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Some Volterra-Fredholm type nonlinear discrete inequalities involving four iterated infinite sums

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Abstract
Some new generalized Volterra-Fredholm type nonlinear discrete inequalities involving four iterated infinite sums are established in this paper. To illustrate the validity of the established inequalities, we present some applications for them, in which new explicit bounds for the solutions of certain infinite sum-difference equations are deduced.
MSC: 26D15

Keywords: nonlinear discrete inequalities; Volterra-Fredholm type inequalities; sum-difference equations; bounds

1 Introduction

In recent years, many researchers have focused on various generalizations of the known Gronwall-Bellman inequality [1, 2], which provide explicit bounds for unknown solutions of certain difference equations, and a lot of such generalized inequalities have been established in the literature [3–20] including the known Ou-Iang inequality [3]. In [21], Ma generalized the discrete version of Ou-Iang's inequality in two variables to a Volterra-Fredholm form for the first time, which has proved to be very useful in the study of qualitative as well as quantitative properties of the solutions of certain Volterra-Fredholm type difference equations. But since then, few results on Volterra-Fredholm type discrete inequalities have been established. Recent results in this direction include the works of Zheng [22], Ma [23], Zheng and Feng [24] to our best knowledge. We notice that the Volterra-Fredholm type discrete inequalities in [22–24] are constructed by an explicit function u^p in the left-hand side (see [22, Theorems 2.5, 2.6], [23, Theorems 2.1, 2.5, 2.6, 2.7], [24, Theorems 5, 8, 10, 11]).

Motivated by the works in [22–24], in this paper, we establish some new generalized Volterra-Fredholm type discrete inequalities involving four iterated infinite sums with the right-hand side denoted by an arbitrary function $\phi(u)$, which are of more general forms. To illustrate the usefulness of the established results, we also present some applications for them and study the boundedness of the solutions of certain Volterra-Fredholm type infinite sum-difference equations.

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$, and $\mathbb{Z}$ denotes the set of integers, while $\mathbb{N}_0$ denotes the set of nonnegative integers. In the next of this paper, let $\Omega := ([m_0, \infty] \times [n_0, \infty]) \cap \mathbb{Z}^2$, where $m_0, n_0 \in \mathbb{Z}$, and let $l_1, l_2 \in \mathbb{Z}$ be two constants. If U is a lattice, then we denote the set of all $\mathbb{R}$-valued functions on U by $\wp(U)$ and denote the set of all $\mathbb{R}_+$-valued functions on U by $\wp_+(U)$. Finally, for a function $f \in \wp_+(U)$, we have $\sum_{s=m_0}^{m_1} f = 0$ provided $m_0 > m_1$.

2 Main results

Lemma 2.1 [22, Lemma 2.1] *Suppose $u, a, b \in \wp_+(\Omega)$. If $a(m,n)$ is nonincreasing in the first variable, then for $(m,n) \in \Omega$,*

$$u(m,n) \leq a(m,n) + \sum_{s=m+1}^{\infty} b(s,n)u(s,n)$$

implies

$$u(m,n) \leq a(m,n) \sum_{s=m+1}^{\infty} \big[1 + b(s,n)\big]. \tag{1}$$

Lemma 2.2 *Suppose $u, a, H \in \wp_+(\Omega)$, $b \in \wp_+(\Omega^2)$, and H, a are nonincreasing in every variable with $H(m,n) > 0$, while b is nonincreasing in the third variable. $\varphi, \phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ are strictly increasing with $\varphi(r) > 0$, $\phi(r) > 0$ for $r > 0$. If for $(m,n) \in \Omega$, $u(m,n)$ satisfies the following inequality*:

$$u(m,n) \leq H(m,n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} b(s,t,m,n)\varphi\big(\phi^{-1}\big(u(s,t) + a(s,t)\big)\big), \tag{2}$$

then we have

$$u(m,n) \leq G^{-1}\left[G\big(H(m,n)\big) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} b(s,t,m,n)\right], \tag{3}$$

where

$$G(z) = \int_{z_0}^{z} \frac{1}{\varphi[\phi^{-1}(z + a(m,n))]}\,dz, \quad z \geq z_0 > 0. \tag{4}$$

Proof Fix $(m_1, n_1) \in \Omega$, and let $(m,n) \in ([m_1, \infty] \times [n_1, \infty]) \cap \Omega$. Then we have

$$u(m,n) \leq H(m_1,n_1) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} b(s,t,m,n)\varphi\big[\phi^{-1}\big(u(s,t) + a(s,t)\big)\big]. \tag{5}$$

Let the right-hand side of (5) be $v(m,n)$. Then

$$u(m,n) \leq v(m,n), \quad (m,n) \in \big([m_1, \infty] \times [n_1, \infty]\big) \cap \Omega, \tag{6}$$

and

$$\begin{aligned} &v(m-1,n) - v(m,n) \\ &\quad = \sum_{s=m}^{\infty} \sum_{t=n+1}^{\infty} b(s,t,m-1,n)\varphi\big[\phi^{-1}\big(u(s,t) + a(s,t)\big)\big] \\ &\qquad - \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} b(s,t,m,n)\varphi\big[\phi^{-1}\big(u(s,t) + a(s,t)\big)\big] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=m}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m-1,n)\varphi\bigl[\phi^{-1}\bigl(u(s,t)+a(s,t)\bigr)\bigr] \\
&\quad - \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m-1,n)\varphi\bigl[\phi^{-1}\bigl(u(s,t)+a(s,t)\bigr)\bigr] \\
&\quad + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m-1,n)\varphi\bigl[\phi^{-1}\bigl(u(s,t)+a(s,t)\bigr)\bigr] \\
&\quad - \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m,n)\varphi\bigl[\phi^{-1}\bigl(u(s,t)+a(s,t)\bigr)\bigr] \\
&= \sum_{t=n+1}^{\infty} b(m,t,m-1,n)\varphi\bigl[\phi^{-1}\bigl(u(m,t)+a(m,t)\bigr)\bigr] \\
&\quad + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} \bigl[b(s,t,m-1,n)-b(s,t,m,n)\bigr]\varphi\bigl[\phi^{-1}\bigl(u(s,t)+a(s,t)\bigr)\bigr] \\
&\leq \sum_{t=n+1}^{\infty} b(m,t,m-1,n)\varphi\bigl[\phi^{-1}\bigl(u(m,t)+a(m,t)\bigr)\bigr] \\
&\quad + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} \bigl[b(s,t,m-1,n)-b(s,t,m,n)\bigr]\varphi\bigl[\phi^{-1}\bigl(v(s,t)+a(s,t)\bigr)\bigr] \\
&\leq \left\{\sum_{t=n+1}^{\infty} b(m,t,m-1,n) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\bigl[b(s,t,m-1,n)-b(s,t,m,n)\bigr]\right\} \\
&\quad \times \varphi\bigl[\phi^{-1}\bigl(v(m,n)+a(m,n)\bigr)\bigr],
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\frac{v(m-1,n)-v(m,n)}{\varphi(\phi^{-1}(v(m,n)+a(m,n)))} \\
&\quad \leq \sum_{t=n+1}^{\infty} b(m,t,m-1,n) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\bigl[b(s,t,m-1,n)-b(s,t,m,n)\bigr] \\
&\quad = \sum_{s=m}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m-1,n) - \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m-1,n) \\
&\qquad + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\bigl[b(s,t,m-1,n)-b(s,t,m,n)\bigr] \\
&\quad = \sum_{s=m}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m-1,n) - \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m,n).
\end{aligned}
\tag{7}
$$

