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Solvability of boundary value problems with Riemann-Stieltjes Δ-integral conditions for second-order dynamic equations on time scales at resonance

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Abstract

In this paper, by making use of the coincidence degree theory of Mawhin, the existence of the nontrivial solution for the boundary value problem with Riemann-Stieltjes Δ-integral conditions on time scales at resonance

$$\begin{cases} x^{\Delta\Delta}(t) = f(t, x(t), x^{\Delta}(t)) + e(t), \text{ a.e. } t \in [0, T]_{\mathbb{T}}, \\ x^{\Delta}(0) = 0, \qquad x(T) = \int\limits_{0}^{T} x^{\sigma}(s)\Delta g(s) \end{cases}$$

is established, where $f : [0, T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions and $e : [0, T]_{\mathbb{T}} \to \mathbb{R}$ is a continuous function and $g : [0, T]_{\mathbb{T}} \to \mathbb{R}$ is an increasing function with $\int_0^T \Delta g(s) = 1$. An example is given to illustrate the main results.

Keywords: boundary value problem with Riemann-Stieltjes Δ?Δ?-integral conditions, resonance, time scales

1 Introduction

Hilger [1] introduced the notion of time scales in order to unify the theory of continuous and discrete calculus. The field of dynamical equations on time scales contains, links and extends the classical theory of differential and difference equations, besides many others. There are more time scales than just $\mathbb{R}$ (corresponding to the continuous case) and $\mathbb{N}$ (corresponding to the discrete case) and hence many more classes of dynamic equations. An excellent resource with an extensive bibliography on time scales was produced by Bohner and Peterson [2,3].

Recently, existence theory for positive solutions of boundary value problems (BVPs) on time scales has attracted the attention of many authors; Readers are referred to, for example, [4-11] and the references therein for the existence theory of some two-point BVPs and [12-17] for three-point BVPs on time scales. For the existence of solutions of m-point BVPs on time scales, we refer the reader to [18-20].

At the same time, we notice that a class of boundary value problems with integral boundary conditions have various applications in chemical engineering, thermo-elasticity, population dynamics, heat conduction, chemical engineering underground water flow, thermo-elasticity and plasma physics. On the other hand, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two-point, three-point, multipoint and nonlocal boundary value problems as special cases [[21-24], and the references therein]. However, very little work has been done to the existence of solutions for boundary value problems with integral boundary conditions on time scales.

Motivated by the statements above, in this paper, we are concerned with the following boundary value problem with integral boundary conditions

$$\begin{cases} x^{\Delta\Delta}(t) = f(t, x(t), x^{\Delta}(t)) + e(t), \text{ a.e. } t \in [0, T]_{\mathbb{T}}, \\ x^{\Delta}(0) = 0, \qquad x(T) = \int_0^T x^{\sigma}(s)\Delta g(s), \end{cases} \tag{1.1}$$

where $f : [0, T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $e : [0, T]_{\mathbb{T}} \to \mathbb{R}$ are continuous functions, $g : [0, T]_{\mathbb{T}} \to \mathbb{R}$ is an increasing function with $\int_0^T \Delta g(s) = 1$, and the integral in (1.1) is a Riemann-Stieltjes on time scales, which is introduced in Section 2 of this paper.

According to the calculus theory on time scales, we can illustrate that boundary value problems with integral boundary conditions on time scales also cover two-point, three-point, ..., n-point boundary problems as the nonlocal boundary value problems do in the continuous case. For instance, in BVPs (1.1), let

$$g(s) = \sum_{i=1}^{k} a_i \chi(s - t_i),$$

where $k \geq 1$ is an integer, $a_i \in [0, \infty)$, $i = 1, ..., k$, $\{t_i\}_{i=1}^{k}$ is a finite increasing sequence of distinct points in $[0, T]_{\mathbb{T}}$, and $\chi(s)$ is the characteristic function, that is,

$$\chi(s) = \begin{cases} 1, s > 0, \\ 0, s \leq 0, \end{cases}$$

then the nonlocal condition in BVPs (1.1) reduces to the k-point boundary condition

$$x(T) = \sum_{i=1}^{k} a_i x(t_i),$$

where t_i, $i = 1, 2, ..., k$ can be determined (see Lemma 2.5 in Section 2).

