_{j+1}^+\right)\right|^{\alpha-1}x^{(2n-1)}\left(t_{j+1}^+\right) \\ &\leq \left(a_j^{[2n-1]}\right)^{\alpha}\left(a_{j+1}^{[2n-1]}\right)^{\alpha}\beta \leq 0. \end{aligned}$$

We can easily prove that, for any positive integer $n \geq 1$ and $t \in (t_{j+n}, t_{j+n+1}]$, we have

$$r(t)\left|x^{(2n-1)}(t)\right|^{\alpha-1}x^{(2n-1)}(t) < \left(a_j^{[2n-1]}a_{j+1}^{[2n-1]}\cdots a_{j+n}^{[2n-1]}\right)^{\alpha}\beta \leq 0.$$

Hence, $x^{(2n-1)}(t) < 0$ for $t \geq t_{j+1}$. By the result (a′) of Remark 2, for sufficiently large t, we have $x^{(2n-2)}(t) < 0$. Using the result (b′) of Remark 2 repeatedly, for all sufficiently large t, we get $x(t) < 0$. This is contrary with $x(t) > 0$ for $t \geq T$. Hence, we have $x^{(2n-1)}(t_k) > 0$ for any $t_k \geq T$. So we get $x^{(2n-1)}(t) > 0$ for all sufficiently large t.

(b) We first prove that $x^{(i-1)}(t_k) > 0$ for any $t_k \geq T$. If this is not true, then there exists some $t_j \geq T$ such that $x^{(i-1)}(t_j) < 0$. Since $x^{(i-1)}(t)$ is strictly monotony decreasing on $(t_{j+n}, t_{j+n+1}]$ for $n = 0, 1, 2, \dots$ and for $t \in (t_j, t_{j+1}]$, we have

$$x^{(i-1)}(t) < x^{(i-1)}\left(t_j^+\right) \leq a_j^{[i-1]}x^{(i-1)}(t_j) \leq 0.$$

Similarly, for $t \in (t_{j+1}, t_{j+2}]$, we have

$$x^{(i-1)}(t) < x^{(i-1)}\left(t_{j+1}^+\right) \leq a_j^{[i-1]}a_{j+1}^{[i-1]}x^{(i-1)}(t_j) \leq 0.$$

We can easily prove that, for any positive integer $n \geq 2$ and $t \in (t_{j+n}, t_{j+n+1}]$, we have

$$x^{(i-1)}(t) < a_j^{[i-1]}a_{j+1}^{[i-1]}\cdots a_{j+n}^{[i-1]}x^{(i-1)}(t_j) \leq 0.$$

Hence, $x^{(i-1)}(t) < 0$ for $t \geq t_{j+1}$. By the result (b′) of Remark 2, for sufficiently large t, we have $x^{(i-2)}(t) < 0$. Similarly, by using the result (b′) of Remark 2 again, we can conclude that for all sufficiently large t, $x(t) < 0$. That is contrary with $x(t) > 0$ for $t \geq T$. Hence, we have $x^{(i-1)}(t_k) > 0$ for any $t_k \geq T$. So we get $x^{(i-1)}(t) > 0$ for all sufficiently large t. The proof of Lemma 3 is complete. □

Lemma 4 *Let $x = x(t)$ be a solution of* (4). *Suppose that $T \geq t_0$ and $x(t) > 0$ for $t \geq T$. If the conditions* (A), (B), (C), *and* (D) *hold, then there exist some $T' \geq T$ and $l \in \{1, 3, \ldots, 2n-1\}$ such that for $t \geq T'$*,

$$\begin{cases} x^{(i)}(t) > 0, & i = 0, 1, \ldots, l; \\ (-1)^{(i-1)} x^{(i)}(t) > 0, & i = l+1, \ldots, 2n-1. \end{cases} \tag{26}$$