On the other hand, according to the mean-value theorem for integrals, there exists ξ such that $v(m,n)\leq\xi\leq v(m-1,n)$, and

$$
\begin{aligned}
\int_{v(m,n)}^{v(m-1,n)} \frac{1}{\varphi(\phi^{-1}(z+a(m,n)))}\,dz &= \frac{v(m-1,n)-v(m,n)}{\varphi(\phi^{-1}(\xi+a(m,n)))} \\
&\leq \frac{v(m-1,n)-v(m,n)}{\varphi(\phi^{-1}(v(m,n)+a(m,n)))}.
\end{aligned}
\tag{8}
$$

So, combining (7) and (8), we have

$$\int_{v(m,n)}^{v(m-1,n)} \frac{1}{\varphi(\phi^{-1}(z+a(m,n)))}\,dz = G\big(v(m-1,n)\big) - G\big(v(m,n)\big)$$
$$\le \sum_{s=m}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m-1,n) - \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m,n), \quad (9)$$

where G is defined in (4). Set $m = \eta$ in (9); a summation with respect to η from $m+1$ to ∞ yields

$$G\big(v(m,n)\big) - G\big(v(\infty,n)\big) \le \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m,n) - 0 = \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m,n).$$

Noticing $v(\infty,n) = H(m_1,n_1)$ and G is increasing, it follows that

$$v(m,n) \le G^{-1}\left[G\big(H(m_1,n_1)\big) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m,n)\right]. \quad (10)$$

Combining (6) and (10), we obtain

$$u(m,n) \le G^{-1}\left[G\big(H(m_1,n_1)\big) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} b(s,t,m,n)\right],$$
$$(m,n) \in \big([m_1,\infty] \times [n_1,\infty]\big) \cap \Omega. \quad (11)$$

Setting $m = m_1$, $n = n_1$ in (11) yields

$$u(m_1,n_1) \le G^{-1}\left[G\big(a(m_1,n_1)\big) + \sum_{s=m_1+1}^{\infty}\sum_{t=n_1+1}^{\infty} b(s,t,m_1,n_1)\right]. \quad (12)$$

Since (m_1,n_1) is selected from Ω arbitrarily, then substituting (m_1,n_1) with (m,n) in (12), we get the desired inequality (3). □

Theorem 2.3 *Suppose $u \in \wp_+(\Omega)$, $b_i, c_i \in \wp_+(\Omega^2)$, $i = 1,2,\ldots,l_1$, $d_i, e_i \in \wp_+(\Omega^2)$, $i = 1,2,\ldots,l_2$ with b_i, c_i, d_i, e_i nonincreasing in the last two variables, and there is at least one function among d_i, e_i, $i = 1,2,\ldots,l_2$ not equivalent to zero, a, φ, ϕ are defined as in Lemma 2.2. If for $(m,n) \in \Omega$, $u(m,n)$ satisfies*

$$\phi\big(u(m,n)\big) \le a(m,n) + \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[b_i(s,t,m,n)\varphi\big(u(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} c_i(\xi,\eta,m,n)\varphi\big(u(\xi,\eta)\big)\right]$$
$$+ \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[d_i(s,t,m,n)\phi\big(u(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_i(\xi,\eta,m,n)\phi\big(u(\xi,\eta)\big)\right], \quad (13)$$

then

$$u(m,n) \le \phi^{-1}\left\{a(m,n) + G^{-1}\left\{G\left(T^{-1}\left[\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1} B(s,t,M,N)\right]\right)\right.\right.$$
$$\left.\left. + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} B(s,t,m,n)\right\}\right\} \tag{14}$$

provided that T is increasing, where G is defined in (4), *and*

$$T(x) = G\left(\frac{x-\mu_1}{\mu_2}\right) - G(x), \quad x \ge 0, \tag{15}$$

$$B(s,t,m,n) = \sum_{i=1}^{l_1}\left[b_i(s,t,m,n) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} c_i(\xi,\eta,m,n)\right], \tag{16}$$

$$J(m,n) = \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[d_i(s,t,m,n)a(s,t) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_i(\xi,\eta,m,n)a(\xi,\eta)\right], \tag{17}$$

$$\mu_1 = J(M,N),$$
$$\mu_2 = \sum_{i=1}^{l_2}\sum_{s=m_0}^{M-1}\sum_{t=n_0}^{N-1}\left[d_i(s,t,M,N) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_i(\xi,\eta,M,N)\right]. \tag{18}$$

Proof Denote

$$v(m,n) = \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[b_i(s,t,m,n)\varphi\big(u(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} c_i(\xi,\eta,m,n)\varphi\big(u(\xi,\eta)\big)\right]$$
$$+ \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[d_i(s,t,m,n)\phi\big(u(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_i(\xi,\eta,m,n)\phi\big(u(\xi,\eta)\big)\right].$$

