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Poly-Cauchy numbers and polynomials of the second kind with umbral calculus viewpoint

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Abstract
In this paper, we consider the poly-Cauchy polynomials and numbers of the second kind which were studied by Komatsu. We note that the poly-Cauchy polynomials of the second kind are the special generalized Bernoulli polynomials of the second kind. The purpose of this paper is to give various identities of the poly-Cauchy polynomials of the second kind which are derived from umbral calculus.

1 Introduction

As is well known, the Bernoulli polynomials of the second kind are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} \quad \text{(see [1, p.130])}. \tag{1}$$

When $x = 0$, $b_n = b_n(0)$ are called the Bernoulli numbers of the second kind (see [1, p.131]).

Let $\mathrm{Lif}_k(x)$ be the polylogarithm factorial function, which is defined by

$$\mathrm{Lif}_k(x) = \sum_{n=0}^{\infty} \frac{x^m}{m!(m+1)^k} \quad \text{(see [2–7])}. \tag{2}$$

The poly-Cauchy polynomials of the second kind $\widehat{c}_n^{(k)}(x)$ ($k \in \mathbf{Z}$, $n \in \mathbf{Z}_{\geq 0}$) are defined by the generating function to be

$$\mathrm{Lif}_k\bigl(-\log(1+t)\bigr)(1+t)^x = \sum_{n=0}^{\infty} \widehat{c}_n^{(k)}(x)\frac{t^n}{n!} \quad \text{(see [2, 3])}. \tag{3}$$

When $x = 0$, $\widehat{c}_n^{(k)} = \widehat{c}_n^{(k)}(0)$ are called the poly-Cauchy numbers of the second kind, defined by

$$\sum_{n=0}^{\infty} \widehat{c}_n^{(k)}\frac{t^n}{n!} = \mathrm{Lif}_k\bigl(-\log(1+t)\bigr). \tag{4}$$

In particular, if we take $k = 1$, then we have

$$\mathrm{Lif}_1\bigl(-\log(1+t)\bigr)(1+t)^x = \frac{t}{(1+t)\log(1+t)}(1+t)^x = \frac{t(1+t)^{x-1}}{\log(1+t)}. \tag{5}$$

Thus, we note that

$$\widehat{c}_n^{(1)}(x) = b_n(x-1) = B_n^{(n)}(x), \tag{6}$$

where $B_n^{(\alpha)}(x)$ are the Bernoulli polynomials of order α (see [8]) as their numbers [9, p.257 and p.259].

When $x = 0, \widehat{c}_n^{(1)} = \widehat{c}_n^{(1)}(0) = b_n(-1) = B_n^{(n)}$, where $B_n^{(\alpha)}$ are the Bernoulli numbers of order α.

The falling factorial is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \tag{7}$$

where $S_1(n,l)$ is the signed Stirling number of the first kind.

For $m \in \mathbf{Z}_{\geq 0}$, it is well known that

$$\begin{aligned}\bigl(\log(1+t)\bigr)^m &= m!\sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!}\\ &= \sum_{l=0}^{\infty} S_1(l+m,m)\frac{m!}{(l+m)!}t^{l+m} \quad \text{(see [10, p.62]).}\end{aligned} \tag{8}$$

For $\lambda \in \mathbf{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!} \quad \text{(see [11–13]).}$$

In this paper, we investigate the properties of the poly-Cauchy numbers and polynomials of the second kind with umbral calculus viewpoint. The purpose of this paper is to give various identities of the poly-Cauchy polynomials of the second kind which are derived from umbral calculus.

2 Umbral calculus

Let $\mathbf{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \,\middle|\, a_k \in \mathbf{C} \right\}. \tag{9}$$

Let $\mathbb{P} = \mathbf{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbf{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\bigl\langle f(t)|x^n\bigr\rangle = a_n \quad (n \geq 0). \tag{10}$$

Then, by (9) and (10), we get

$$\langle t^k | x^n \rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{11}$$

where $\delta_{n,k}$ is Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra. The order $O(f(t))$ of a power series $f(t)$ ($\neq 0$) is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a delta series; if $O(f(t)) = 0$, then $f(t)$ is called an invertible series (see [10, 14, 15]). For $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k | s_n(x)\rangle = n!\delta_{n,k}$ for $n,k \geq 0$. The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t),f(t))$ which is denoted by $s_n(x) \sim (g(t),f(t))$ (see [10, 15]).