Proof Let $x(t) > 0$ for $t \geq T$. We first prove that $x^{(2n-1)}(t_k) > 0$ for any $t_k \geq T$. If this is not true, then there exists some $t_j \geq T$ such that $x^{(2n-1)}(t_j) \leq 0$. By (4) and the condition (A), for $t \in (t_{j+m-1}, t_{j+m}]$, $m = 1, 2, \ldots$, we have

$$\left(\left|x^{(2n-1)}(t)\right|^{\alpha-1} x^{(2n-1)}(t)\right)' + \frac{r'(t) + q(t)}{r(t)}\left|x^{(2n-1)}(t)\right|^{\alpha-1} x^{(2n-1)}(t) + \frac{f(t, x(t), x(t-\tau))}{r(t)} = 0,$$

that is,

$$\begin{aligned} &\left(\left|x^{(2n-1)}(t)\right|^{\alpha-1} x^{(2n-1)}(t) \exp \int_{t_j}^{t} \frac{r'(s) + q(s)}{r(s)}\, ds\right)' \\ &\quad = -\frac{f(t, x(t), x(t-\tau))}{r(t)} \exp \int_{t_j}^{t} \frac{r'(s) + q(s)}{r(s)}\, ds \\ &\quad \leq -\frac{p(t)\varphi(x(t-\tau))}{r(t)} \exp \int_{t_j}^{t} \frac{r'(s) + q(s)}{r(s)}\, ds \leq 0. \end{aligned} \tag{27}$$

Let $s(t) = |x^{(2n-1)}(t)|^{\alpha-1} x^{(2n-1)}(t) \exp \int_{t_j}^{t} \frac{r'(s)+q(s)}{r(s)}\, ds$, we have $s'(t) \leq 0$, $s(t)$ is monotonically decreasing on $(t_{j+m-1}, t_{j+m}]$, $m = 1, 2, \ldots$.

For $t \in (t_j, t_{j+1}]$, we have

$$s(t) \leq s\left(t_j^+\right) \leq \left(a_j^{[2n-1]}\right)^{\alpha} s(t_j) \leq 0,$$

particularly, we have

$$s(t_{j+1}) \leq \left(a_j^{[2n-1]}\right)^{\alpha} s(t_j) \leq 0.$$

Similarly, for $t \in (t_{j+1}, t_{j+2}]$, we have

$$s(t) \leq s\left(t_{j+1}^+\right) \leq \left(a_{j+1}^{[2n-1]}\right)^{\alpha} s(t_{j+1}) \leq \left(a_{j+1}^{[2n-1]}\right)^{\alpha} \left(a_j^{[2n-1]}\right)^{\alpha} s(t_j) \leq 0.$$

By induction, for $t \in (t_{j+m-1}, t_{j+m}]$, $m = 1, 2, \ldots$, we obtain

$$\begin{aligned} s(t) &\leq s\left(t_{j+m-1}^+\right) \leq \left(a_{j+m-1}^{[2n-1]}\right)^{\alpha} \cdots \left(a_{j+1}^{[2n-1]}\right)^{\alpha} \left(a_j^{[2n-1]}\right)^{\alpha} s(t_j) \\ &= \prod_{t_j < t_k < t} \left(a_k^{[2n-1]}\right)^{\alpha} s(t_j) \leq 0. \end{aligned} \tag{28}$$

Since $s(t) \leq 0$, $s'(t) \leq 0$, $s(t)$ is not always equal to 0 on any interval $[t, \infty)$, we have $s(t) < 0$ for sufficiently large t, therefore, we get $x^{(2n-1)}(t) < 0$ for sufficiently large t, without loss of generality, we may let $x^{(2n-1)}(t) < 0$ for $t \geq t_j$. Let $s(t_j) = -\gamma^\alpha$ ($\gamma > 0$), using (28), we have

$$\left|x^{(2n-1)}(t)\right|^{\alpha-1} x^{(2n-1)}(t) \exp \int_{t_j}^{t} \frac{r'(s)+q(s)}{r(s)}\,ds \leq \prod_{t_j<t_k<t} \left(a_k^{[2n-1]}\right)^\alpha s(t_j).$$

By the above equality, we obtain

$$\left|x^{(2n-1)}(t)\right|^{\alpha-1} x^{(2n-1)}(t) \leq -\gamma^\alpha \prod_{t_j<t_k<t} \left(a_k^{[2n-1]}\right)^\alpha \exp\left(-\int_{t_j}^{t} \frac{r'(s)+q(s)}{r(s)}\,ds\right).$$

Noting that $x^{(2n-1)}(t) < 0$ for $t \geq t_j$, we can get

$$-\left|x^{(2n-1)}(t)\right|^{\alpha} \leq -\gamma^\alpha \prod_{t_j<t_k<t} \left(a_k^{[2n-1]}\right)^\alpha \exp\left(-\int_{t_j}^{t} \frac{r'(s)+q(s)}{r(s)}\,ds\right).$$