The effect of resonance in a mechanical equation is very important to scientists. Nearly every mechanical equation will exhibit some resonance and can, with the application of even a very small external pulsed force, be stimulated to do just that. Scientists usually work hard to eliminate resonance from a mechanical equation, as they perceive it to be counter-productive. In fact, it is impossible to prevent all resonance. Mathematicians have provided more theory of resonance from equations. For the case where ordinary differential equation is at resonance, most studies have tended to the equation $x''(t) = f(t, x(t), x'(t)) + e(t)$. For example, Feng and Webb [25] studied the following boundary value problem

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), \ t \in (0, 1), \\ x(0) = 0, \qquad x(1) = \alpha x(\xi), \end{cases}$$

when $\alpha\xi = 1(\xi \in (0, 1))$ is at resonance.

It is easy to see that $x_1(t) \equiv c(c \in \mathbb{R})$ and $x_2(t) = pt(p \in \mathbb{R})$ are a fundamental set of solutions of the linear mapping $Lx(t) = x^{\Delta\Delta}(t) = 0$. Let $U_1(x) = x^{\Delta}(0)$ and $U_2(x) = x(T) - \int_0^T x^{\sigma}(s)\Delta g(s)$. Since $\int_0^T \Delta g(s) = 1$, we have that

$$Q(x) = \begin{pmatrix} U_1(x_1) & U_1(x_2) \\ (U_2(x_1) & U_2(x_2) \end{pmatrix} = \begin{pmatrix} 0 & p \\ 0 & pT - p\int_0^T \sigma(s)\Delta g(s) \end{pmatrix}.$$

Thus, det $Q(x) = 0$, which implies that BVPs (1.1) is at resonance. By applying coincidence degree theorem of Mawhin to integral boundary value problems on time scales at resonance, this paper will establish some sufficient conditions for the existence of at least one solution to BVPs (1.1).

The rest of this paper is organized as follows. Section 2 introduces the Riemann-Stieltjes integral on time scales. Some lemmas and criterion for the existence of at least one solution to BVPs (1.1) are established in Section 3, and examples are given to illustrate our main results in Section 4.

2 Preliminaries

This section includes two parts. In the first part, we shall recall some basic definitions and lemmas of the calculus on time scales, which will be used in this paper. For more details, we refer to books by Bohner and Peterson [2,3]. In the second part, we introduce the Riemann-Stieltjes Δ-integral and ∇-integral on time scales, which was first established by Mozyrska et al. in [26].

2.1 The basic calculus on time scales

Definition 2.1. [3] A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$.

The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) := \sigma(t) - t.$$

The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense or right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$, respectively. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum m_1, define $\mathbb{T}^k = \mathbb{T} - \{m_1\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m_2, define $\mathbb{T}_k = \mathbb{T} - \{m_2\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$.

Definition 2.2. [3] A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous (rd-continuous is short for right-dense continuous) provided it is continuous at each right-dense point in $\mathbb{T}$ and has a left-sided limit at each left-dense point in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.3. [3] If $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^k$, then the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

Definition 2.4. [3] For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by Banach space), the (delta) derivative is defined at point t by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is not right-scattered, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

provided this limit exists.

Definition 2.5. [3] If $F^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

Lemma 2.1. [3]*Let* $a \in \mathbb{T}^k$, $b \in \mathbb{T}$ *and assume that* $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ *is continuous at* (t, t), *where* $t \in \mathbb{T}^k$ *with* $t > a$. *Also assume that* $f^{\Delta}(t, \cdot)$ *is rd-continuous on* $[a, \sigma(t)]$. *Suppose that for each* $\varepsilon > 0$ *there exists a neighborhood U of t, independent of* $\tau \in [a, \sigma(t)]$, *such that*

$$|f(\sigma(t), \tau) - f(s, \tau) - f^{\Delta}(t, \tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \text{for all } s \in U,$$

where f^{Δ} *denotes the derivative of f with respect to the first variable. Then*

(1) $g(t) := \int_a^t f(t, \tau)\Delta\tau$ *implies* $g^{\Delta}(t) = \int_a^t f^{\Delta}(t, \tau)\Delta\tau + f(\sigma(t), t)$;

(2) $h(t) := \int_t^b f(t, \tau)\Delta\tau$ *implies* $h^{\Delta}(t) = \int_t^b f^{\Delta}(t, \tau)\Delta\tau - f(\sigma(t), t)$.

