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Blow-up for an evolution p-laplace system with nonlocal sources and inner absorptions

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Abstract

This paper investigates the blow-up properties of positive solutions to the following system of evolution p-Laplace equations with nonlocal sources and inner absorptions

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \int_\Omega v^m \mathrm{d}x - \alpha u^r, & x \in \Omega, \quad t > 0, \\ v_t - \operatorname{div}(|\nabla v|^{q-2}\nabla v) = \int_\Omega u^n \mathrm{d}x - \beta v^s, & x \in \Omega, \quad t > 0 \end{cases}$$

with homogeneous Dirichlet boundary conditions in a smooth bounded domain $\Omega \in R^N (N \geq 1)$, where $p, q > 2$, $m, n, r, s \geq 1$, $\alpha, \beta > 0$. Under appropriate hypotheses, the authors discuss the global existence and blow-up of positive weak solutions by using a comparison principle.

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1 Introduction

In this paper, we deal with the blow-up properties of positive solutions to an evolution p-Laplace system of the form

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \int_\Omega v^m \mathrm{d}x - \alpha u^r, & x \in \Omega, t > 0, \\ v_t - \operatorname{div}(|\nabla v|^{q-2}\nabla v) = \int_\Omega u^n \mathrm{d}x - \beta v^s, & x \in \Omega, t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $p, q > 2$, $m, n, r, s \geq 1$, $\alpha, \beta > 0$, Ω is a bounded domain in $R^N (N \geq 1)$ with a smooth boundary $\partial\Omega$, the initial data $u_0(x) \in C(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$, $v_0(x) \in C(\overline{\Omega}) \cap W_0^{1,q}(\Omega)$ and $\frac{\partial u_0(x)}{\partial \nu} < 0$, $\frac{\partial v_0(x)}{\partial \nu} < 0$, where ν denotes the unit outer normal vector on $\partial\Omega$.

System (1.1) is the classical reaction-diffusion system of Fujita-type for $p = q = 2$. If $p \neq 2$, $q \neq 2$, (1.1) appears in the theory of non-Newtonian fluids [1,2] and in nonlinear filtration theory [3]. In the non-Newtonian fluids theory, the pair (p, q) is a characteristic quantity of the medium. Media with $(p, q) > (2, 2)$ are called dilatant fluids and those with $(p, q) < (2, 2)$ are called pseudoplastics. If $(p, q) = (2, 2)$, they are Newtonian fluids.

System (1.1) has been studied by many authors. For $p = q = 2$, Escobedo and Herrero [4] considered the following problem

$$\begin{cases} u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q, & x \in \Omega, \quad t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.2}$$

where p, $q > 0$. Their main results read as follows. (i) If $pq \le 1$, every solution of (1.2) is global in time. (ii) If $pq > 1$, some solutions are global while some others blow up in finite time.

In the last three decades, many authors studied the following degenerate parabolic problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), & x \in \overline{\Omega}. \end{cases} \tag{1.3}$$

under different conditions (see [5,6] for nonlinear boundary conditions; see [7-10] for local nonlinear reaction terms; see [11] for nonlocal nonlinear reaction terms). In [12], the existence, uniqueness, and regularity of solutions were obtained. When $f(u) = -u^q$, $q > 0$ or $f(u) \equiv 0$ extinction phenomenon of the solution may appear [13-15]; However, if $f(u) = u^q$, $q > 1$ the solution may blow up in finite time [7-10,14].

Especially, in [11], Li and Xie dealt with the following p-Laplace equation

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \int_\Omega u^q(x,t)\mathrm{d}x, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), & x \in \overline{\Omega}. \end{cases} \tag{1.4}$$

Under appropriate hypotheses, they established the local existence and uniqueness of its solution. Furthermore, they obtained that the solution u exists globally if $q < p - 1$; u blows up in finite time if $q > p - 1$ and $u_0(x)$ is large enough.

Recently, in [16], Li generalized (1.4) to system and studied the following problem

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \alpha \int_\Omega v^m \mathrm{d}x, & x \in \Omega, t > 0, \\ v_t - \operatorname{div}(|\nabla v|^{q-2}\nabla v) = \beta \int_\Omega u^n \mathrm{d}x, & x \in \Omega, t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \overline{\Omega}. \end{cases} \tag{1.5}$$

Similar to [11], he proved that whether the solution blows up in finite time depends on the initial data, constants α, β, and the relations between mn and $(p - 1)(q - 1)$.

For other works on parabolic system like (1.1), we refer readers to [17-30] and the references therein.

When $p = q$, $m = n$, $r = s$, $\alpha = \beta$, $u_0(x) = v_0(x)$, system (1.1) is then reduced to a single p-Laplace equation

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \int_\Omega u^m \mathrm{d}x - \alpha u^r. \tag{1.6}$$

However, to the authors' best knowledge, there is little literature on the study of the global existence and blow-up properties for problems (1.1) and (1.6). Motivated by the above works, in this paper, we investigate the blow-up properties of solutions of the problem (1.1) and extend the results of [4,11,16,19] to more generalized cases.

In order to state our results, we introduce some useful symbols. Throughout this paper, we let $\phi(x)$, $\psi(x)$ be the unique solution of the following elliptic problem

$$-\operatorname{div}(|\nabla\varphi|^{p-2}\nabla\varphi) = 1, \quad x \in \Omega; \quad \varphi(x) = 0, \quad x \in \partial\Omega \tag{1.7}$$

and

$$-\operatorname{div}(|\nabla\psi|^{q-2}\nabla\psi) = 1, \quad x \in \Omega; \quad \psi(x) = 0, \quad x \in \partial\Omega, \tag{1.8}$$

respectively. For convenience, we denote

$$m_1 = \min_{\overline{\Omega}} \varphi(x), \quad M_1 = \max_{\overline{\Omega}} \varphi(x), \quad m_2 = \min_{\overline{\Omega}} \psi(x), \quad M_2 = \max_{\overline{\Omega}} \psi(x).$$

Before starting the main results, we introduce a pair of parameters (μ, γ) solving the following characteristic algebraic system

$$\begin{pmatrix} -\mu & m \\ n & -\gamma \end{pmatrix}\begin{pmatrix} \tau \\ \theta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

namely,

$$\tau = \frac{m+\gamma}{mn-\mu\gamma}, \quad \theta = \frac{n+\mu}{mn-\mu\gamma}$$

with

$$\mu = \max\{p-1, r\}, \quad \gamma = \max\{q-1, s\}.$$

It is obvious that $1/\tau$ and $1/\theta$ share the same signs. We claim that the critical exponent of problem (1.1) should be $(1/\tau, 1/\theta) = (0, 0)$, described by the following theorems.