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle, \tag{12}$$

and

$$f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k\rangle \frac{t^k}{k!}, \qquad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x)\rangle \frac{x^k}{k!}. \tag{13}$$

Thus, by (13), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad \text{and} \quad e^{yt}p(x) = p(x+y). \tag{14}$$

Let us assume that $s_n(x) \sim (g(t),f(t))$. Then the generating function of $s_n(x)$ is given by

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad \text{for all } x \in \mathbf{C}, \tag{15}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $\bar{f}(f(t)) = t$ (see [10, 15]).

For $s_n(x) \sim (g(t),f(t))$, we have the following equation:

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \tag{16}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j | x^n \rangle x^j, \tag{17}$$

and

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \tag{18}$$

where $p_n(x) = g(t)s_n(x)$ (see [10, p.21]).

Let us assume that $p_n(x) \sim (1, f(t))$, $q_n(x) \sim (1, g(t))$. Then the transfer formula is given by

$$q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x) \quad (n \geq 0) \text{ (see [10, p.51])}.$$

For $s_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$, let us assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_n(x) \quad (n \geq 0). \tag{19}$$

Then we have

$$C_{n,m} = \frac{1}{m!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \middle| x^n \right\rangle \quad \text{(see [10, p.132])}. \tag{20}$$

3 Poly-Cauchy numbers and polynomials of the second kind

From (3), we note that $\widehat{c}_n^{(k)}(x)$ is the Sheffer sequence for the pair

$$\left(g(t) = \frac{1}{\operatorname{Lif}_k(-t)}, f(t) = e^t - 1\right),$$

that is,

$$\widehat{c}_n^{(k)}(x) \sim \left(\frac{1}{\operatorname{Lif}_k(-t)}, e^t - 1\right). \tag{21}$$

Because for $\bar{f}(t) = \log(1 + t)$, using the formula (15), we get

$$\operatorname{Lif}_k\bigl(-\log(1+t)\bigr)(1+t)^x = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}$$

which is the generating function of $\widehat{c}_n^{(k)}(x)$ in (3).

From (21), we have

$$\frac{1}{\operatorname{Lif}_k(-t)}\widehat{c}_n^{(k)}(x) \sim \bigl(1, e^t - 1\bigr), \tag{22}$$

and

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \sim \bigl(1, e^t - 1\bigr). \tag{23}$$

By (22) and (23), we get

$$\begin{aligned}
\widehat{c}_n^{(k)}(x) &= \operatorname{Lif}_k(-t)(x)_n = \sum_{m=0}^{n} S_1(n,m)\operatorname{Lif}_k(-t)x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\sum_{a=0}^{m} \frac{(-1)^a}{a!(a+1)^k} t^a x^m
\end{aligned}$$

$$= \sum_{m=0}^{n} \sum_{a=0}^{m} S_1(n,m) \frac{(-1)^a \binom{m}{a}}{(a+1)^k} x^{m-a}$$
$$= \sum_{m=0}^{n} \sum_{j=0}^{m} S_1(n,m) \frac{(-1)^{m-j} \binom{m}{j}}{(m-j+1)^k} x^j$$
$$= \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} S_1(n,m) \frac{(-1)^{m-j} \binom{m}{j}}{(m-j+1)^k} \right\} x^j. \tag{24}$$

By (17) and (21), we get

$$\widehat{c}_n^{(k)}(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle \mathrm{Lif}_k\bigl(-\log(1+t)\bigr)\bigl(\log(1+t)\bigr)^j \middle| x^n \right\rangle x^j. \tag{25}$$

Now, we observe that

$$\left\langle \mathrm{Lif}_k\bigl(-\log(1+t)\bigr)\bigl(\log(1+t)\bigr)^j \middle| x^n \right\rangle$$
$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)^k} \left\langle \bigl(\log(1+t)\bigr)^{m+j} \middle| x^n \right\rangle$$
$$= \sum_{m=0}^{n-j} \frac{(-1)^m}{m!(m+1)^k} \sum_{l=0}^{n-j-m} \frac{S_1(l+m+j, m+j)}{(l+m+j)!} (m+j)! \left\langle t^{m+j+l} \middle| x^n \right\rangle$$
$$= \sum_{m=0}^{n-j} \frac{(-1)^m}{m!(m+1)^k} \sum_{l=0}^{n-m-j} \frac{S_1(l+m+j, m+j)}{(l+m+j)!} (m+j)! n! \delta_{n,l+m+j}$$
$$= \sum_{m=0}^{n-j} \frac{(-1)^m (m+j)!}{m!(m+1)^k} S_1(n, m+j). \tag{26}$$