That is,

$$x^{(2n-1)}(t) \leq -\gamma \prod_{t_j<t_k<t} a_k^{[2n-1]} \exp\left(-\int_{t_j}^{t} \frac{r'(s)+q(s)}{\alpha r(s)}\,ds\right) < 0, \tag{29}$$

by Lemma 3, we have $x^{(2n-2)}(t) > 0$ for sufficiently large t, without loss of generality, let $x^{(2n-2)}(t) > 0$, $t \geq t_j$. In view of the condition (B), we have

$$x^{(2n-2)}\left(t_k^+\right) \leq b_k^{[2n-2]} x^{(2n-2)}(t_k), \quad k = j+1, j+2, \ldots. \tag{30}$$

By (29) and (30), applying Lemma 1, we obtain

$$\begin{aligned}
x^{(2n-2)}(t) &\leq x^{(2n-2)}\left(t_j^+\right) \prod_{t_j<t_k<t} b_k^{[2n-2]} \\
&\quad - \gamma \int_{t_0}^{t} \prod_{s<t_k<t} b_k^{[2n-2]} \prod_{t_j<t_k<s} a_k^{[2n-1]} \exp\left(-\int_{t_j}^{s} \frac{r'(v)+q(v)}{\alpha r(v)}\,dv\right) ds \\
&= \prod_{t_j<t_k<t} b_k^{[2n-2]} \Bigg[x^{(2n-2)}\left(t_j^+\right) \\
&\quad - \gamma \int_{t_0}^{t} \prod_{t_j<t_k<s} \frac{a_k^{[2n-1]}}{b_k^{[2n-2]}} \exp\left(-\int_{t_j}^{s} \frac{r'(v)+q(v)}{\alpha r(v)}\,dv\right) ds \Bigg],
\end{aligned} \tag{31}$$

letting $t \to \infty$, applying (31) and the condition (D), we get $x^{(2n-2)}(t) < 0$, which is contracted with $x^{(2n-2)}(t) > 0$, $t \geq t_j$. So we have $x^{(2n-1)}(t_k) > 0$ for any $t_k \geq T$. Since $x^{(2n-1)}(t) > 0$ for $t \geq t_j$, here, without loss of generality, we may let $x^{(2n-1)}(t) > 0$ for $t \geq t_0$. Then $x^{(2n-2)}(t)$ is strictly increasing on $(t_k, t_{k+1}]$. If for any t_k, $x^{(2n-2)}(t_k) < 0$, then $x^{(2n-2)}(t) < 0$ for $t \geq T_1$. If there exists some t_j such that $x^{(2n-2)}(t_j) \geq 0$, since $x^{(2n-2)}(t)$ is strictly monotony increasing and $a_k^{[2n-2]} > 0$, then $x^{(2n-2)}(t) > 0$ for $t > t_j$. Thus there exists $T_2 \geq T_1$ such that $x^{(2n-2)}(t) > 0$ for $t \geq T_2$. So one of the following statements holds:

(A_1) $x^{(2n-1)}(t) > 0$, $x^{(2n-2)}(t) > 0$, $t \geq T_2$;
(B_1) $x^{(2n-1)}(t) > 0$, $x^{(2n-2)}(t) < 0$, $t \geq T_2$.

If (A_1) holds, by the result (b) of Lemma 2, $x^{(2n-3)}(t) > 0$ for all sufficiently large t. Using the result (b) of Lemma 2 repeatedly, for all sufficiently large t, we can conclude that

$$x^{(2n-1)}(t) > 0, \qquad x^{(2n-2)}(t) > 0, \qquad \ldots, \qquad x'(t) > 0, \qquad x(t) > 0.$$

If (B_1) holds, by Lemma 3, we have for all sufficiently large t. Similarly, there exists some $T_3 \geq T_2$ such that one of the following statements holds:

(A_2) $x^{(2n-3)}(t) > 0$, $x^{(2n-4)}(t) > 0$, $t \geq T_3$;
(B_2) $x^{(2n-3)}(t) > 0$, $x^{(2n-4)}(t) < 0$, $t \geq T_3$.

Repeating the discussion above, we can see eventually that there exist some $T' \geq T$ and $l \in \{1, 3, \ldots, 2n-1\}$ such that for $t \geq T'$,

$$\begin{cases} x^{(i)}(t) > 0, & i = 0, 1, \ldots, l; \\ (-1)^{(i-1)} x^{(i)}(t) > 0, & i = l+1, \ldots, 2n-1. \end{cases}$$

The proof is complete. □

Remark 4 We may prove in a similar manner the following statements.