Theorem 1.1. *Assume that* $(1/\tau, 1/\theta) < (0, 0)$, *then there exist solutions of* (1.1) *being globally bounded.*

Theorem 1.2. *Assume that* $(1/\tau, 1/\theta) > (0, 0)$, *then the nonnegative solution of (1.1) blows up in finite time for sufficiently large initial values and exists globally for sufficiently small initial values.*

Theorem 1.3. *Assume that* $(1/\tau, 1/\theta) = (0, 0)$, $\phi(x)$ *and* $\psi(x)$ *are defined in* (1.7) *and* (1.8), *respectively.*

(i) *Suppose that* $r > p - 1$ *and* $s > q - 1$. *If* $\alpha^n\beta^r \geq |\Omega|^{n+r}$, *then the solutions are globally bounded for small initial data; if* $\int_\Omega \psi^m \mathrm{d}x > \alpha\varphi^r$, $\int_\Omega \varphi^n \mathrm{d}x > \beta\psi^s$, *then the solutions blow up in finite time for large data.*

(ii) *Suppose that* $p - 1 > r$ *and* $q - 1 > s$. *If* $\left(\int_\Omega \varphi^n \mathrm{d}x\right)^{\frac{1}{q-1}}\left(\int_\Omega \psi^m \mathrm{d}x\right)^{\frac{1}{m}} \leq 1$, *then the solutions are globally bounded for small initial data; if* $\int_\Omega \psi^m \mathrm{d}x > 1$, $\int_\Omega \varphi^n \mathrm{d}x > 1$*then the solutions blow up in finite time for large data.*

(iii) *Suppose that* $p - 1 > r$ *and* $s > q - 1$. *If* $\int_\Omega \varphi^n \mathrm{d}x \leq |\Omega|^{-\frac{1}{m}}\beta^{\frac{1}{s}}$, *then the solutions are globally bounded for small initial data; if* $\int_\Omega \psi^m \mathrm{d}x > 1$, $\int_\Omega \varphi^n \mathrm{d}x > \beta\psi^s$, *then the solutions blow up in finite time for large data.*

(iv) *Suppose that* $r > p - 1$ *and* $q - 1 > s$. *If* $\int_\Omega \psi^m \mathrm{d}x \leq |\Omega|^{-\frac{1}{n}}\alpha^{\frac{1}{r}}$, *then the solutions are globally bounded for small initial data; if* $\int_\Omega \varphi^n \mathrm{d}x > 1$, $\int_\Omega \psi^m \mathrm{d}x > \alpha\varphi^r$, *then the solutions blow up in finite time for sufficiently large data.*

The rest of this paper is organized as follows. In Section 2, we shall establish the comparison principle and local existence theorem for problem (1.1). Theorems 1.1 and

1.2 will be proved in Section 3 and Section 4, respectively. Finally, we will give the proof of Theorem 1.3 in Section 5.

2 Preliminaries

Since the equations in (1.1) are degenerate at points where ǔu = 0 or ǔv = 0, there is no classical solution in general, and we therefore consider its weak solutions. Let $\Omega_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$ and $\overline{\Omega}_T = \overline{\Omega} \times [0, T)$. We begin with the precise definition of a weak solution of problem (1.1).

Definition 2.1 A pair of functions $(u(x, t), v(x, t))$ is called a weak solution of problem (1.1) in $\overline{\Omega}_T \times \overline{\Omega}_T$ if and only if

(i) (u, v) is in the space $(C(0,T;L^\infty(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))) \times (C(0,T;L^\infty(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega)))$ and $(u_t, v_t) \in L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega))$.

(ii) the following equalities

$$\int\int_{\Omega_T} u_t\phi_1 \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} |\nabla u|^{p-2}\nabla u \cdot \nabla\phi_1 \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \phi_1 (\int_\Omega v^m \mathrm{d}x - \alpha u^r)\mathrm{d}x\mathrm{d}t$$

and

$$\int\int_{\Omega_T} v_t\phi_2 \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} |\nabla v|^{q-2}\nabla v \cdot \nabla\phi_2 \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_T} \phi_2 (\int_\Omega u^n \mathrm{d}x - \beta v^s)\mathrm{d}x\mathrm{d}t$$

hold for all φ_1, φ_2, which belong to the class of test functions

$$\Theta_1 \equiv \left\{\Psi \in C^{1,1}(\overline{\Omega}_T); \Psi(x,T) = 0; \Psi(x,t) = 0 \text{ on } S_T\right\}.$$

(iii) $u(x, t)|_{t=0} = u_0(x)$, $v(x, t)|_{t=0} = v_0(x)$ for all $x \in \overline{\Omega}$.

In a natural way, the notion of a weak subsolution for (1.1) is given as follows.

Definition 2.2 A pair of functions $(\underline{u}(x, t), \underline{v}(x, t))$ is called a weak subsolution of problem (1.1) in $\overline{\Omega}_T \times \overline{\Omega}_T$ if and only if

(i) $(\underline{u}, \underline{v})$ is in the space $(C(0,T;L^\infty(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))) \times (C(0,T;L^\infty(\Omega)) \cap L^q(0,T;W_0^{1,q}(\Omega)))$ and $(\underline{u}_t, \underline{v}_t) \in L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega))$.

(ii) the following inequalities

$$\int\int_{\Omega_T} \underline{u}_t\phi_1 \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} \left|\nabla \underline{u}\right|^{p-2}\nabla \underline{u} \cdot \nabla\phi_1 \mathrm{d}x\mathrm{d}t \le \int\int_{\Omega_T} \phi_1 (\int_\Omega \underline{v}^m \mathrm{d}x - \alpha \underline{u}^r)\mathrm{d}x\mathrm{d}t$$

and

$$\int\int_{\Omega_T} \underline{v}_t\phi_2 \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} \left|\nabla \underline{v}\right|^{q-2}\nabla \underline{v} \cdot \nabla\phi_2 \mathrm{d}x\mathrm{d}t \le \int\int_{\Omega_T} \phi_2 (\int_\Omega \underline{u}^n \mathrm{d}x - \beta \underline{v}^s)\mathrm{d}x\mathrm{d}t$$

hold for any φ_1, φ_2, which belong to the class of test functions

$$\Theta_2 \equiv \{\Psi \in C^{1,1}(\overline{\Omega}_T); \Psi(x,t) \ge 0; \Psi(x,T) = 0; \Psi(x,t) = 0 \text{ on } S_T\}.$$

(iii) $\underline{u}(x, t)|_{t=0} \le u_0(x)$, $\underline{v}(x, t)|_{t=0} \le v_0(x)$ for all $x \in \overline{\Omega}$.