Then we have

$$u(m,n) \le \phi^{-1}\big(a(m,n) + v(m,n)\big). \tag{19}$$

So,

$$v(m,n) \le \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left\{b_i(s,t,m,n)\varphi\left[\phi^{-1}\big(a(s,t) + v(s,t)\big)\right]\right.$$
$$\left. + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} c_i(\xi,\eta,m,n)\varphi\left[\phi^{-1}\big(a(\xi,\eta) + v(\xi,\eta)\big)\right]\right\}$$
$$+ \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[d_i(s,t,m,n)\big(a(s,t) + v(s,t)\big)\right.$$
$$\left. + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_i(\xi,\eta,m,n)\big(a(\xi,\eta) + v(\xi,\eta)\big)\right]$$

$$\begin{aligned}&= H(m,n) + \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Bigg\{b_i(s,t,m,n)\varphi\big[\phi^{-1}\big(a(s,t)+v(s,t)\big)\big]\\&\quad+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi\big[\phi^{-1}\big(a(\xi,\eta)+v(\xi,\eta)\big)\big]\Bigg\},\end{aligned}\tag{20}$$

where $H(m,n) = J(m,n) + \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}[d_i(s,t,m,n)v(s,t) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)v(\xi,\eta)]$, and $J(m,n)$ is defined in (17). Then using $H(m,n)$ is nonincreasing in every variable, we obtain

$$\begin{aligned}v(m,n) &\le H(M,N) + \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\big\{b_i(s,t,m,n)\varphi\big[\phi^{-1}\big(a(s,t)+v(s,t)\big)\big]\big\}\\&\quad+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi\big[\phi^{-1}\big(a(\xi,\eta)+v(\xi,\eta)\big)\big]\\&\le H(M,N) + \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Bigg[b_i(s,t,m,n)+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\Bigg]\\&\quad\times\varphi\big[\phi^{-1}\big(a(s,t)+v(s,t)\big)\big]\\&= H(M,N) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}B(s,t,m,n)\varphi\big[\phi^{-1}\big(a(s,t)+v(s,t)\big)\big],\end{aligned}\tag{21}$$

where $B(s,t,m,n)$ is defined in (16).

Since there is at least one function among d_i, e_i, $i = 1,2,\dots,l_2$ not equivalent to zero, then $H(M,N) > 0$. On the other hand, as $b_i(s,t,m,n)$, $c_i(s,t,m,n)$ are both nonincreasing in the last two variables, then $B(s,t,m,n)$ is also nonincreasing in the last two variables, and by a suitable application of Lemma 2.2, we obtain

$$v(m,n) \le G^{-1}\Bigg[G\big(H(M,N)\big) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}B(s,t,m,n)\Bigg].\tag{22}$$

Furthermore, by the definitions of $H(m,n)$, μ_1, μ_2 and (22), we have

$$\begin{aligned}H(M,N) &= J(M,N) + \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Bigg\{d_i(s,t,M,N)v(s,t)\\&\quad+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,M,N)v(\xi,\eta)\Bigg\}\\&\le J(M,N) + v(M,N)\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Bigg\{d_i(s,t,M,N)+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,M,N)\Bigg\}\\&= \mu_1 + \mu_2 v(M,N)\\&\le \mu_1 + \mu_2 G^{-1}\Bigg[G\big(H(M,N)\big) + \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}B(s,t,M,N)\Bigg],\end{aligned}$$

and

$$G\left(\frac{H(M,N)-\mu_1}{\mu_2}\right) \le G\big(H(M,N)\big) + \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty} B(s,t,M,N),$$

which is rewritten as

$$T\big(H(M,N)\big) \le \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty} B(s,t,M,N),$$

where T is defined in (15). By T is increasing, we have

$$H(M,N) \le T^{-1}\left[\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty} B(s,t,M,N)\right]. \tag{23}$$

Combining (19), (22) and (23), we get the desired result. □

Corollary 2.4 *Suppose* $g_{1i}, g_{2i}, b_{1i}, c_{1i} \in \wp_+(\Omega)$, $i = 1,2,\ldots,l_1$ *with* g_{1i}, g_{2i} *nonincreasing in every variable*, $d_{1i}, e_{1i} \in \wp_+(\Omega)$, $i = 1,2,\ldots,l_2$, u, a, φ, ϕ *are defined as in Theorem* 2.3. *If for* $(m,n) \in \Omega$, $u(m,n)$ *satisfies*

$$\begin{aligned}\phi\big(u(m,n)\big) \le{}& a(m,n) + \sum_{i=1}^{l_1} g_{1i}(m,n) \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Bigg[b_{1i}(s,t)\varphi\big(u(s,t)\big) \\ &+ \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} c_{1i}(\xi,\eta)\varphi\big(u(\xi,\eta)\big)\Bigg] \\ &+ \sum_{i=1}^{l_2} g_{2i}(m,n) \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Bigg[d_{1i}(s,t)\phi\big(u(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_{1i}(\xi,\eta)\phi\big(u(\xi,\eta)\big)\Bigg],\end{aligned}$$

then

$$\begin{aligned}u(m,n) \le{}& \phi^{-1}\Bigg\{a(m,n) + G^{-1}\Bigg\{G\Bigg(T^{-1}\Bigg[\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty} B(s,t,M,N)\Bigg]\Bigg) \\ &+ \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} B(s,t,m,n)\Bigg\}\Bigg\}\end{aligned}$$

provided that T *is increasing, where* G, T *are defined in Theorem* 2.3, *and*

$$B(s,t,m,n) = \sum_{i=1}^{l_1} g_{1i}(m,n)\left[b_i(s,t) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} c_i(\xi,\eta)\right],$$

$$J(m,n) = \sum_{i=1}^{l_2} g_{2i}(m,n)\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[d_i(s,t)a(s,t) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_i(\xi,\eta)a(\xi,\eta)\right],$$

$$\mu_1 = J(M,N), \qquad \mu_2 = \sum_{i=1}^{l_2} g_{2i}(m,n)\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[d_i(s,t) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_i(\xi,\eta)\right].$$

The proof for Corollary 2.4 can be completed by setting $b_i(s,t,m,n) = g_{1i}(m,n)b_{1i}(s,t)$, $c_i(s,t,m,n) = g_{1i}(m,n)c_{1i}(s,t)$, $d_i(s,t,m,n) = g_{2i}(m,n)d_{1i}(s,t)$, $e_i(s,t,m,n) = g_{2i}(m,n)e_{1i}(s,t)$ in Theorem 2.3.