From (25) and (26), we have

$$\widehat{c}_n^{(k)}(x) = \sum_{j=0}^{n} \frac{1}{j!} \sum_{m=0}^{n-j} \frac{(-1)^m (m+j)!}{m!(m+1)^k} S_1(n,m+j) x^j = \sum_{j=0}^{n} \left\{ \sum_{m=0}^{n-j} \frac{(-1)^m \binom{m+j}{m}}{(m+1)^k} S_1(n,m+j) \right\} x^j$$
$$= \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} \frac{(-1)^{m-j} \binom{m}{j}}{(m-j+1)^k} S_1(n,m) \right\} x^j, \tag{27}$$

which is the same as the expression in (24). From (1), we note that

$$\frac{1}{\mathrm{Lif}_k(-t)} \widehat{c}_n^{(k)}(x) \sim \bigl(1, e^t - 1\bigr), \quad x^n \sim (1, t). \tag{28}$$

For $n \geq 1$, by (19) and (28), we get

$$\frac{1}{\mathrm{Lif}_k(-t)} \widehat{c}_n^{(k)}(x) = x \left(\frac{t}{e^t - 1} \right)^n x^{-1} x^n = x \left(\frac{t}{e^t - 1} \right)^n x^{n-1}$$
$$= x B_{n-1}^{(n)}(x) = \sum_{l=0}^{n-1} \binom{n-1}{l} B_{n-1-l}^{(n)} x^{l+1}. \tag{29}$$

Thus, by (29), we see that

$$\begin{aligned}\widehat{c}_n^{(k)}(x) &= \sum_{l=0}^{n-1}\binom{n-1}{l}B_{n-1-l}^{(n)}\operatorname{Lif}_k(-t)x^{l+1} \\ &= \sum_{l=0}^{n-1}\sum_{m=0}^{l+1}(-1)^m\binom{n-1}{l}\binom{l+1}{m}\frac{B_{n-1-l}^{(n)}}{(m+1)^k}x^{l+1-m} \\ &= \sum_{l=0}^{n-1}\sum_{j=0}^{l+1}(-1)^{l+1-j}\binom{n-1}{l}\binom{l+1}{j}\frac{B_{n-1-l}^{(n)}}{(l+2-j)^k}x^j \\ &= \sum_{l=0}^{n-1}(-1)^{l+1}\binom{n-1}{l}\frac{B_{n-1-l}^{(n)}}{(l+2)^k} \\ &\quad + \sum_{j=1}^{n}\left\{\sum_{l=j-1}^{n-1}(-1)^{l+1-j}\binom{n-1}{l}\binom{l+1}{j}\frac{B_{n-1-l}^{(n)}}{(l+2-j)^k}\right\}x^j. \end{aligned} \tag{30}$$

Therefore, by (27) and (30), we obtain the following theorem.

Theorem 1 *For $n \geq 1$, $1 \leq j \leq n$, we have*

$$\sum_{m=j}^{n}\frac{(-1)^{m-j}\binom{m}{j}}{(m-j+1)^k}S_1(n,m) = \sum_{l=j-1}^{n-1}(-1)^{l+1-j}\binom{n-1}{l}\binom{l+1}{j}\frac{B_{n-1-l}^{(n)}}{(l+2-j)^k}.$$

In addition, for $n \geq 1$, we have

$$\widehat{c}_n^{(k)} = \sum_{m=0}^{n}S_1(n,m)\frac{(-1)^m}{(m+1)^k} = \sum_{l=0}^{n-1}(-1)^{l+1}\binom{n-1}{l}\frac{B_{n-1-l}^{(n)}}{(l+2)^k}.$$

From (18), we note that

$$\widehat{c}_n^{(k)}(x+y) = \sum_{j=0}^{n}\binom{n}{j}\widehat{c}_j^{(k)}(x)p_{n-j}(y), \tag{31}$$

where $p_n(y) = \frac{1}{\operatorname{Lif}_k(-t)}\widehat{c}_n^{(k)}(y) \sim (1, e^t - 1)$.