Similarly, a pair of functions $(\overline{u}(x,t), \overline{v}(x,t))$ is a weak supersolution of (1.1) if the reversed inequalities hold in Definition 2.2. A weak solution of (1.1) is both a weak subsolution and a weak supersolution of (1.1).

We shall use the following comparison principle to prove our global and nonglobal existence results.

Proposition 2.3 Let $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ be a nonnegative subsolution and supersolution of (1.1), respectively, with $(\underline{u}(x,0), \underline{v}(x,0)) \le (\overline{u}(x,0), \overline{v}(x,0))$ for all $x \in \overline{\Omega}$. Then, $(\underline{u}, \underline{v}) \le (\overline{u}, \overline{v})$ a.e. in $\overline{\Omega}_T \times \overline{\Omega}_T$.

Proof. From the definitions of weak subsolution and supersolution, for any φ_1, $\varphi_2 \in \Theta_2$, we could obtain that

$$\begin{aligned}&\int\int_{\Omega_T} (\underline{u}_t - \overline{u}_t)\phi_1 \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} \left(|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla \overline{u}|^{p-2}\nabla \overline{u}\right)\cdot \nabla\phi_1 \mathrm{d}x\mathrm{d}t \\ &\le \int\int_{\Omega_T} \phi_1 \left[\int_\Omega (\underline{v}^m - \overline{v}^m)\mathrm{d}x - \alpha(\underline{u}^r - \overline{u}^r)\right]\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&\int\int_{\Omega_T} (\underline{v}_t - \overline{v}_t)\phi_2 \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_T} \left(|\nabla \underline{v}|^{q-2}\nabla \underline{v} - |\nabla \overline{v}|^{q-2}\nabla \overline{v}\right)\cdot \nabla\phi_2 \mathrm{d}x\mathrm{d}t \\ &\le \int\int_{\Omega_T} \phi_2 \left[\int_\Omega (\underline{u}^n - \overline{u}^n)\mathrm{d}x - \beta(\underline{v}^s - \overline{v}^s)\right]\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.2}$$

In addition, inequalities (2.1) and (2.2) remain true for any subcylinder of the form $\Omega_\tau = \Omega \times (0, \tau) \subset \Omega_T$ and corresponding lateral boundary $S_\tau = \partial\Omega \times (0, \tau) \subset S_T$. Taking a special test function $\phi_1 = \chi_{[0,\tau]}(\underline{u} - \overline{u})_+$ in (2.1), where $\chi_{[0,\tau]}$ is the characteristic function defined on $[0, \tau]$ and $s_+ = \max\{s, 0\}$, we find that

$$\begin{aligned}&\int\int_{\Omega_\tau} (\underline{u}_t - \overline{u}_t)(\underline{u} - \overline{u})_+ \mathrm{d}x\mathrm{d}t + \int\int_{\Omega_\tau} \left(|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla \overline{u}|^{p-2}\nabla \overline{u}\right)\cdot \nabla(\underline{u} - \overline{u})_+ \mathrm{d}x\mathrm{d}t \\ &\le m|\Omega|\widehat{M}^{m-1}\int\int_{\Omega_\tau} (\underline{v} - \overline{v})_+(\underline{u} - \overline{u})_+\mathrm{d}x\mathrm{d}t + \alpha r\widehat{M}^{r-1}\int\int_{\Omega_\tau}(\underline{u} - \overline{u})_+^2\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{2.3}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω and

$$\widehat{M} = \max\left\{\|\underline{u}\|_{L^\infty(\Omega_T)}, \|\overline{u}\|_{L^\infty(\Omega_T)}, \|\underline{v}\|_{L^\infty(\Omega_T)}, \|\overline{v}\|_{L^\infty(\Omega_T)}\right\}.$$

Next, our task is to estimate the first term on the right-side of (2.3). In view of Cauchy's inequality, we see that

$$\begin{aligned}&m|\Omega|\widehat{M}^{m-1}\int\int_{\Omega_\tau} (\underline{v} - \overline{v})_+(\underline{u} - \overline{u})_+\mathrm{d}x\mathrm{d}t \\ &\le \frac{1}{2}m|\Omega|\widehat{M}^{m-1}\left(\int\int_{\Omega_\tau}(\underline{v} - \overline{v})_+^2\mathrm{d}x\mathrm{d}t + \int\int_{\Omega_\tau}(\underline{u} - \overline{u})_+^2\mathrm{d}x\mathrm{d}t\right).\end{aligned} \tag{2.4}$$

Furthermore, by Lemma 1.4.4 in [12], we know that there exists $\delta > 0$ such that

$$\left(|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla \overline{u}|^{p-2}\nabla \overline{u}\right)\cdot \nabla\chi_{[0,\tau]}(\underline{u} - \overline{u}) \ge \min\left\{0, \delta\left|\nabla(\underline{u} - \overline{u})_+\right|^p\right\}. \tag{2.5}$$

Combining now (2.3)-(2.5), we deduce that

$$\int_\Omega (\underline{u} - \overline{u})_+^2\mathrm{d}x \le C_1\int\int_{\Omega_\tau}(\underline{u} - \overline{u})_+^2\mathrm{d}x\mathrm{d}t + C_2\int\int_{\Omega_\tau}(\underline{v} - \overline{v})_+^2\mathrm{d}x\mathrm{d}t, \tag{2.6}$$

here $C_1 = \frac{1}{2}m|\Omega|\widehat{M}^{m-1} + \alpha r\widehat{M}^{r-1}$, $C_2 = \frac{1}{2}m|\Omega|\widehat{M}^{m-1}$.