Theorem 2.5 *Suppose $w \in \wp_+(\Omega)$, u, a, b_i, c_i, d_i, e_i, φ, ϕ are defined as in Theorem 2.3. Furthermore, assume $\varphi \circ \phi^{-1}$ is submultiplicative, that is, $\varphi(\phi^{-1}(\alpha\beta)) \le \varphi(\phi^{-1}(\alpha))\varphi(\phi^{-1}(\beta))$ $\forall \alpha, \beta \in \mathbb{R}_+$. If for $(m,n) \in \Omega$, $u(m,n)$ satisfies*

$$\begin{aligned}\phi\bigl(u(m,n)\bigr) \le{}& a(m,n) + \sum_{s=m+1}^{\infty} w(s,n)\phi\bigl(u(s,n)\bigr)\\&+ \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Biggl[b_i(s,t,m,n)\varphi\bigl(u(s,t)\bigr) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} c_i(\xi,\eta,m,n)\varphi\bigl(u(\xi,\eta)\bigr)\Biggr]\\&+ \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Biggl[d_i(s,t,m,n)\phi\bigl(u(s,t)\bigr)\\&+ \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_i(\xi,\eta,m,n)\phi\bigl(u(\xi,\eta)\bigr)\Biggr],\end{aligned} \tag{24}$$

then

$$\begin{aligned}u(m,n) \le \phi^{-1}\Biggl\{\Biggl\{a(m,n) + G^{-1}\Biggl\{G\Biggl(T^{-1}\Biggl[\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\overline{B}(s,t,M,N)\Biggr]\Biggr)\\+ \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\overline{B}(s,t,m,n)\Biggr\}\Biggr\}\overline{w}(m,n)\Biggr\}\end{aligned} \tag{25}$$

provided that T is increasing, where G is defined in (4), *and*

$$T(x) = G\biggl(\frac{x-\overline{\mu}_1}{\overline{\mu}_2}\biggr) - G(x), \quad x \ge 0, \tag{26}$$

$$\overline{B}(s,t,m,n) = \sum_{i=1}^{l_1}\Biggl[\overline{b}_i(s,t,m,n) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\overline{c}_i(\xi,\eta,m,n)\Biggr], \tag{27}$$

$$\overline{J}(m,n) = \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Biggl[\overline{d}_i(s,t,m,n)a(s,t) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\overline{e}_i(\xi,\eta,m,n)a(\xi,\eta)\Biggr], \tag{28}$$

$$\begin{aligned}&\overline{b}_i(s,t,m,n) = b_i(s,t,m,n)\varphi\bigl[\phi^{-1}\bigl(\overline{w}(s,t)\bigr)\bigr],\\&\overline{c}_i(s,t,m,n) = c_i(s,t,m,n)\varphi\bigl[\phi^{-1}\bigl(\overline{w}(s,t)\bigr)\bigr], \quad i = 1,\ldots,l_1,\end{aligned} \tag{29}$$

$$\begin{aligned}&\overline{d}_i(s,t,m,n) = d_i(s,t,m,n)\overline{w}(s,t),\\&\overline{e}_i(s,t,m,n) = e_i(s,t,m,n)\overline{w}(s,t), \quad i = 1,2,\ldots,l_2,\end{aligned} \tag{30}$$

$$\overline{w}(m,n) = \prod_{s=m+1}^{\infty}\bigl[1 + w(s,n)\bigr], \tag{31}$$

$$\overline{\mu}_1 = \overline{J}(M,N), \qquad \overline{\mu}_2 = \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Biggl[\overline{d}_i(s,t,M,N) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\overline{e}_i(\xi,\eta,M,N)\Biggr]. \tag{32}$$

Proof Denote

$$z(m,n) = a(m,n) + \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Bigg[b_i(s,t,m,n)\varphi\big(u(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi\big(u(\xi,\eta)\big)\Bigg] + \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Bigg[d_i(s,t,m,n)\phi\big(u(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)\phi\big(u(\xi,\eta)\big)\Bigg].$$

Then we have

$$\phi\big(u(m,n)\big) \le z(m,n) + \sum_{s=m+1}^{\infty}w(s,n)\phi\big(u(s,n)\big). \tag{33}$$

Obviously, $z(m,n)$ is nonincreasing in the first variable. So, by Lemma 2.1, we obtain

$$\phi\big(u(m,n)\big) \le z(m,n)\prod_{s=m+1}^{\infty}\big[1 + w(s,n)\big] = z(m,n)\overline{w}(m,n),$$

where $\overline{w}(m,n)$ is defined in (31). Define

$$v(m,n) = \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Bigg[b_i(s,t,m,n)\varphi\big(u(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi\big(u(\xi,\eta)\big)\Bigg] + \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Bigg[d_i(s,t,m,n)\phi\big(u(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)\phi\big(u(\xi,\eta)\big)\Bigg].$$

Then we obtain

$$u(m,n) \le \phi^{-1}\big[\big(a(m,n) + v(m,n)\big)\overline{w}(m,n)\big], \tag{34}$$

and furthermore, using $\varphi\circ\phi^{-1}$ is submultiplicative, (34) and Lemma 2.2, we have