By (22) and (23), we get

$$(y)_n = p_n(y) \sim \left(1, e^t - 1\right). \tag{32}$$

Thus, from (31) and (32), we have

$$\widehat{c}_n^{(k)}(x+y) = \sum_{j=0}^{n}\binom{n}{j}\widehat{c}_j^{(k)}(x)(y)_{n-j}. \tag{33}$$

By (14), (16), and (21), we get

$$\widehat{c}_n^{(k)}(x+1) - \widehat{c}_n^{(k)}(x) = \left(e^t - 1\right)\widehat{c}_n^{(k)}(x) = n\widehat{c}_{n-1}^{(k)}(x).$$

For $s_n(x) \sim (g(t), f(t))$, the recurrence formula for $s_n(x)$ is given by

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x) \quad \text{(see [10])}. \tag{34}$$

By (21) and (34), we get

$$\begin{aligned} \widehat{c}_{n+1}^{(k)}(x) &= \left(x - \frac{\mathrm{Lif}'_k(-t)}{\mathrm{Lif}_k(-t)}\right)e^{-t}\widehat{c}_n^{(k)}(x) \\ &= x\widehat{c}_n^{(k)}(x-1) - e^{-t}\frac{\mathrm{Lif}'_k(-t)}{\mathrm{Lif}_k(-t)}\widehat{c}_n^{(k)}(x). \end{aligned} \tag{35}$$

We observe that

$$\begin{aligned} \frac{\mathrm{Lif}'_k(-t)\,\mathrm{Lif}'_k(-t)}{\mathrm{Lif}_k(-t)}\widehat{c}_n^{(k)}(x) = \mathrm{Lif}'_k(-t)\frac{1}{\mathrm{Lif}_k(-t)}\widehat{c}_n^{(k)}(x) &= \mathrm{Lif}'_k(-t)(x)_n \\ &= \sum_{l=0}^{n} S_1(n,l)\,\mathrm{Lif}'_k(-t)x^l \\ &= \sum_{l=0}^{n} S_1(n,l)\sum_{m=0}^{l}\frac{(-1)^m\binom{l}{m}}{(m+2)^k}x^{l-m} \\ &= \sum_{j=0}^{n}\left\{\sum_{l=j}^{n}\frac{(-1)^{l-j}\binom{l}{j}}{(l-j+2)^k}S_1(n,l)\right\}x^j. \end{aligned} \tag{36}$$

Therefore, by (35) and (36), we obtain the following theorem.

Theorem 2 *For $n \geq 0$, we have*

$$\widehat{c}_{n+1}^{(k)}(x) = x\widehat{c}_n^{(k)}(x-1) - \sum_{j=0}^{n}\left\{\sum_{l=j}^{n} S_1(n,l)\frac{(-1)^{l-j}}{(l-j+2)^k}\binom{l}{j}\right\}(x-1)^j.$$

From (11), we note that

$$\begin{aligned} \widehat{c}_n^{(k)}(y) &= \left\langle \sum_{l=0}^{\infty}\widehat{c}_l^{(k)}(y)\frac{t^l}{l!}\middle| x^n\right\rangle = \left\langle \mathrm{Lif}_k\big(-\log(1+t)\big)(1+t)^y \middle| x^n\right\rangle \\ &= \left\langle \mathrm{Lif}_k\big(-\log(1+t)\big)(1+t)^y \middle| xx^{n-1}\right\rangle \\ &= \left\langle \partial_t\big(\mathrm{Lif}_k\big(-\log(1+t)\big)(1+t)^y\big) \middle| x^{n-1}\right\rangle \\ &= \left\langle \partial_t\big(\mathrm{Lif}_k\big(-\log(1+t)\big)\big)(1+t)^y \middle| x^{n-1}\right\rangle \\ &\quad + \left\langle \mathrm{Lif}_k\big(-\log(1+t)\big)\partial_t(1+t)^y \middle| x^{n-1}\right\rangle \\ &= \left\langle \partial_t\big(\mathrm{Lif}_k\big(-\log(1+t)\big)\big)(1+t)^y \middle| x^{n-1}\right\rangle + y\widehat{c}_{n-1}^{(k)}(y-1), \end{aligned} \tag{37}$$

where $\partial_t f(t) = \frac{df(t)}{dt}$.