Likewise, taking test function $\phi_2 = \chi_{[0,\tau]}(\underline{v} - \overline{v})_+$ in (2.2), we have that

$$\int_\Omega (\underline{v} - \overline{v})_+^2 \mathrm{d}x \le C_3 \int \int_{\Omega_\tau} (\underline{u} - \overline{u})_+^2 \mathrm{d}x\mathrm{d}t + C_4 \int \int_{\Omega_\tau} (\underline{v} - \overline{v})_+^2 \mathrm{d}x\mathrm{d}t, \tag{2.7}$$

where C_3, C_4 denote some positive constants. Moreover, there exists a large enough constant C, such that

$$\int_\Omega \left[(\underline{u} - \overline{u})_+^2 + (\underline{v} - \overline{v})_+^2\right] \mathrm{d}x \le C \int \int_{\Omega_\tau} \left[(\underline{u} - \overline{u})_+^2 + (\underline{v} - \overline{v})_+^2\right] \mathrm{d}x\mathrm{d}t. \tag{2.8}$$

Now, we write

$$y(\tau) = (\underline{u} - \overline{u})_+^2 + (\underline{v} - \overline{v})_+^2,$$

then, (2.8) implies that

$$y(\tau) \le C \int_0^\tau y(t)\mathrm{d}t \quad \text{for a.e.} \quad 0 \le \tau \le T. \tag{2.9}$$

By Gronwall's inequality, we know that $y(\tau) = 0$, for any $\tau \in [0, T]$. Thus, $(\underline{u} - \overline{u})_+ = (\underline{v} - \overline{v})_+ = 0$, this means that $\underline{u} \le \overline{u}$, $\underline{v} \le \overline{v}$ in $\overline{\Omega}_T$ as desired. The proof of Proposition 2.3 is complete. □

With the above established comparison principle in hand, we are able to show the basic existence theorem of weak solutions. Here, we only state the local existence theorem, and its proof is standard [12, 16, for more details].

Theorem 2.1 *Given* $(0, 0) \le (u_0, v_0) \in (C(\overline{\Omega}) \cap W_0^{1,p}) \times (C(\overline{\Omega}) \cap W_0^{1,q})$, *there is some* $T_0 > 0$ *such that the problem* (1.1) *admits a nonnegative unique weak solution* (u, v) *for each* $t < T_0$, *and* $(u, v) \in (C(0, T_0; L^\infty(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))) \times (C(0, T; L^\infty(\Omega)) \cap L^q(0, T_0; W_0^{1,q}(\Omega)))$. *Furthermore, either* $T_0 = \infty$ *or*

$$\lim_{t \to T_0^-} \sup(\| u(x, t)||_\infty + \| v(x, t)||_\infty) = \infty.$$

3 Proof of Theorem 1.1

Proof of Theorem 1.1. Notice that $(1/\tau, 1/\theta) < (0, 0)$ implies

$$mn < \mu\gamma = \max\{p - 1, r\} \max\{q - 1, s\}.$$

We will prove Theorem 1.1 in four subcases.

(a) For $\mu = r$, $\gamma = s$, we then have $mn < rs$. Let $(\overline{u}, \overline{v}) = (A, B)$, where $A \ge \max_{x \in \overline{\Omega}} u_0(x)$, $B \ge \max_{x \in \overline{\Omega}} v_0(x)$ will be determined later. After a simple computation, we have

$$\overline{u}_t - \mathrm{d}iv(|\nabla \overline{u}|^{p-2} \nabla \overline{u}) - \int_\Omega \overline{v}^m \mathrm{d}x + \alpha \overline{u}^r = \alpha A^r - |\Omega| B^m,$$

and

$$\overline{v}_t - \mathrm{div}(|\nabla \overline{v}|^{p-2} \nabla \overline{v}) - \int_\Omega \overline{u}^n \mathrm{d}x + \beta \overline{v}^s = \beta B^s - |\Omega| A^n.$$

So, $(\overline{u}, \overline{v})$ is a time-independent supersolution of problem (1.1) if

$$\alpha A^r \ge |\Omega| B^m \text{ and } \beta B^s \ge |\Omega|\, A^n,$$

i.e.,

$$B^{\frac{m}{r}}\left(\frac{|\Omega|}{\alpha}\right)^{\frac{1}{r}} \leq A \leq B^{\frac{s}{n}}\left(\frac{|\Omega|}{\beta}\right)^{\frac{1}{n}}. \tag{3.1}$$

(b) For $\mu = p - 1$, $\gamma = q - 1$, we then have $mn < (p - 1)(q - 1)$. Let

$$(\overline{u}, \overline{v}) = (A(\varphi + 1), B(\psi + 1)),$$

where ϕ, ψ satisfying (1.7) and (1.8), respectively. Taking

$$A \geq \max\left\{\max_{\overline{\Omega}} u_0(x), \left((m_1 + 1)^{\frac{mn}{q-1}}(M_2 + 1)^m|\Omega|^{\frac{m+q-1}{q-1}}\right)^{\frac{q-1}{(p-1)(q-1)-mn}}\right\},$$

and

$$B \geq \max\left\{\max_{\overline{\Omega}} v_0(x), \left((m_1 + 1)^n(M_2 + 1)^{\frac{mn}{p-1}}|\Omega|^{\frac{n+p-1}{q-1}}\right)^{\frac{p-1}{(p-1)(q-1)-mn}}\right\},$$

then it is easy to verify that $(\overline{u}, \overline{v})$ is a global supersolution for system (1.1).

(c) For $\mu = r$, $\gamma = q - 1$, we then have $mn < r(q - 1)$. Choose $A \geq \max_{x\in\overline{\Omega}} u_0(x)$ and $B \geq \max_{x\in\overline{\Omega}} v_0(x)$ satisfy

$$(|\Omega|A^n)^{\frac{1}{q-1}} \leq B \leq \left(\frac{\alpha}{|\Omega|}A^r(M_2 + 1)^{-m}\right)^{\frac{1}{m}}.$$

Let $(\overline{u}, \overline{v}) = (A, B(\psi + 1))$ with ψ defined by (1.8). By direct Computation, we arrive at

$$\overline{u}_t - \operatorname{div}(|\nabla\overline{u}|^{p-2}\nabla\overline{u}) - \int_\Omega \overline{v}^m \mathrm{d}x + \alpha\overline{u}^r \geq 0, \tag{3.2}$$

and

$$\overline{v}_t - \operatorname{div}(|\nabla\overline{v}|^{p-2}\nabla\overline{v}) - \int_\Omega \overline{u}^n \mathrm{d}x + \beta\overline{v}^s \geq 0. \tag{3.3}$$

(d) For $\mu = p - 1$, $\gamma = s$, we then have $mn < r(q - 1)$. Let $(\overline{u}, \overline{v}) = (A(\varphi + 1), B)$ with ϕ defined by (1.7), where $A \geq \max_{x\in\overline{\Omega}} u_0(x)$ and $B \geq \max_{x\in\overline{\Omega}} v_0(x)$. Then, (3.2) and (3.3) hold if

$$(|\Omega|B^m)^{\frac{1}{p-1}} \leq A \leq \left(\frac{\beta}{|\Omega|}B^s(M_1 + 1)^{-n}\right)^{\frac{1}{n}}.$$