$$\begin{aligned} v(m,n) \le{}& \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Bigg\{b_i(s,t,m,n)\varphi\big[\phi^{-1}\big(\big(a(s,t) + v(s,t)\big)\overline{w}(s,t)\big)\big] \\ &+ \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi\big[\phi^{-1}\big(\big(a(\xi,\eta) + v(\xi,\eta)\big)\overline{w}(\xi,\eta)\big)\big]\Bigg\} \\ &+ \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Bigg\{d_i(s,t,m,n)\big[a(s,t) + v(s,t)\big]\overline{w}(s,t) \\ &+ \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)\big[a(\xi,\eta) + v(\xi,\eta)\big]\overline{w}(\xi,\eta)\Bigg\} \\ \le{}& \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Bigg\{b_i(s,t,m,n)\varphi\big[\phi^{-1}\big(a(s,t) + v(s,t)\big)\big]\varphi\big[\phi^{-1}\big(\overline{w}(s,t)\big)\big] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi\bigl[\phi^{-1}\bigl(a(\xi,\eta)+v(\xi,\eta)\bigr)\bigr]\varphi\bigl[\phi^{-1}\bigl(\overline{w}(\xi,\eta)\bigr)\bigr]\Biggr\}\\
&+\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Biggl\{d_i(s,t,m,n)\bigl[a(s,t)+v(s,t)\bigr]\overline{w}(s,t)\\
&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)\bigl[a(\xi,\eta)+v(\xi,\eta)\bigr]\overline{w}(\xi,\eta)\Biggr\}\\
=&\sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Biggl\{\overline{b}_i(s,t,m,n)\phi^{-1}\bigl[a(s,t)+v(s,t)\bigr]\\
&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\overline{c}_i(\xi,\eta,m,n)\phi^{-1}\bigl[a(\xi,\eta)+v(\xi,\eta)\bigr]\Biggr\}\\
&+\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Biggl\{\overline{d}_i(s,t,m,n)\bigl[a(s,t)+v(s,t)\bigr]\\
&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\overline{e}_i(\xi,\eta,m,n)\bigl[a(\xi,\eta)+v(\xi,\eta)\bigr]\Biggr\}\\
=&\overline{H}(m,n)+\sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Biggl\{\overline{b}_i(s,t,m,n)\varphi\bigl[\phi^{-1}\bigl(a(s,t)+v(s,t)\bigr)\bigr]\\
&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi\bigl[\phi^{-1}\bigl(a(\xi,\eta)+v(\xi,\eta)\bigr)\bigr]\Biggr\},
\end{aligned}
\tag{35}
$$

where $\overline{H}(m,n)=\overline{J}(m,n)+\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\{\overline{d}_i(s,t,m,n)v(s,t)+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\overline{e}_i(\xi,\eta,m,n)v(\xi,\eta)\}$, and $\overline{J}(m,n)$ is defined in (28). Then similar to the process of (21)-(23), we obtain

$$
v(m,n)\le G^{-1}\Biggl[G\bigl(\overline{H}(M,N)\bigr)+\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\overline{B}(s,t,m,n)\Biggr],
\tag{36}
$$

and

$$
\overline{H}(M,N)\le T^{-1}\Biggl[\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\overline{B}(s,t,M,N)\Biggr].
\tag{37}
$$

Combining (34), (36) and (37), we get the desired result. □

Theorem 2.6 *Suppose $u, a, b_i, c_i, d_i, e_i, \varphi, \phi$ are defined as in Theorem 2.3. $L_{1i}, L_{2i}, T_{1i}, T_{2i}: \Omega\times\mathbb{R}_+\to\mathbb{R}_+$, $i=1,2,\ldots,l_2$ satisfies $0\le L_{ji}(m,n,u)-L_{ji}(m,n,v)\le T_{ji}(m,n,v)(u-v)$, $j=1,2$ for $u\ge v\ge 0$. If for $(m,n)\in\Omega$, $u(m,n)$ satisfies*

$$
\begin{aligned}
\phi\bigl(u(m,n)\bigr)\le a(m,n)+\sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Biggl[&b_i(s,t,m,n)\varphi\bigl(u(s,t)\bigr)\\
&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi\bigl(u(\xi,\eta)\bigr)\Biggr]
\end{aligned}
$$

$$+\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Bigg[d_i(s,t,m,n)L_{1i}\big(s,t,\phi\big(u(s,t)\big)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)L_{2i}\big(\xi,\eta,\phi\big(u(\xi,\eta)\big)\big)\Bigg], \tag{38}$$

then

$$u(m,n)\le\phi^{-1}\Bigg\{a(m,n)+G^{-1}\Bigg\{G\Bigg(T^{-1}\Bigg[\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\widehat{B}(s,t,M,N)\Bigg]\Bigg) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\widehat{B}(s,t,m,n)\Bigg\}\Bigg\} \tag{39}$$

provided that T is increasing, where G is defined in (4), *and*

$$T(x)=G\left(\frac{x-\widehat{\mu}_1}{\widehat{\mu}_2}\right)-G(x),\quad x\ge 0, \tag{40}$$

$$\widehat{B}(s,t,m,n)=\sum_{i=1}^{l_1}\Bigg[\widehat{b}_i(s,t,m,n)+\sum_{\xi=m_0}^{s}\sum_{\eta=n_0}^{t}\widehat{c}_i(\xi,\eta,m,n)\Bigg], \tag{41}$$

$$\widehat{J}(m,n)=\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Bigg[d_i(s,t,m,n)L_{1i}\big(s,t,a(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)L_{2i}\big(\xi,\eta,a(\xi,\eta)\big)\Bigg], \tag{42}$$

$$\widehat{b}_i(s,t,m,n)=b_i(s,t,m,n),\qquad \widehat{c}_i(s,t,m,n)=c_i(s,t,m,n),\quad i=1,2,\ldots,l_1, \tag{43}$$

$$\begin{aligned}&\widehat{d}_i(s,t,m,n)=d_i(s,t,m,n)T_{1i}\big(s,t,a(s,t)\big),\\&\widehat{e}_i(s,t,m,n)=e_i(s,t,m,n)T_{2i}\big(s,t,a(s,t)\big),\quad i=1,\ldots,l_2,\end{aligned} \tag{44}$$

$$\begin{aligned}&\widehat{\mu}_1=\widehat{J}(M,N),\\&\widehat{\mu}_2=\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Bigg[\widehat{d}_i(s,t,M,N)+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\widehat{e}_i(\xi,\eta,M,N)\Bigg].\end{aligned} \tag{45}$$

Proof Denote

$$\begin{aligned}v(m,n)={}&\sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\Bigg[b_i(s,t,m,n)\varphi\big(u(s,t)\big)+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi\big(u(\xi,\eta)\big)\Bigg]\\&+\sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\Bigg[d_i(s,t,m,n)L_{1i}\big(s,t,\phi\big(u(s,t)\big)\big)\\&+\sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)L_{2i}\big(\xi,\eta,\phi\big(u(\xi,\eta)\big)\big)\Bigg].\end{aligned}$$