If we replace the condition in Lemma 4 '$x(t) > 0$ for $t \geq T$ ' with '$x(t) < 0$ for $t \geq T$', and under the conditions (A), (B), (C), and (D), then there exist some $T' \geq T$ and $l \in \{1, 3, \ldots, 2n-1\}$ such that, for $t \geq T'$,

$$\begin{cases} x^{(i)}(t) < 0, & i = 0, 1, \ldots, l; \\ (-1)^{(i-1)} x^{(i)}(t) < 0, & i = l+1, \ldots, 2n-1. \end{cases} \tag{32}$$

5 Proofs of main theorems

We now turn to the proofs of Theorem 1 and Theorem 2.

Proof of Theorem 1 If (4) has a nonoscillatory solution $x = x(t)$, without loss of generality, let $x(t) > 0$ ($t \geq t_0$). By Lemma 4, there exist $T \geq t_0$ and an integer $l \in \{1, 3, \ldots, 2n-1\}$ such that for $t \geq T$,

$$x(t) > 0, \qquad x'(t) > 0, \qquad x^{(2n-1)}(t) > 0. \tag{33}$$

Let

$$u(t) = \frac{r(t)|x^{(2n-1)}(t)|^{\alpha-1} x^{(2n-1)}(t)}{\varphi(x(t-\tau))}. \tag{34}$$

We see that $u(t_k^+) \geq 0$ ($k = 1, 2, \ldots$), $u(t) > 0$ for $t \geq T$. By (4), (33), and the condition (A), we get

$$\begin{aligned} u'(t) = {} & \frac{-q(t)|x^{(2n-1)}(t)|^{\alpha-1} x^{(2n-1)}(t) - f(t, x(t), x(t-\tau))}{\varphi(x(t-\tau))} \\ & - \frac{r(t)|x^{(2n-1)}(t)|^{\alpha-1} x^{(2n-1)}(t) \varphi'(x(t-\tau)) x'(t-\tau)}{\varphi^2(x(t-\tau))} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{-q(t)|x^{(2n-1)}(t)|^{\alpha-1}x^{(2n-1)}(t) - f(t,x(t),x(t-\tau))}{\varphi(x(t-\tau))} \\
&\leq -\frac{q(t)}{r(t)}u(t) - p(t), \quad t \neq t_{0,w}.
\end{aligned}
\tag{35}
$$

It follows from the conditions (B), $a_k^{(0)} \geq 1$, and $\varphi'(x) \geq 0$ that

$$
\begin{aligned}
u(t_k^+) &= \frac{r(t_k^+)|x^{(2n-1)}(t_k^+)|^{\alpha-1}x^{(2n-1)}(t_k^+)}{\varphi(x(t_k-\tau)^+)} \\
&\leq \begin{cases}
\frac{(b_k^{[2n-1]})^\alpha r(t_k)|x^{(2n-1)}(t_k)|^{\alpha-1}x^{(2n-1)}(t_k)}{\varphi(x(t_k-\tau))} \\
\quad = (b_k^{[2n-1]})^\alpha u(t_k), \quad t_k - \tau \neq t_m \ (0 < m < k), \\
\frac{(b_k^{[2n-1]})^\alpha r(t_k)|x^{(2n-1)}(t_k)|^{\alpha-1}x^{(2n-1)}(t_k)}{\varphi(a_m^{[0]}x(t_k-\tau))} \\
\quad \leq \frac{(b_k^{[2n-1]})^\alpha r(t_k)|x^{(2n-1)}(t_k)|^{\alpha-1}x^{(2n-1)}(t_k)}{\varphi(x(t_k-\tau))} \\
\quad = (b_k^{[2n-1]})^\alpha u(t_k), \quad t_k - \tau = t_m \ (0 < m < k),
\end{cases}
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
u((t_k+\tau)^+) &= \frac{r((t_k+\tau)^+)|x^{(2n-1)}((t_k+\tau)^+)|^{\alpha-1}x^{(2n-1)}((t_k+\tau)^+)}{\varphi(x(t_k^+))} \\
&\leq \begin{cases}
\frac{r(t_k+\tau)|x^{(2n-1)}(t_k+\tau)|^{\alpha-1}x^{(2n-1)}(t_k+\tau)}{\varphi(a_k^{[0]}x(t_k))} \\
\quad \leq \frac{r(t_k+\tau)|x^{(2n-1)}(t_k+\tau)|^{\alpha-1}x^{(2n-1)}(t_k+\tau)}{\varphi(x(t_k))} \\
\quad = u(t_k+\tau), \quad t_k + \tau \neq t_m \ (k < m), \\
\frac{r(t_m)|x^{(2n-1)}(t_m^+)|^{\alpha-1}x^{(2n-1)}(t_m^+)}{\varphi(a_k^{[0]}x(t_k))} \\
\quad \leq \frac{(b_m^{[2n-1]})^\alpha r(t_m)|x^{(2n-1)}(t_m)|^{\alpha-1}x^{(2n-1)}(t_m)}{\varphi(a_k^{[0]}x(t_k))} \\
\quad \leq \frac{(b_m^{[2n-1]})^\alpha r(t_k+\tau)|x^{(2n-1)}(t_k+\tau)|^{\alpha-1}x^{(2n-1)}(t_k+\tau)}{\varphi(x(t_k))} \\
\quad = (b_m^{[2n-1]})^\alpha u(t_k+\tau), \quad t_k + \tau = t_m \ (k < m).
\end{cases}
\end{aligned}
\tag{37}
$$