The proof of Theorem 1.1 is complete. □

4 Proof of Theorem 1.2

Proof of Theorem 1.2. Observe that $1/\tau$, $1/\theta > 0$ implies

$$pq > \mu\gamma = \max\{p - 1, r\}\max\{q - 1, s\}.$$

For $\mu = r$, $\gamma = s$. Choosing

$$B = \left(\frac{\alpha^n\beta^r}{|\Omega|^{n+r}}\right)^{\frac{1}{mn-rs}} \text{ and } A = \frac{1}{2}\left[\left(\frac{|\Omega|}{\alpha}\right)^{\frac{1}{r}}B^{\frac{m}{r}} + \left(\frac{\beta}{|\Omega|}\right)^{\frac{1}{n}}B^{\frac{s}{n}}\right],$$

then $(\overline{u},\overline{v}) = (A, B)$ is a global supersolution for problem (1.1) provided that $A \geq \max_{x\in\overline{\Omega}} u_0(x)$ and $B \geq \max_{x\in\overline{\Omega}} v_0(x)$.

For $\mu = p - 1$, $\gamma = q - 1$. Let $(\overline{u},\overline{v}) = (A(\varphi + 1), B(\psi + 1))$, where ϕ and ψ satisfying (1.7) and (1.8), respectively. Choosing

$$A = \frac{1}{2}\left(|\Omega|^{\frac{1}{p-1}}(M_2 + 1)^{\frac{m}{p-1}} B^{\frac{m}{p-1}} + \frac{1}{m_1 + 1}|\Omega|^{-\frac{1}{n}} B^{\frac{q-1}{n}}\right),$$

and

$$B = \left(|\Omega|^{n+p-1}(m_1 + 1)^{n(p-1)}(M_2 + 1)^{mn}\right)^{-\frac{1}{mn-(p-1)(q-1)}},$$

therefore, $(\overline{u},\overline{v})$ is a global supersolution for system (1.1) if $A \geq \max_{x\in\overline{\Omega}} u_0(x)$ and $B \geq \max_{x\in\overline{\Omega}} v_0(x)$.

For other cases, the solutions of (1.1) should be global due to the above discussion.

Next, we begin to prove our blow-up conclusion under large enough initial data. Due to the requirement of the comparison principle, we will construct blow-up subsolutions in some subdomain of Ω in which u, $v > 0$. We use an idea from Souplet [31] and apply it to degenerate equations. Since problem (1.1) does not make sense for negative values of (u, v), we actually consider the following problem

$$\begin{cases} Pu(x,t) \equiv u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) - \int_\Omega v_+^m \mathrm{d}x + \alpha u_+^r = 0, & x \in \Omega, t > 0, \\ Qv(x,t) \equiv v_t - \mathrm{div}(|\nabla v|^{q-2}\nabla v) - \int_\Omega u_+^n \mathrm{d}x + \beta v_+^s = 0, & x \in \Omega, t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \overline{\Omega}, \end{cases} \tag{4.1}$$

where $u_+ = \max\{0, u\}$, $v_+ = \max\{0, v\}$. Let $\varpi(x)$ be a nontrivial nonnegative continuous function and vanish on $\partial\Omega$. Without loss of generality, we may assume that $0 \in \Omega$ and $\varpi(0) > 0$. We shall construct a self-similar blow-up subsolution to complete our proof.

Set

$$\underline{u}(x,t) = \frac{W(y_1)}{(T-t)^{l_1}}, \quad \underline{v}(x,t) = \frac{W(y_2)}{(T-t)^{l_2}}, \tag{4.2}$$

here

$$y_i = \frac{|x|}{(T-t)^{\sigma_i}} \geq 0, \quad W(y_i) = 1 - {y_i}^2, \quad i = 1, 2,$$

and l_i, $\sigma_i > 0 (i = 1, 2)$, $0 < T < 1$ are to be determined later. Notice the fact that

$$\begin{aligned} &\mathrm{supp}\underline{u}(x,t)_+ = \overline{B(0,(T-t)^{\sigma_1})} \subset \overline{B(0,T^{\sigma_1})} \subset \Omega, \\ &\mathrm{supp}\underline{v}(x,t)_+ = \overline{B(0,(T-t)^{\sigma_2})} \subset \overline{B(0,T^{\sigma_2})} \subset \Omega \end{aligned} \tag{4.3}$$

for sufficiently small $T > 0$.

Calculating directly, we obtain

$$\begin{aligned} \underline{u}_t &= \frac{l_1 W(y_1) + \sigma_1 y_1 W'(y_1)}{(T-t)^{l_1+1}}, \quad -\Delta\underline{u} = \frac{2N}{(T-t)^{l_1+2\sigma_1}}, \\ \underline{v}_t &= \frac{l_2 W(y_2) + \sigma_2 y_2 W'(y_2)}{(T-t)^{l_2+1}}, \quad -\Delta\underline{v} = \frac{2N}{(T-t)^{l_2+2\sigma_2}}, \end{aligned}$$

and

$$\int_\Omega \underline{v}_+^m \mathrm{d}x = \frac{1}{(T-t)^{ml_2}} \int_{B(0,(T-t)^{\sigma_2})} W^m\Big(\frac{|x|}{(T-t)^{\sigma_2}}\Big)\mathrm{d}x \geq \frac{S_1}{(T-t)^{ml_2-N\sigma_2}},$$
$$\int_\Omega \underline{u}_+^n \mathrm{d}x = \frac{1}{(T-t)^{nl_1}} \int_{B(0,(T-t)^{\sigma_1})} W^n\Big(\frac{|x|}{(T-t)^{\sigma_1}}\Big)\mathrm{d}x \geq \frac{S_2}{(T-t)^{nl_1-N\sigma_1}},$$

where

$$S_1 = \int_{B(0,1)} W^m(|\xi|)\mathrm{d}\xi, \quad S_2 = \int_{B(0,1)} W^n(|\xi|)\mathrm{d}\xi.$$