Then we have

$$u(m,n) \le \phi^{-1}\bigl(a(m,n) + v(m,n)\bigr). \tag{46}$$

So,

$$\begin{aligned}
v(m,n) &\le \sum_{i=1}^{l_1} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \Biggl\{ b_i(s,t,m,n)\varphi\bigl[\phi^{-1}\bigl(a(s,t)+v(s,t)\bigr)\bigr] \\
&\quad + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} c_i(\xi,\eta,m,n)\varphi\bigl[\phi^{-1}\bigl(a(\xi,\eta)+v(\xi,\eta)\bigr)\bigr]\Biggr\} \\
&\quad + \sum_{i=1}^{l_2} \sum_{s=M+1}^{\infty} \sum_{t=N+1}^{\infty} \Biggl\{ d_i(s,t,m,n) L_{1i}\bigl(s,t,a(s,t)+v(s,t)\bigr) \\
&\quad + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_i(\xi,\eta,m,n) L_{2i}\bigl(\xi,\eta,a(\xi,\eta)+v(\xi,\eta)\bigr)\Biggr\} \\
&= \sum_{i=1}^{l_1} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \Biggl\{ b_i(s,t,m,n)\varphi\bigl[\phi^{-1}\bigl(a(s,t)+v(s,t)\bigr)\bigr] \\
&\quad + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} c_i(\xi,\eta,m,n)\varphi\bigl[\phi^{-1}\bigl(a(\xi,\eta)+v(\xi,\eta)\bigr)\bigr]\Biggr\} \\
&\quad + \sum_{i=1}^{l_2} \sum_{s=M+1}^{\infty} \sum_{t=N+1}^{\infty} \Biggl\{ d_i(s,t,m,n)\bigl[L_{1i}\bigl(s,t,a(s,t)+v(s,t)\bigr) \\
&\quad - L_{1i}\bigl(s,t,a(s,t)\bigr) + L_{1i}\bigl(s,t,a(s,t)\bigr)\bigr] \\
&\quad + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_i(\xi,\eta,m,n)\bigl[L_{2i}\bigl(\xi,\eta,a(\xi,\eta)+v(\xi,\eta)\bigr) \\
&\quad - L_{2i}\bigl(\xi,\eta,a(\xi,\eta)\bigr) + L_{2i}\bigl(\xi,\eta,a(\xi,\eta)\bigr)\bigr]\Biggr\} \\
&\le \sum_{i=1}^{l_1} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \Biggl\{ b_i(s,t,m,n)\varphi\bigl[\phi^{-1}\bigl(a(s,t)+v(s,t)\bigr)\bigr] \\
&\quad + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} c_i(\xi,\eta,m,n)\varphi\bigl[\phi^{-1}\bigl(a(\xi,\eta)+v(\xi,\eta)\bigr)\bigr]\Biggr\} \\
&\quad + \sum_{i=1}^{l_2} \sum_{s=M+1}^{\infty} \sum_{t=N+1}^{\infty} \Biggl\{ d_i(s,t,m,n)\bigl[T_{1i}\bigl(s,t,a(s,t)\bigr)v(s,t) + L_{1i}\bigl(s,t,a(s,t)\bigr)\bigr] \\
&\quad + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} e_i(\xi,\eta,m,n)\bigl[T_{2i}\bigl(\xi,\eta,a(\xi,\eta)\bigr)v(\xi,\eta) + L_{2i}\bigl(\xi,\eta,a(\xi,\eta)\bigr)\bigr]\Biggr\} \\
&= H(m,n) + \sum_{i=1}^{l_1} \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \Biggl\{ b_i(s,t,m,n)\varphi\bigl[\phi^{-1}\bigl(a(s,t)+v(s,t)\bigr)\bigr] \\
&\quad + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} c_i(\xi,\eta,m,n)\varphi\bigl[\phi^{-1}\bigl(a(\xi,\eta)+v(\xi,\eta)\bigr)\bigr]\Biggr\},
\end{aligned} \tag{47}$$

where $\widehat{H}(m,n) = \widehat{J}(m,n) + \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\{\widehat{d}_i(s,t,m,n)v(s,t) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\widehat{e}_i(\xi,\eta,m,n)v(\xi,\eta)\}$, and $\widehat{J}(m,n)$ is defined in (42). Then similar to the process of (21)-(23), we obtain

$$v(m,n) \le G^{-1}\left[G\big(\widehat{H}(M,N)\big) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\widehat{B}(s,t,m,n)\right], \tag{48}$$

and

$$H(M,N) \le T^{-1}\left[\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\widehat{B}(s,t,M,N)\right]. \tag{49}$$

Combining (46), (48) and (49), we get the desired result. □

Theorem 2.7 *Suppose $w \in \wp_+(\Omega)$, u, a, b_i, c_i, d_i, e_i, φ, ϕ are defined as in Theorem* 2.3, *and L_{ji}, T_{ji}, $j = 1,2$, $i = 1,2,\ldots,l_2$ are defined as in Theorem* 2.6. *If for $(m,n) \in \Omega$, $u(m,n)$ satisfies*

$$\begin{aligned}\phi\big(u(m,n)\big) \le{}& a(m,n) + \sum_{s=m+1}^{\infty} w(s,n)\phi\big(u(m,n)\big)\\ &+ \sum_{i=1}^{l_1}\sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[b_i(s,t,m,n)\varphi\big(u(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}c_i(\xi,\eta,m,n)\varphi\big(u(\xi,\eta)\big)\right]\\ &+ \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[d_i(s,t,m,n)L_{1i}\big(s,t,\phi\big(u(s,t)\big)\big)\right.\\ &\left.+ \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)L_{2i}\big(\xi,\eta,\phi\big(u(\xi,\eta)\big)\big)\right],\end{aligned}$$

then

$$\begin{aligned}u(m,n) \le \phi^{-1}\Bigg\{&a(m,n) + G^{-1}\Bigg\{G\left(T^{-1}\left[\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\widetilde{B}(s,t,M,N)\right]\right)\\ &+ \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\widetilde{B}(s,t,m,n)\Bigg\}\Bigg\}\end{aligned}$$