Since $t\,\mathrm{Lif}'_k(t) = \mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t)$, we get

$$\mathrm{Lif}'_k(t) = \frac{\mathrm{Lif}_{k-1}(t) - \mathrm{Lif}_k(t)}{t}. \tag{38}$$

By (37) and (38), we see that

$$\begin{aligned}\widehat{c}_n^{(k)}(y) &= y\widehat{c}_{n-1}^{(k)}(y-1)\\ &\quad+\left\langle \frac{\mathrm{Lif}_{k-1}(-\log(1+t))-\mathrm{Lif}_k(-\log(1+t))}{(1+t)\log(1+t)}(1+t)^y \middle| x^{n-1}\right\rangle\\ &= y\widehat{c}_{n-1}^{(k)}(y-1)\\ &\quad+\left\langle \frac{\mathrm{Lif}_{k-1}(-\log(1+t))-\mathrm{Lif}_k(-\log(1+t))}{t(1+t)}(1+t)^y \middle| \frac{t}{\log(1+t)}x^{n-1}\right\rangle. \end{aligned} \tag{39}$$

From (1), (6), and (38), we note that

$$\begin{aligned}\widehat{c}_n^{(k)}(y) &= y\widehat{c}_{n-1}^{(k)}(y-1)+\sum_{l=0}^{n-1}\frac{B_l^{(l)}(1)}{l!}(n-1)_l\\ &\quad\times\left\langle \frac{\mathrm{Lif}_{k-1}(-\log(1+t))-\mathrm{Lif}_k(-\log(1+t))}{t}(1+t)^{y-1} \middle| x^{n-l-1}\right\rangle\\ &= y\widehat{c}_{n-1}^{(k)}(y-1)+\sum_{l=0}^{n-1}\frac{B_l^{(l)}(1)}{l!}(n-1)_l\\ &\quad\times\left\langle \frac{\mathrm{Lif}_{k-1}(-\log(1+t))-\mathrm{Lif}_k(-\log(1+t))}{t}(1+t)^{y-1} \middle| t\frac{x^{n-l}}{n-l}\right\rangle\\ &= y\widehat{c}_{n-1}^{(k)}(y-1)+\sum_{l=0}^{n-1}\binom{n-1}{l}\frac{B_l^{(l)}(1)}{n-l}\left\{\widehat{c}_{n-l}^{(k-1)}(y-1)-\widehat{c}_{n-l}^{(k)}(y-1)\right\}\\ &= y\widehat{c}_{n-1}^{(k)}(y-1)+\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}B_l^{(l)}(1)\left\{\widehat{c}_{n-l}^{(k-1)}(y-1)-\widehat{c}_{n-l}^{(k)}(y-1)\right\}. \end{aligned} \tag{40}$$

It is not difficult to show that $\widehat{c}_0^{(k)}(y-1)=\widehat{c}_0^{(k-1)}(y-1)$. Since $\hat{c}_0^{(k)}(y-1)=\widehat{c}_0^{(k-1)}(y-1)$, by (40), we obtain the following theorem.

Theorem 3 *For $n\geq 1$, we have*

$$\widehat{c}_n^{(k)}(x) = x\hat{c}_{n-1}^{(k)}(x-1)+\frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}B_l^{(l)}(1)\left\{\widehat{c}_{n-l}^{(k-1)}(x-1)-\widehat{c}_{n-l}^{(k)}(x-1)\right\}.$$

For $n\geq m\geq 1$, we compute

$$\left\langle \left(\log(1+t)\right)^m \mathrm{Lif}_k\left(-\log(1+t)\right) | x^n\right\rangle$$

in two different ways.

On the one hand,

$$\begin{aligned}&\left\langle \left(\log(1+t)\right)^m \mathrm{Lif}_k\left(-\log(1+t)\right) | x^n\right\rangle\\ &\quad= \left\langle \mathrm{Lif}_k\left(-\log(1+t)\right) \middle| \sum_{l=0}^{\infty}\frac{m!}{(l+m)!}S_1(l+m,m)t^{l+m}x^n\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m} \langle \mathrm{Lif}_k\bigl(-\log(1+t)\bigr) \,|\, x^{n-l-m} \rangle \\
&= \sum_{l=0}^{n-m} m! \binom{n}{l+m} S_1(l+m,m) \hat{c}_{n-l-m}^{(k)}. \qquad (41)
\end{aligned}
$$