On the other hand, we know

$$\begin{aligned} \operatorname{div}(|\nabla\underline{u}|^{p-2}\nabla\underline{u}) &= |\nabla\underline{u}|^{p-2}\Delta\underline{u} + (p-2)|\nabla\underline{u}|^{p-4}(\nabla\underline{u})'(H_x(\underline{u}))\nabla\underline{u} \\ &= |\nabla\underline{u}|^{p-2}\Delta\underline{u} + (p-2)|\nabla\underline{u}|^{p-4}\sum_{j=1}^N\sum_{i=1}^N \frac{\partial\underline{u}}{\partial x_i}\frac{\partial^2\underline{u}}{\partial x_i\partial x_j}\frac{\partial\underline{u}}{\partial x_j}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \operatorname{div}(|\nabla\underline{v}|^{q-2}\nabla\underline{v}) &= |\nabla\underline{v}|^{q-2}\Delta\underline{v} + (q-2)|\nabla\underline{v}|^{q-4}(\nabla\underline{v})'(H_x(\underline{v}))\nabla\underline{v} \\ &= |\nabla\underline{v}|^{q-2}\Delta\underline{v} + (q-2)|\nabla\underline{v}|^{q-4}\sum_{j=1}^N\sum_{i=1}^N \frac{\partial\underline{v}}{\partial x_i}\frac{\partial^2\underline{v}}{\partial x_i\partial x_j}\frac{\partial\underline{v}}{\partial x_j}, \end{aligned} \tag{4.5}$$

here $H_x(\underline{u})$, $H_x(\underline{v})$ denotes the Hessian matrix of $\underline{u}(x, t)$, $\underline{v}(x, t)$ respect to x, respectively. Use the notation $\mathrm{d}(\Omega) = \mathrm{diam}(\Omega)$, then from (4.4) and (4.5), it follows that

$$\begin{aligned} \left|\operatorname{div}(|\nabla\underline{u}|^{p-2}\nabla\underline{u})\right| &\leq \frac{2N}{(T-t)^{l_1+2\sigma_1}}\Big(\frac{\mathrm{d}(\Omega)}{(T-t)^{l_1+2\sigma_1}}\Big)^{p-2} \\ &\quad + \frac{2N(p-2)}{(T-t)^{l_1+2\sigma_1}}\Big(\frac{\mathrm{d}(\Omega)}{(T-t)^{l_1+2\sigma_1}}\Big)^{p-4}\Big(\frac{\mathrm{d}(\Omega)}{(T-t)^{l_1+2\sigma_1}}\Big)^2 \\ &= \frac{2N(p-1)\mathrm{d}(\Omega)^{p-2}}{(T-t)^{(l_1+2\sigma_1)(p-1)}}, \\ \left|\operatorname{div}(|\nabla\underline{v}|^{q-2}\nabla\underline{v})\right| &\leq \frac{2N}{(T-t)^{l_2+2\sigma_2}}\Big(\frac{\mathrm{d}(\Omega)}{(T-t)^{l_2+2\sigma_2}}\Big)^{q-2} \\ &\quad + \frac{2N(q-2)}{(T-t)^{l_2+2\sigma_2}}\Big(\frac{\mathrm{d}(\Omega)}{(T-t)^{l_2+2\sigma_2}}\Big)^{q-4}\Big(\frac{\mathrm{d}(\Omega)}{(T-t)^{l_2+2\sigma_2}}\Big)^2 \\ &= \frac{2N(q-1)\mathrm{d}(\Omega)^{q-2}}{(T-t)^{l_2+2\sigma_2(q-1)}}. \end{aligned}$$

Further, we have

$$\begin{aligned} P\underline{u}(x,t) &\leq \frac{l_1}{(T-t)^{l_1+1}} + \frac{2N(p-1)\mathrm{d}(\Omega)^{p-2}}{(T-t)^{(l_1+2\sigma_1)(p-1)}} + \frac{\alpha}{(T-t)^{rl_1}} \\ &\quad - \frac{S_1}{(T-t)^{ml_2-N\sigma_2}}, \end{aligned} \tag{4.6}$$

and

$$Q\underline{v}(x,t) \le \frac{l_2}{(T-t)^{l_2+1}} + \frac{2N(q-1)\mathrm{d}(\Omega)^{q-2}}{(T-t)^{(l_2+2\sigma_2)(q-1)}} + \frac{\beta}{(T-t)^{sl_2}} - \frac{S_2}{(T-t)^{nl_1-N\sigma_1}}. \tag{4.7}$$

Since $1/\tau$, $1/\theta < 0$, we see that $\mu\gamma < mn$. In addition, it is clear that

$$\frac{\mu}{m} < \frac{n+1}{m+1} \text{ or } \frac{\gamma}{n} < \frac{m+1}{n+1}. \tag{4.8}$$

For $\frac{\mu}{m} < \frac{n+1}{m+1}$, we choose l_1 and l_2 such that

$$\frac{\mu}{m} < \frac{l_2}{l_1} < \min\left\{\frac{n+1}{m+1}, \frac{n}{\gamma}\right\} \text{ and } \mu < \frac{1+l_1}{l_1} < \frac{ml_2}{l_1}. \tag{4.9}$$

Recall that $\mu = \max\{p - 1, r\}$ and $\gamma = \max\{q - 1, s\}$, then (4.9) implies

$$ml_2 > rl_1, \quad ml_2 > l_1(p-1), \quad ml_2 > l_1 + 1,$$

and

$$nl_1 > sl_2, \quad nl_1 > l_2(q-1), \quad nl_1 > l_2 + 1.$$

Next, we can choose positive constants σ_1, σ_2 sufficiently small such that

$$\sigma_1 = \sigma_2 < \min\left\{\frac{ml_2-(l_1+1)}{N}, \frac{ml_2-rl_1}{N}, \frac{ml_2-l_1(p-1)}{N+2(p-1)}, \frac{nl_1-(l_2+1)}{N}, \frac{nl_1-sl_2}{N}, \frac{nl_1-l_2(q-1)}{N+2(q-1)}\right\},$$

consequently, we have

$$\begin{aligned} ml_2 - N\sigma_2 &> \max\left\{l_1+1, (l_1+2\sigma_1)(p-1), rl_1\right\}, \\ nl_1 - N\sigma_1 &> \max\left\{l_2+1, (l_2+2\sigma_2)(q-1), sl_2\right\}. \end{aligned} \tag{4.10}$$