provided that T is increasing, where G is defined in (4), *and*

$$T(x) = G\left(\frac{x-\widetilde{\mu}_1}{\widetilde{\mu}_2}\right) - G(x), \quad x \ge 0,$$

$$\widetilde{B}(s,t,m,n) = \sum_{i=1}^{l_1}\left[\widetilde{b}_i(s,t,m,n) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\widetilde{c}_i(\xi,\eta,m,n)\right],$$

$$\begin{aligned}\widetilde{J}(m,n) ={}& \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[d_i(s,t,m,n)L_{1i}\big(s,t,a(s,t)\widetilde{\omega}(s,t)\big)\right.\\ &\left.+ \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}e_i(\xi,\eta,m,n)L_{2i}\big(\xi,\eta,a(\xi,\eta)\widetilde{\omega}(\xi,\eta)\big)\right],\end{aligned}$$

$$\widetilde{b}_i(s,t,m,n) = b_i(s,t,m,n)\varphi\big[\phi^{-1}\big(\widetilde{w}(s,t)\big)\big],$$

$$\widetilde{c}_i(s,t,m,n) = c_i(s,t,m,n)\varphi\big[\phi^{-1}\big(\widetilde{w}(s,t)\big)\big], \quad i = 1,2,\ldots,l_1,$$

$$\widetilde{d}_i(s,t,m,n) = d_i(s,t,m,n)\widetilde{w}(s,t)T_{1i}\big(s,t,a(s,t)\widetilde{w}(s,t)\big), \quad i = 1,2,\ldots,l_2,$$

$$\widetilde{e}_i(s,t,m,n) = e_i(s,t,m,n)\widetilde{w}(s,t)T_{2i}\big(s,t,a(s,t)\widetilde{w}(s,t)\big), \quad i = 1,2,\ldots,l_2,$$

$$\widetilde{w}(m,n) = \prod_{s=m+1}^{\infty}\big[1 + w(s,n)\big],$$

$$\widetilde{\mu}_1 = \widetilde{J}(M,N), \qquad \widetilde{\mu}_2 = \sum_{i=1}^{l_2}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[\widetilde{d}_i(s,t,M,N) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\widetilde{e}_i(\xi,\eta,M,N)\right].$$

The proof for Theorem 2.7 is similar to the combination of Theorem 2.5 and Theorem 2.6, and we omit the details here.

Remark 2.8 We note that the inequalities established in Theorems 2.3, 2.5-2.7 are essentially different from the results in [22–24] as the left-hand side of the inequalities established here is an arbitrary function $\phi(u)$. Furthermore, if we set $\phi(u) = u^p$, $a(m,n) = 0$, then Theorem 2.5 reduces to [22, Theorem 2.5].

3 Applications

In this section, we present some applications for the results established above. Similar to the applications in [22–24], we research a certain Volterra-Fredholm sum-difference equation and derive some new bounds for its solutions.

Example Consider the following Volterra-Fredholm type infinite sum-difference equation:

$$\begin{aligned} u^p(m,n) = {} & \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[F_1\big(s,t,m,n,u(s,t)\big) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}F_2\big(\xi,\eta,m,n,u(\xi,\eta)\big)\right] \\ & + \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[G_1\big(s,t,m,n,u(s,t)\big)\right. \\ & \left. + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}G_2\big(\xi,\eta,m,n,u(\xi,\eta)\big)\right], \end{aligned} \tag{50}$$

where $u \in \wp(\Omega)$, $p \geq 1$ is an odd number, $F_i, G_i : \Omega^2 \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$.

Theorem 3.1 *Suppose $u(m,n)$ is a solution of* (50), *and* $|F_1(s,t,m,n,u)| \leq f_1(s,t,m,n)|u|^{\frac{p}{2}}$, $|F_2(s,t,m,n,u)| \leq f_2(s,t,m,n)|u|^{\frac{p}{2}}$, $|G_1(s,t,m,n,u)| \leq g_1(s,t,m,n)|u|^p$, $|G_2(s,t,m,n,u)| \leq g_2(s,t,m,n)|u|^p$, $f_i, g_i \in \wp_+(\Omega^2)$, $i = 1,2$, f_i, g_i *are nondecreasing in the last two variables, and there is at least one function among* g_1, g_2 *not equivalent to zero, then we have*

$$u(m,n) \leq 4^{-\frac{1}{p}}\left\{\frac{\sqrt{\mu}}{1-\sqrt{\mu}}\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}B(s,t,M,N) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}B(s,t,m,n)\right\}^{\frac{2}{p}} \tag{51}$$

provided that $\mu < 1$, *where*

$$B(s,t,m,n) = f_1(s,t,m,n) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} f_2(\xi,\eta,m,n),$$

$$\mu = \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[g_1(s,t,M,N) + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} g_2(\xi,\eta,M,N)\right].$$

Proof From (50) we have

$$\begin{aligned}
\left|u(m,n)\right|^p &\le \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[\left|F_1\big(s,t,m,n,u(s,t)\big)\right| + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\left|F_2\big(\xi,\eta,m,n,u(\xi,\eta)\big)\right|\right] \\
&\quad + \sum_{s=m_0}^{M-1}\sum_{t=n_0}^{M-1}\left[\left|G_1\big(s,t,m,n,u(s,t)\big)\right| + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty}\left|G_2\big(\xi,\eta,m,n,u(\xi,\eta)\big)\right|\right] \\
&\le \left|a(m,n)\right| + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\left[f_1(s,t,m,n)\left|u(s,t)\right|^{\frac{p}{2}}\right. \\
&\quad \left. + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} f_2(\xi,\eta,m,n)\left|u(\xi,\eta)\right|^{\frac{p}{2}}\right] \\
&\quad + \sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\left[g_1(s,t,m,n)\left|u(s,t)\right|^p\right. \\
&\quad \left. + \sum_{\xi=s}^{\infty}\sum_{\eta=t}^{\infty} g_2(\xi,\eta,m,n)\left|u(\xi,\eta)\right|^p\right].
\end{aligned} \tag{52}$$