On the other hand, we get

$$
\begin{aligned}
&\bigl\langle \bigl(\log(1+t)\bigr)^m \mathrm{Lif}_k\bigl(-\log(1+t)\bigr) \,|\, x^n \bigr\rangle \\
&\quad = \bigl\langle \bigl(\log(1+t)\bigr)^m \mathrm{Lif}_k\bigl(-\log(1+t)\bigr) \,|\, x x^{n-1} \bigr\rangle \\
&\quad = \bigl\langle \partial_t \bigl(\bigl(\log(1+t)\bigr)^m \mathrm{Lif}_k\bigl(-\log(1+t)\bigr)\bigr) \,|\, x^{n-1} \bigr\rangle. \qquad (42)
\end{aligned}
$$

Now, we observe that

$$
\begin{aligned}
&\partial_t \bigl(\bigl(\log(1+t)\bigr)^m \mathrm{Lif}_k\bigl(-\log(1+t)\bigr)\bigr) \\
&\quad = m\bigl(\log(1+t)\bigr)^{m-1} \frac{1}{1+t} \mathrm{Lif}_k\bigl(-\log(1+t)\bigr) \\
&\qquad + \bigl(\log(1+t)\bigr)^m \frac{\mathrm{Lif}_{k-1}(-\log(1+t)) - \mathrm{Lif}_k(-\log(1+t))}{(1+t)\log(1+t)} \\
&\quad = \bigl(\log(1+t)\bigr)^{m-1} (1+t)^{-1} \bigl\{ m \,\mathrm{Lif}_k\bigl(-\log(1+t)\bigr) \\
&\qquad + \mathrm{Lif}_{k-1}\bigl(-\log(1+t)\bigr) - \mathrm{Lif}_k\bigl(-\log(1+t)\bigr) \bigr\}. \qquad (43)
\end{aligned}
$$

By (42) and (43), we get

$$
\begin{aligned}
&\bigl\langle \bigl(\log(1+t)\bigr)^m \mathrm{Lif}_k\bigl(-\log(1+t)\bigr) \,|\, x^n \bigr\rangle \\
&\quad = \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1,m-1) \\
&\qquad \times \bigl\{ (m-1) \langle \mathrm{Lif}_k\bigl(-\log(1+t)\bigr)(1+t)^{-1} \,|\, t^{l+m-1} x^{n-1} \rangle \\
&\qquad + \langle \mathrm{Lif}_{k-1}\bigl(-\log(1+t)\bigr)(1+t)^{-1} \,|\, t^{l+m-1} x^{n-1} \rangle \bigr\} \\
&\quad = (m-1) \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1,m-1)(n-1)_{l+m-1} \\
&\qquad \times \langle \mathrm{Lif}_k\bigl(-\log(1+t)\bigr)(1+t)^{-1} \,|\, x^{n-m-l} \rangle \\
&\qquad + \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_1(l+m-1,m-1)(n-1)_{l+m-1} \\
&\qquad \times \langle \mathrm{Lif}_{k-1}\bigl(-\log(1+t)\bigr)(1+t)^{-1} \,|\, x^{n-m-l} \rangle \\
&\quad = \sum_{l=0}^{n-m} (m-1)! \binom{n-1}{l+m-1} S_1(l+m-1,m-1) \\
&\qquad \times \bigl\{ (m-1)\hat{c}_{n-l-m}^{(k)}(-1) + \hat{c}_{n-l-m}^{(k-1)}(-1) \bigr\}. \qquad (44)
\end{aligned}
$$

Therefore, by (41) and (44), we obtain the following theorem.

Theorem 4 *For $n \geq m \geq 1$, we have*

$$\begin{aligned}&\sum_{l=0}^{n-m} m!\binom{n}{l+m} S_1(l+m,m)\widehat{c}_{n-l-m}^{(k)} \\ &\quad= \sum_{l=0}^{n-m}(m-1)!\binom{n-1}{l+m-1} S_1(l+m-1,m-1) \\ &\qquad\times \bigl\{(m-1)\widehat{c}_{n-l-m}^{(k)}(-1)+\widehat{c}_{n-l-m}^{(k-1)}(-1)\bigr\}.\end{aligned}$$

In particular, if we take $m = 1$, then we get

$$\widehat{c}_n^{(k-1)}(-1) = \sum_{l=0}^{n-1}(-1)^l l!\binom{n}{l+1}\widehat{c}_{n-l-1}^{(k)}.$$

Remark For $s_n(x) \sim (g(t), f(t))$, it is known that

$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\langle \bar{f}(t)|x^{n-l}\rangle s_l(x) \quad \text{(see [10, p.108])}. \tag{45}$$

By (21) and (45), we easily show that

$$\frac{d}{dx}\widehat{c}_n^{(k)}(x) = (-1)^n n!\sum_{l=0}^{n-1}\frac{(-1)^{l-1}}{(n-l)l!}\widehat{c}_l^{(k)}(x),$$

which is a special case of Proposition 2 in [4].