For $\frac{\gamma}{n} < \frac{m+1}{n+1}$, we fix l_1 and l_2 to satisfy

$$\frac{\gamma}{n} < \frac{l_1}{l_2} < \min\left\{\frac{m+1}{n+1}, \frac{m}{\mu}\right\} \text{ and } \gamma < \frac{1+l_2}{l_2} < \frac{nl_1}{l_2}, \tag{4.11}$$

then we can also select σ_1, σ_2 small enough such that (4.10) holds.
From (4.6), (4.7) and (4.10), for sufficiently small $T > 0$, it follows that

$$P\underline{u}(x,t) \le 0, \quad Q\underline{v}(x,t) \le 0 \text{ in } \overline{\Omega}_T. \tag{4.12}$$

Since $\varpi(0) > 0$ and $\varpi(x)$ are continuous, there exist two positive constants ρ and ε such that $\varpi(x) \ge \varepsilon$ for all $x \in B(0, \rho) \subset \Omega$. Choose T small enough to insure $B(0,T^{\sigma_1}) \subset B(0,\rho)$, hence $\underline{u} \le 0$, $\underline{v} \le 0$ on S_T. From (4.1) and (4.2), it follows that $\underline{v}(x,0) \le \overline{M}\varpi(x)$, $\underline{v}(x,0) \le \overline{M}\varpi(x)$ for sufficiently large $\overline{M}$. By comparison principle, we have $(\underline{u}, \underline{v}) \le (u, v)$ provided that $u_0(x) \ge \overline{M}\varpi(x)$ and $v_0(x) \ge \overline{M}\varpi(x)$. It shows that (u, v) blows up in finite time. The proof of Theorem 1.2 is complete. □

5 Proof of Theorem 1.3

Proof of Theorem 1.3. In the critical case of $(1/\tau, 1/\theta) = (0, 0)$, we have $mn = \mu\gamma$.

(i) For $r > p - 1$, $s > q - 1$, we know $mn = rs$. Thanks to $\alpha^n\beta^r \geq |\Omega|^{n+r}$, we can choose A and B sufficiently large such that $A \geq \max_{x\in\overline{\Omega}} u_0(x)$, $B \geq \max_{x\in\overline{\Omega}} v_0(x)$ and

$$B^{\frac{m}{r}}\left(\frac{|\Omega|}{\alpha}\right)^{\frac{1}{r}} \leq A \leq B^{\frac{s}{n}}\left(\frac{|\Omega|}{\beta}\right)^{\frac{1}{n}}.$$

Clearly, $(\overline{u}, \overline{v}) = (A, B)$ is a supersolution of problem (1.1), then by comparison principle, the solution of (1.1) should be global.

Next, we begin to prove our blow-up conclusion. Since $mn = rs$, we can choose constants $l_1, l_2 > 1$ such that

$$\frac{q-2}{r-1} < \frac{s}{n} = \frac{l_1}{l_2} = \frac{m}{r} < \frac{s-1}{p-2}. \tag{5.1}$$

According to Proposition 2.3, we only need to construct a suitable blow-up subsolution of problem (1.1) on $\overline{\Omega}_T \times \overline{\Omega}_T$. Let $y(t)$ be the solution of the following ordinary differential equation

$$\begin{cases} y'(t) = c_1 y^{\delta_1} - c_2 y^{\delta_2}, & t > 0, \\ y(0) = y_0 > 0, \end{cases}$$

where

$$c_1 = \min\left\{\frac{\int_\Omega \psi^m \mathrm{d}x - \alpha\varphi^r}{l_1\varphi}, \frac{\int_\Omega \varphi^n \mathrm{d}x - \beta\psi^s}{l_2\psi}\right\}, \quad c_2 = \max\left\{\frac{1}{l_1\varphi}, \frac{1}{l_2\psi}\right\},$$
$$\delta_1 = \min\left\{(r-1)l_1 + 1, (s-1)l_2 + 1\right\}, \quad \delta_2 = \max\left\{(p-2)l_1 + 1, (q-2)l_2 + 1\right\}.$$

Since $\int_\Omega \psi^m \mathrm{d}x > \alpha\varphi^r$ and $\int_\Omega \varphi^n \mathrm{d}x > \beta\psi^s$, we have $c_1 > 0$. On the other hand, by virtue of (5.1), it is easy to see that $\delta_1 > \delta_2$. Then, it is obvious that there exists a constant $0 < T' < +\infty$ such that

$$\lim_{t\to T'} y(t) = +\infty.$$

Construct

$$(\underline{u}(x,t), \underline{v}(x,t)) = (y^{l_1}(t)\varphi(x), y^{l_2}(t)\psi(x)),$$

where ϕ, ψ satisfying (1.7) and (1.8), respectively. Moreover, by the assumptions on initial data, we can take small enough constant y_0 such that

$$u_0(x) \geq y_0^{l_1} M_1 \text{ and } v_0(x) \geq y_0^{l_2} M_2 \text{ for all } x \in \Omega. \tag{5.2}$$

Now, we begin to verify that ($\underline{u}(x, t)$, $\underline{v}(x, t)$) is a blow-up subsolution of the problem (1.1) on $\overline{\Omega}_T \times \overline{\Omega}_T$, $T < T'$. In fact, $\forall (x, t) \in \Omega_T \times (0, T)$, a series of computations show

$$\begin{aligned} P\underline{u}(x,t) &\equiv \underline{u}_t - \operatorname{div}(|\nabla\underline{u}|^{p-2}\nabla\underline{u}) - \int_\Omega \underline{v}^m \mathrm{d}x + \alpha\underline{u}^r \\ &= l_1\varphi y^{l_1-1}y'(t) + y^{l_1(p-1)} - y^{ml_2}\int_\Omega \psi^m \mathrm{d}x + \alpha y^{rl_1}\varphi^r \\ &= l_1\varphi y^{l_1-1}\left(y'(t) + \frac{1}{l_1\varphi}y^{(p-2)l_1+1} - \frac{\int_\Omega \psi^m \mathrm{d}x - \alpha\varphi^r}{l_1\varphi}y^{l_1(r-1)+1}\right) \\ &\leq 0. \end{aligned} \tag{5.3}$$

Similarly, we also have

$$\begin{aligned} Q\underline{v}(x,t) &\equiv \underline{v}_t - \operatorname{div}\left(|\nabla \underline{v}|^{q-2}\nabla \underline{v}\right) - \int_\Omega \underline{u}^n \mathrm{d}x + \beta \underline{v}^s \\ &= l_2\psi \gamma^{l_2-1}\gamma'(t) + \gamma^{l_2(q-1)} - \gamma^{nl_1}\int_\Omega \varphi^n \mathrm{d}x + \beta\gamma^{sl_2}\psi^s \\ &= l_2\psi\gamma^{l_2-1}\left(\gamma'(t) + \frac{1}{l_2\psi}\gamma^{(q-2)l_2+1} - \frac{\int_\Omega \varphi^n \mathrm{d}x - \beta\psi^s}{l_2\psi}\gamma^{l_2(s-1)+1}\right) \\ &\le 0. \end{aligned} \tag{5.4}$$