Define $\phi(u) = u^p$, $\varphi(u) = u^{\frac{p}{2}}$, and

$$G(z) = \int_{z_0}^{z}\frac{1}{\sqrt{z}}\,dz = 2\sqrt{z} - 2\sqrt{z_0}, \quad z \ge z_0 > 0, \tag{53}$$

$$T(x) = G\left(\frac{x}{\mu}\right) - G(x) = 2\sqrt{\frac{x}{\mu}} - 2\sqrt{x}, \quad x \ge 0. \tag{54}$$

Then by $\mu < 1$, we have T is strictly increasing, and a suitable application of Theorem 2.3 (with $a(m,n) = 0$ and $l_1 = l_2 = 1$) to (52) yields

$$\begin{aligned}
u(m,n) &\le \phi^{-1}\left\{G^{-1}\left\{G\left(T^{-1}\left[\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty} B(s,t,M,N)\right]\right)\right.\right. \\
&\quad \left.\left. + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} B(s,t,m,n)\right\}\right\}.
\end{aligned} \tag{55}$$

Combining (53)-(55), we can deduce the desired result. □

Theorem 3.2 *Suppose* $u(m,n)$ *is a solution of* (50), *and* $|F_1(s,t,m,n,u)| \le f_1(s,t,m,n)|u|^{\frac{p}{3}}$, $|F_2(s,t,m,n,u)| \le f_2(s,t,m,n)|u|^{\frac{p}{3}}$, $|G_1(s,t,m,n,u)| \le g_1(s,t,m,n)L_1(s,t,|u|^p)$, $|G_2(s,t,m,n,u)| \le g_2(s,t,m,n)L_2(s,t,|u|^p)$, *where* f_i, g_i, $i = 1,2$ *are defined as in Theorem* 3.1, L_1, L_2, T_1,

$T_2 : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $0 \le L_i(m,n,u) - L_i(m,n,v) \le T_i(m,n,v)(u-v)$ for $u \ge v \ge 0$ and $L_i(m,n,0) = 0$, $i = 1,2$, then we have

$$u(m,n) \le \left\{ \frac{2}{3} \left[\frac{\widehat{\mu}^{\frac{2}{3}}}{1-\widehat{\mu}^{\frac{2}{3}}} \sum_{s=M+1}^{\infty} \sum_{t=N+1}^{\infty} \widehat{B}(s,t,M,N) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \widehat{B}(s,t,m,n) \right] \right\}^{\frac{3}{2p}} \tag{56}$$

provided that $\widehat{\mu} < 1$, where

$$\widehat{B}(s,t,m,n) = f_1(s,t,m,n) + \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} f_2(\xi,\eta,m,n),$$

$$\widehat{\mu} = \sum_{s=M+1}^{\infty} \sum_{t=N+1}^{\infty} \left[g_1(s,t,M,N)T_1(s,t,0) + \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} g_2(\xi,\eta,M,N)T_2(\xi,\eta,0) \right].$$

Proof From (50) we have

$$\begin{aligned}
\left|u(m,n)\right|^p \le{}& \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[\left|F_1\big(s,t,m,n,u(s,t)\big)\right| + \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} \left|F_2\big(\xi,\eta,m,n,u(\xi,\eta)\big)\right| \right] \\
&+ \sum_{s=M+1}^{\infty} \sum_{t=N+1}^{\infty} \left[\left|G_1\big(s,t,m,n,u(s,t)\big)\right| + \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} \left|G_2\big(\xi,\eta,m,n,u(\xi,\eta)\big)\right| \right] \\
\le{}& \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[f_1(s,t,m,n)\left|u(s,t)\right|^{\frac{p}{3}} + \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} f_2(\xi,\eta,m,n)\left|u(\xi,\eta)\right|^{\frac{p}{3}} \right] \\
&+ \sum_{s=M+1}^{\infty} \sum_{t=N+1}^{\infty} \left[g_1(s,t,m,n)L_1\big(s,t,\left|u(s,t)\right|^p\big) \right. \\
&\left. + \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} g_2(\xi,\eta,m,n)L_2\big(\xi,\eta,\left|u(\xi,\eta)\right|^p\big) \right] \\
\le{}& \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} \left[f_1(s,t,m,n)\left|u(s,t)\right|^{\frac{p}{3}} + \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} f_2(\xi,\eta,m,n)\left|u(\xi,\eta)\right|^{\frac{p}{3}} \right] \\
&+ \sum_{s=M+1}^{\infty} \sum_{t=N+1}^{\infty} \left[g_1(s,t,m,n)T_1(s,t,0)\left|u(s,t)\right|^p \right. \\
&\left. + \sum_{\xi=s}^{\infty} \sum_{\eta=t}^{\infty} g_2(\xi,\eta,m,n)T_2(\xi,\eta,0)\left|u(\xi,\eta)\right|^p \right].
\end{aligned} \tag{57}$$

Define $\phi(u) = u^p$, $\varphi(u) = u^{\frac{p}{3}}$, and

$$G(z) = \int_{z_0}^{z} \frac{1}{z^{\frac{1}{3}}}\,dz = \frac{3}{2}\left[z^{\frac{2}{3}} - z_0^{\frac{2}{3}}\right], \quad z \ge z_0 > 0, \tag{58}$$

$$T(x) = G\left(\frac{x}{\widehat{\mu}}\right) - G(x) = \frac{3}{2}\left(\frac{1-\widehat{\mu}^{\frac{2}{3}}}{\widehat{\mu}^{\frac{2}{3}}}\right)x^{\frac{2}{3}}, \quad x \ge 0. \tag{59}$$

Then by $\widehat{\mu} < 1$, we have T is strictly increasing, and a suitable application of Theorem 2.6 (with $\phi(u) = u^p$, $\varphi(u) = u^{\frac{p}{3}}$, $a(m,n) = 0$ and $l_1 = l_2 = 1$) to (57) yields

$$u(m,n) \le \phi^{-1}\left\{G^{-1}\left\{G\left(T^{-1}\left[\sum_{s=M+1}^{\infty}\sum_{t=N+1}^{\infty}\widehat{B}(s,t,M,N)\right]\right) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty}\widehat{B}(s,t,m,n)\right\}\right\}. \tag{60}$$

Combining (58)-(60), we can deduce the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BZ carried out the main part of this article. All authors read and approved the final manuscript.

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