Let us consider the following two Sheffer sequences:

$$\widehat{c}_n^{(k)}(x) \sim \left(\frac{1}{\mathrm{Lif}_k(-t)}, e^t - 1\right), \tag{46}$$

and

$$B_n^{(r)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^r, t\right).$$

Suppose that

$$\widehat{c}_n^{(k)}(x) = \sum_{m=0}^{n} C_{n,m}B_m^{(r)}(x). \tag{47}$$

By (20), we see that

$$\begin{aligned}C_{n,m} &= \frac{1}{m!}\left\langle \frac{\left(\frac{t}{\log(1+t)}\right)^r}{\frac{1}{\mathrm{Lif}_k(-\log(1+t))}}\bigl(\log(1+t)\bigr)^m \Big| x^n\right\rangle \\ &= \frac{1}{m!}\left\langle \mathrm{Lif}_k\bigl(-\log(1+t)\bigr)\left(\frac{t}{\log(1+t)}\right)^r\bigl(\log(1+t)\bigr)^m \Big| x^n\right\rangle\end{aligned}$$

$$= \frac{1}{m!} \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m}$$

$$\times \left\langle \mathrm{Lif}_k\bigl(-\log(1+t)\bigr) \left(\frac{t}{\log(1+t)} \right)^r \middle| x^{n-l-m} \right\rangle$$

$$= \frac{1}{m!} \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m} \sum_{a=0}^{n-l-m} B_a^{(a-r+1)} \frac{1}{a!}$$

$$\times \bigl\langle \mathrm{Lif}_k\bigl(-\log(1+t)\bigr) \big| t^a x^{n-l-m} \bigr\rangle$$

$$= \sum_{l=0}^{n-m} \binom{n}{l+m} S_1(l+m,m) \sum_{a=0}^{n-l-m} B_a^{(a-r+1)} \frac{(n-l-m)_a}{a!}$$

$$\times \bigl\langle \mathrm{Lif}_k\bigl(-\log(1+t)\bigr) \big| x^{n-l-m-a} \bigr\rangle$$

$$= \sum_{l=0}^{n-m} \sum_{a=0}^{n-l-m} \binom{n}{l+m} \binom{n-m-l}{a} S_1(l+m,m) B_a^{(a-r+1)}(1) \widehat{c}_{n-l-m-a}^{(k)}. \tag{48}$$

Therefore, by (47) and (48), we obtain the following theorem.

Theorem 5 *For $n \geq 0$, we have*

$$\widehat{c}_n^{(k)}(x) = \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \binom{n}{l+m} \binom{n-m-l}{a} S_1(l+m,m) B_a^{(a-r+1)}(1) \widehat{c}_{n-m-l-a}^{(k)} \right\} B_m^{(r)}(x).$$

Remark The Narumi polynomials of order a are defined by the generating function to be

$$\sum_{k=0}^{\infty} \frac{N_k^{(a)}(x)}{k!} t^k = \left(\frac{t}{\log(1+t)} \right)^{-a} (1+t)^x \quad \text{(see [10, p.127])}. \tag{49}$$

Indeed, $N_a^{(k)}(x) = B_k^{(k+a+1)}(x+1)$, $N_k^{(a)}(x) \sim ((\frac{e^t-1}{t})^a, e^t - 1)$.

By (48) and (49), we get

$$C_{n,m} = \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \binom{n}{l+m} \binom{n-l-m}{a} S_1(l+m,m) N_a^{(-r)} \widehat{c}_{n-l-m-a}^{(k)}. \tag{50}$$

From (47) and (50), we have

$$\widehat{c}_n^{(k)}(x) = \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \binom{n}{l+m} \binom{n-l-m}{a} \right.$$

$$\left. \times S_1(l+m,m) N_a^{(-r)} \widehat{c}_{n-l-m-a}^{(k)} \right\} B_m^{(r)}(x). \tag{51}$$

By (1), we easily show that

$$C_{n,m} = \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{a_1+\cdots+a_r=a} \binom{n}{l+m} \binom{n-l-m}{a} \binom{a}{a_1,\ldots,a_r} \times S_1(l+m,m) b_{a_1} \cdots b_{a_r} \widehat{c}_{n-m-l-a}^{(k)}. \tag{52}$$