So we get

$$
\begin{aligned}
&u'(t) \leq -\frac{q(t)}{r(t)}u(t) - p(t), \quad t \neq t_{0,w}, \\
&u(t_{0,w}^+) \leq \theta_{0,w}u(t_{0,w}),
\end{aligned}
$$

where $t_{0,w} = t_k$ or $t_k + \tau$ ($t_1 = t_{0,1} < t_{0,2} < \cdots < t_{0,w} < t_{0,w+1} < \cdots$) and $\theta_{0,w}$ is defined by (6). Applying Lemma 1, we obtain

$$
\begin{aligned}
u(t) &\leq u(T^+) \prod_{T<t_k<t} \theta_{0,w} \exp\left(\int_T^t -\frac{q(s)}{r(s)}\,ds\right) - \int_T^t \prod_{s<t_k<t} \theta_{0,w} p(s) \exp\left(\int_s^t -\frac{q(\nu)}{r(\nu)}\,d\nu\right) ds \\
&\leq \prod_{T<t_k<t} \theta_{0,w} \exp\left(\int_T^t -\frac{q(s)}{r(s)}\,ds\right) \\
&\quad \times \left[u(T^+) - \int_T^t \prod_{T<t_k<s} \frac{1}{\theta_{0,w}} p(s) \exp\left(\int_T^s \frac{q(\nu)}{r(\nu)}\,d\nu\right) ds\right].
\end{aligned}
\tag{38}
$$

It is easy to see from (5) and (38) that $u(t) < 0$ for sufficiently large t. This is contrary to $u(t) > 0$ for $t \geq T$. Thus every solution of (4) is oscillatory. The proof of Theorem 1 is complete. □

Proof of Theorem 2 If (4) has a nonoscillatory solution $x = x(t)$, without loss of generality, let $x(t) > 0$ ($t \geq t_0$). By Lemma 4, there exists $T \geq t_0$ and an integer $l \in \{1, 3, \ldots, 2n-1\}$ such that for $t \geq T$,

$$x(t) > 0, \qquad x'(t) > 0, \qquad x^{(2n-1)}(t) > 0.$$

Let $u(t)$ be defined by (34), then $u(t_k^+) \geq 0$ ($k = 1, 2, \ldots$), $u(t) > 0$ for $t \geq T$.

By (4), and the condition (A), we also can get

$$u'(t) \leq -\frac{q(t)}{r(t)} u(t) - p(t), \quad t \neq t_{0,w}. \tag{39}$$

It follows from the conditions (B), $\varphi(ab) \geq \varphi(a)\varphi(b)$ ($ab > 0$), and $\varphi'(x) \geq 0$ that

$$u\bigl(t_k^+\bigr) = \frac{r(t_k^+)|x^{(2n-1)}(t_k^+)|^{\alpha-1}x^{(2n-1)}(t_k^+)}{\varphi(x(t_k-\tau)^+)} \leq \begin{cases} \frac{(b_k^{[2n-1]})^\alpha r(t_k)|x^{(2n-1)}(t_k)|^{\alpha-1}x^{(2n-1)}(t_k)}{\varphi(x(t_k-\tau))} \\ \quad = (b_k^{[2n-1]})^\alpha u(t_k), \quad t_k - \tau \neq t_m \ (0 < m < k), \\ \frac{(b_k^{[2n-1]})^\alpha r(t_k)|x^{(2n-1)}(t_k)|^{\alpha-1}x^{(2n-1)}(t_k)}{\varphi(a_m^{[0]}x(t_k-\tau))} \\ \quad \leq \frac{(b_k^{[2n-1]})^\alpha r(t_k)|x^{(2n-1)}(t_k)|^{\alpha-1}x^{(2n-1)}(t_k)}{\varphi(a_m^{[0]})\varphi(x(t_k-\tau))} \\ \quad = \frac{(b_k^{[2n-1]})^\alpha}{\varphi(a_m^{[0]})} u(t_k), \quad t_k - \tau = t_m \ (0 < m < k), \end{cases} \tag{40}$$