The construction of the Δ-measure on $\mathbb{T}$ and the following concepts can be found in [3].

(i) For each $t_0 \in \mathbb{T} \backslash \{\max \mathbb{T}\}$, the single-point set t_0 is Δ-measurable, and its Δ-measure is given by

$$\mu_{\Delta}(\{t_0\}) = \sigma(t_0) - t_0 = \mu(t_0).$$

(ii) If $a, b \in \mathbb{T}$ and $a \leq b$, then

$$\mu_{\Delta}([a, b)) = b - a \text{ and } \mu_{\Delta}((a, b)) = b - \sigma(a).$$

(iii) If $a, b \in \mathbb{T} \backslash \{\max \mathbb{T}\}$ and $a \leq b$, then

$$\mu_{\Delta}((a, b]) = \sigma(b) - \sigma(a) \text{ and } \mu_{\Delta}([a, b]) = \sigma(b) - a.$$

The Lebesgue integral associated with the measure μ_Δ on $\mathbb{T}$ is called the Lebesgue delta integral. For a (measurable) set $E \subset \mathbb{T}$ and a function $f : E \to \mathbb{R}$, the corresponding integral of f on E is denoted by $\int_E f(t)\Delta t$. All theorems of the general Lebesgue integration theory hold also for the Lebesgue delta integral on $\mathbb{T}$.

2.2 The Riemann-Stieltjes integral on time scales

Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$, $a < b$, and $I = [a, b]_{\mathbb{T}}$. A partition of I is any finite-ordered

subset

$$P = \{t_0, t_1, \ldots, t_n\} \subset [a, b]_{\mathbb{T}}, \text{where } a = t_0 < t_1 < \cdots < t_n = b.$$

Let g be a real-valued increasing function on I. Each partition $P = \{t_0, t_1, ..., t_n\}$ of I decomposes I into subintervals $I_{\Delta_j} = [t_{j-1}, \rho(t_j)]_{\mathbb{T}} := [t_{j-1}, t_j]_\Delta$, $j = 1, 2, ..., n$, such that $I_{\Delta_j} \cap I_{\Delta_k} = \emptyset$ for any $k \neq j$. By $\Delta t_j = t_j - t_{j-1}$, we denote the length of the jth subinterval in the partition P; by $\mathcal{P}(I)$ the set of all partitions of I.

Let P_m, $P_n \in \mathcal{P}(I)$. If $P_m \subset P_n$, we call P_n a refinement of P_m. If P_m, P_n are independently chosen, then the partition $P_m \bigcup P_n$ is a common refinement of P_m and P_n.

Let us now consider an increasing real-valued function g on the interval I. Then, for the partition P of I, we define

$$g(P) = \{g(a) = g(t_0), g(t_1,) \ldots, g(t_{n-1}), g(t_n)\} \subset g(I),$$

where $\Delta g_j = g(t_j) - g(t_{j-1})$. We note that Δg_j is positive and $\sum_{j=1}^{n} \Delta g_j = g(b) - g(a)$. Moreover, $g(P)$ is a partition of $[g(a), g(b)]_{\mathbb{R}}$. In what follows, for the particular case $g(t) = t$ we obtain the Riemann sums for delta integral. We note that for a general g the image $g(I)$ is not necessarily an interval in the classical sense, even for rd-continuous function g, because our interval I may contain scattered points. From now on, let g be always an increasing real function on the considered interval $I = [a, b]_{\mathbb{T}}$.