From (47) and (52), we can derive the following equation:

$$\widehat{c}_n^{(k)}(x) = \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{a_1+\cdots+a_r=a} \binom{n}{l+m} \binom{n-l-m}{a} \binom{a}{a_1,\ldots,a_r} \times S_1(l+m,m) \left(\prod_{i=1}^{r} b_{a_i} \right) \widehat{c}_{n-m-l-a}^{(k)} \right\} B_m^{(r)}(x). \tag{53}$$

For (20) and (24), let

$$\widehat{c}_n^{(k)}(x) = \sum_{m=0}^{n} C_{n,m} H_m^{(r)}(x|\lambda), \tag{54}$$

where, by (20), we get

$$\begin{aligned} C_{n,m} &= \frac{1}{m!(1-\lambda)^r} \left\langle \operatorname{Lif}_k\bigl(-\log(1+t)\bigr)(1+t-\lambda)^r \middle| \bigl(\log(1+t)\bigr)^m x^n \right\rangle \\ &= \frac{1}{m!(1-\lambda)^r} \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_{l+m} \\ &\quad \times \left\langle \operatorname{Lif}_k\bigl(-\log(1+t)\bigr)(1+t-\lambda)^r \middle| x^{n-l-m} \right\rangle. \end{aligned} \tag{55}$$

We observe that

$$\begin{aligned} &\left\langle \operatorname{Lif}_k\bigl(-\log(1+t)\bigr)(1+t-\lambda)^r \middle| x^{n-l-m} \right\rangle \\ &\quad = \sum_{a=0}^{r} \binom{r}{a} (1-\lambda)^{r-a} \left\langle \operatorname{Lif}_k\bigl(-\log(1+t)\bigr) \middle| t^a x^{n-l-m} \right\rangle \\ &\quad = \sum_{a=0}^{r} \binom{r}{a} (1-\lambda)^{r-a} (n-m-l)_a \left\langle \operatorname{Lif}_k\bigl(-\log(1+t)\bigr) \middle| x^{n-l-m-a} \right\rangle \\ &\quad = \sum_{a=0}^{r} \binom{r}{a} (1-\lambda)^{r-a} (n-m-l)_a \widehat{c}_{n-l-m-a}^{(k)}. \end{aligned} \tag{56}$$

Thus, by (55) and (56), we get

$$C_{n,m} = \sum_{l=0}^{n-m} \sum_{a=0}^{r} \binom{n}{l+m} \binom{r}{a} (n-m-l)_a (1-\lambda)^{-a} S_1(l+m,m) \widehat{c}_{n-m-l-a}^{(k)}. \tag{57}$$

Therefore, by (54) and (57), we obtain the following theorem.

Theorem 6 *For $n \geq 0$, we have*

$$\widehat{c}_n^{(k)}(x) = \sum_{m=0}^{n} \left\{ \sum_{l=0}^{n-m} \sum_{a=0}^{r} \binom{n}{l+m} \binom{r}{a} (n-m-l)_a (1-\lambda)^{-a} S_1(l+m,m) \times \widehat{c}_{n-m-l-a}^{(k)} \right\} H_m^{(r)}(x|\lambda).$$

For $\widehat{c}_n^{(k)}(x) \sim (\frac{1}{\mathrm{Lif}_k(-t)}, e^t - 1)$, and $(x)_n \sim (1, e^t - 1)$, let us assume that

$$\widehat{c}_n^{(k)}(x) = \sum_{m=0}^{n} C_{n,m}(x)_m. \tag{58}$$

From (20), we note that

$$\begin{aligned} C_{n,m} &= \frac{1}{m!} \langle \mathrm{Lif}_k(-\log(1+t)) t^m | x^n \rangle \\ &= \frac{1}{m!} \langle \mathrm{Lif}_k(-\log(1+t)) | t^m x^n \rangle \\ &= \binom{n}{m} \langle \mathrm{Lif}_k(-\log(1+t)) | x^{n-m} \rangle \\ &= \binom{n}{m} \widehat{c}_{n-m}^{(k)}. \end{aligned} \tag{59}$$

Therefore, by (58) and (59), we obtain the following theorem.

Theorem 7 *For $n \geq 0$, we have*

$$\widehat{c}_n^{(k)}(x) = \sum_{m=0}^{n} \binom{n}{m} \widehat{c}_{n-m}^{(k)}(x)_m.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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