$$u\bigl((t_k+\tau)^+\bigr) = \frac{r((t_k+\tau)^+)|x^{(2n-1)}((t_k+\tau)^+)|^{\alpha-1}x^{(2n-1)}((t_k+\tau)^+)}{\varphi(x(t_k^+))} \leq \begin{cases} \frac{r(t_k+\tau)|x^{(2n-1)}(t_k+\tau)|^{\alpha-1}x^{(2n-1)}(t_k+\tau)}{\varphi(a_k^{[0]}x(t_k))} \\ \quad \leq \frac{r(t_k+\tau)|x^{(2n-1)}(t_k+\tau)|^{\alpha-1}x^{(2n-1)}(t_k+\tau)}{\varphi(a_k^{[0]})\varphi(x(t_k))} \\ \quad = \frac{1}{\varphi(a_k^{[0]})} u(t_k+\tau), \quad t_k + \tau \neq t_m \ (k < m), \\ \frac{r(t_m)|x^{(2n-1)}(t_m^+)|^{\alpha-1}x^{(2n-1)}(t_m^+)}{\varphi(a_k^{[0]}x(t_k))} \\ \quad \leq \frac{(b_m^{[2n-1]})^\alpha r(t_m)|x^{(2n-1)}(t_m)|^{\alpha-1}x^{(2n-1)}(t_m)}{\varphi(a_k^{[0]}x(t_k))} \\ \quad \leq \frac{(b_m^{[2n-1]})^\alpha r(t_k+\tau)|x^{(2n-1)}(t_k+\tau)|^{\alpha-1}x^{(2n-1)}(t_k+\tau)}{\varphi(a_k^{[0]})\varphi(x(t_k))} \\ \quad = \frac{(b_m^{[2n-1]})^\alpha}{\varphi(a_k^{[0]})} u(t_k+\tau), \quad t_k + \tau = t_m \ (k < m). \end{cases} \tag{41}$$

So we have

$$u'(t) \leq -\frac{q(t)}{r(t)} u(t) - p(t), \quad t \neq t_{0,w},$$

$$u\bigl(t_{0,w}^+\bigr) \leq \mu_{0,w} u(t_{0,w}),$$

where $t_{0,w} = t_k$ or $t_k + \tau$ ($t_1 = t_{0,1} < t_{0,2} < \cdots < t_{0,w} < t_{0,w+1} < \cdots$) and $\mu_{0,w}$ is defined by (8). Applying Lemma 1, we obtain

$$\begin{aligned} u(t) &\le u(T^+) \prod_{T<t_k<t} \mu_{0,w} \exp\left(\int_T^t -\frac{q(s)}{r(s)}\,ds\right) - \int_T^t \prod_{s<t_k<t} \mu_{0,w} p(s) \exp\left(\int_s^t -\frac{q(v)}{r(v)}\,dv\right) ds \\ &\le \prod_{T<t_k<t} \mu_{0,w} \exp\left(\int_T^t -\frac{q(s)}{r(s)}\,ds\right) \\ &\quad \times \left[u(T^+) - \int_T^t \prod_{T<t_k<s} \frac{1}{\mu_{0,w}} p(s) \exp\left(\int_T^s \frac{q(v)}{r(v)}\,dv\right) ds\right]. \end{aligned} \tag{42}$$

It is easy to see from (7) and (42) that $u(t) < 0$ for sufficiently large t. This is contrary to $u(t) > 0$ for $t \ge T$. Thus every solution of (4) is oscillatory. The proof of Theorem 2 is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally and read and approved the final manuscript.

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Acknowledgements
The authors wish to express their sincere thanks to the anonymous referees and the handling editor for many constructive comments leading to the improved version of this paper.

Received: 31 December 2014 Accepted: 26 June 2015 Published online: 28 August 2015

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