On the other hand, $\forall t \in [0, T]$, we have

$$\underline{u}(x,t)|_{x\in\partial\Omega} = \gamma^{l_1}(t)\varphi(x)|_{x\in\partial\Omega} = 0, \tag{5.5}$$

and

$$\underline{v}(x,t)|_{x\in\partial\Omega} = \gamma^{l_2}(t)\psi(x)|_{x\in\partial\Omega} = 0. \tag{5.6}$$

Combining now (5.2)-(5.6), we see that $(\underline{u}, \underline{v})$ is a subsolution of (1.1) and $(\underline{u}, \underline{v}) < (u, v)$ on $\overline{\Omega}_T \times \overline{\Omega}_T$ by comparison principle, thus (u, v) must blow up in finite time since $(\underline{u}, \underline{v})$ does.

(ii) For $p - 1 > r$, $q - 1 > s$, we know $mn = (p - 1)(q - 1)$. Under the assumption $(\int_\Omega \varphi^n \mathrm{d}x)^{\frac{1}{q-1}}(\int_\Omega \psi^m \mathrm{d}x)^{\frac{1}{m}} \le 1$, we can choose A, B such that

$$A^{\frac{n}{q-1}}\left(\int_\Omega \varphi^n \mathrm{d}x\right)^{\frac{1}{q-1}} \le B \le A^{\frac{p-1}{m}}\left(\int_\Omega \psi^m \mathrm{d}x\right)^{-\frac{1}{m}}.$$

Then, $(\overline{u}, \overline{v}) = (A\varphi, B\psi)$ is a global supersolution of (1.1).

Since $mn = (p - 1)(q - 1)$, we can choose constants $l_1, l_2 > 1$ such that

$$\frac{s-1}{p-2} < \frac{q-1}{n} = \frac{l_1}{l_2} = \frac{m}{p-1} < \frac{q-2}{r-1}. \tag{5.7}$$

Next, we consider the following ordinary differential equation

$$\begin{cases} \gamma'(t) = c_1\gamma^{\delta_1} - c_2\gamma^{\delta_2}, & t > 0, \\ \gamma(0) = \gamma_0 > 0, \end{cases}$$

where

$$c_1 = \min\left\{\int_\Omega \psi^m \mathrm{d}x - 1, \int_\Omega \varphi^n \mathrm{d}x - 1\right\}, \quad c_2 = \max\left\{\frac{\alpha\varphi^{r-1}}{l_1}, \frac{\beta\psi^{s-1}}{l_2}\right\},$$
$$\delta_1 = \min\left\{(p-2)l_1 + 1, (q-2)l_2 + 1\right\}, \quad \delta_2 = \max\left\{(r-1)l_1 + 1, (s-1)l_2 + 1\right\}.$$

Since $\int_\Omega \psi^m \mathrm{d}x > 1$, $\int_\Omega \varphi^n \mathrm{d}x > 1$, we have $c_1 > 0$. On the other hand, in light of (5.7), it is easy to show that $\delta_1 > \delta_2$. Then, it is clear that $y(t)$ will become infinite in a finite time $T^* < +\infty$.

Let

$$(\underline{u}(x,t), \underline{v}(x,t)) = (\gamma^{l_1}(t)\varphi(x), \gamma^{l_2}(t)\psi(x)),$$

where $\phi(x)$, $\psi(x)$ satisfies (1.7) and (1.8), respectively. Similar to the arguments for the case $r > p - 1$, $s > q - 1$, we can prove that $(\underline{u}(x, t), \underline{v}(x, t))$ is a blow-up subsolution of the problem (1.1) on $\overline{\Omega}_T \times \overline{\Omega}_T$, $T < T^*$. Then, the solution (u, v) of (1.1) blows up in finite time.

(iii) For $p - 1 > r$, $s > q - 1$, we know $mn = s(p - 1)$. Since $\int_\Omega \varphi^n \mathrm{d}x \le |\Omega|^{-\frac{1}{m}} \beta^{\frac{1}{s}}$, we can choose A, B such that

$$\beta^{-\frac{1}{s}} A^{\frac{n}{s}} \int_\Omega \varphi^n \mathrm{d}x \le B \le |\Omega|^{-\frac{1}{m}} A^{\frac{p-1}{m}}.$$

We can check $(\overline{u}, \overline{v}) = (A\varphi, B)$ is a global supersolution of (1.1).

Thanks to $mn = s(p - 1)$, we can choose constants l_1, $l_2 > 1$ such that

$$\frac{q-1}{n} < \frac{s}{n} = \frac{l_1}{l_2} = \frac{m}{p-1} < \frac{m}{r}. \tag{5.8}$$

Let

$$(\underline{u}(x,t), \underline{v}(x,t)) = (\gamma^{l_1}(t)\varphi(x), \gamma^{l_2}(t)\psi(x)),$$

where $\phi(x)$, $\psi(x)$ are defined in (1.7) and (1.8), respectively, and $y(t)$ satisfies the following Cauchy problem

$$\begin{cases} \gamma'(t) = c_1 \gamma^{\delta_1} - c_2 \gamma^{\delta_2}, & t > 0, \\ \gamma(0) = \gamma_0 > 0, \end{cases}$$

where

$$c_1 = \min\left\{ \int_\Omega \psi^m \mathrm{d}x - 1, \frac{\int_\Omega \varphi^n \mathrm{d}x - \beta\psi^s}{l_2 \psi} \right\}, \quad c_2 = \max\left\{ \frac{\alpha\varphi^{r-1}}{l_1}, \frac{1}{l_2\psi} \right\},$$
$$\delta_1 = \min\left\{ (p-2)l_1 + 1, (s-1)l_2 + 1 \right\}, \quad \delta_2 = \max\left\{ (r-1)l_1 + 1, (q-2)l_2 + 1 \right\}.$$

Then, the left arguments are the same as those for the case $r > p - 1$, $s > q - 1$, so we omit them.

(iv) The proof of this case is parallel to (iii). The proof of Theorem 1.3 is complete. □

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Authors' contributions
The authors declare that the work was realized in collaboration with same responsibility. All authors read and approved the final manuscript.

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