Lemma 2.2. [26]*Let* $I = [a, b]_{\mathbb{T}}$ *be a closed (bounded) interval in* $\mathbb{T}$ *and let g be a continuous increasing function on I. For every* $\delta > 0$, *there is a partition* $P_\delta = \{t_0, t_1, \ldots, t_n\} \in \mathcal{P}(I)$ *such that for each* $j \in \{1, 2, ..., n\}$, *one has*

$$\Delta g_j = g(t_j) - g(t_{j-1}) \leq \delta \quad \text{or} \quad \Delta g_j > \delta \quad \wedge \quad \rho(t_j) = t_{j-1}.$$

Let f be a real-valued and bounded function on the interval I. Let us take a partition $P = \{t_0, t_1, ..., t_n\}$ of I. Denote $I_{\Delta_j} = [t_{j-1}, t_j]_\Delta$, $j = 1, 2, ..., n$, and

$$m_{\Delta_j} = \inf_{t \in I_{\Delta_j}} f(t), \quad M_{\Delta_j} = \sup_{t \in I_{\Delta_j}} f(t).$$

The upper Darboux-Stieltjes Δ-sum of f with respect the partition P, denoted by $U_\Delta(P, f, g)$, is defined by

$$U_\Delta(P, f, g) = \sum_{j=1}^{n} M_{\Delta_j} \Delta g_j,$$

while the lower Darboux-Stieltjes Δ-sum of f with respect the partition P, denoted by $L_\Delta(P, f, g)$, is defined by

$$L_\Delta(P,f,g) = \sum_{j=1}^{n} m_{\Delta_j}\Delta g_j.$$

Definition 2.6. [26] Let $I = [a,b]_\mathbb{T}$, where $a,b \in \mathbb{T}$. Let g be continuous on I. The upper Darboux-Stieltjes Δ-integral from a to b with respect to function g is defined by

$$\overline{\int_a^b} f(t)\Delta g(t) = \inf_{P\in\mathcal{P}(I)} U_\Delta(P,f,g);$$

the lower Darboux-Stieltjes Δ-integral from a to b with respect to function g is defined by

$$\underline{\int_a^b} f(t)\Delta g(t) = \sup_{P\in\mathcal{P}(I)} U_\Delta(P,f,g).$$

If $\overline{\int_a^b} f(t)\Delta g(t) = \underline{\int_a^b} f(t)\Delta g(t)$, then we say that f is Δ-integrable with respect to g on I, and the common value of the integrals, denoted by $\int_a^b f(t)\Delta g(t) = \int_a^b f\Delta g$, is called the Riemann-Stieltjes Δ-integral of f with respect to g on I.

The set of all functions that are Δ-integrable with respect to g in the Riemann-Stieltjes sense will be denoted by $\mathcal{R}_\Delta(g,I)$.

Theorem 2.1. [26]*Let f be a bounded function on* $I = [a,b]_\mathbb{T}$, $a,b \in \mathbb{T}$, $m \le f(t) \le M$ *for all* $t \in I$, *and g be a function defined and monotonically increasing on I. Then*

$$m(g(b) - g(a)) \le \underline{\int_a^b} f(t)\Delta g(t) \le \overline{\int_a^b} f(t)\Delta g(t) \le M(g(b) - g(a)).$$

If $f \in \mathcal{R}_\Delta(g,I)$, *then*

$$m(g(b) - g(a)) \le \int_a^b f(t)\Delta g(t) \le M(g(b) - g(a)).$$

Theorem 2.2. [26] (Integrability criterion) *Let f be a bounded function on* $I = [a,b]_\mathbb{T}$, $a,b \in \mathbb{T}$. *Then,* $f \in \mathcal{R}_\Delta(g,I)$ *if and only if for every* $\varepsilon > 0$, *there exists a partition* $P \in \mathcal{P}(I)$ *such that*

$$U_\Delta(P,f,g) - L_\Delta(P,f,g) < \varepsilon.$$

Theorem 2.3. [26]*Let* $I = [a,b]_\mathbb{T}$, $a,b \in \mathbb{T}$. *Then, the condition* $f \in \mathcal{R}_\Delta(g,I)$ *is equivalent to each one of the following items:*

(i) *f is a monotonic function on I;*
(ii) *f is a continuous function on I;*
(iii) *f is regulated on I;*
(iv) *f is a bounded and has a finite number of discontinuity points on I.*

In the following, we state some algebraic properties of the Riemann-Stieltjes integral on time scales as well. The properties are valid for an arbitrary time scale $\mathbb{T}$ with at least two points. We define $\int_a^a f(t)\Delta g(t) = 0$ and $\int_a^b f(t)\Delta g(t) = -\int_b^a f(t)\Delta g(t)$ for $a > b$.

Theorem 2.4. [26]*Let* $I = [a, b]_{\mathbb{T}}$, $a, b \in \mathbb{T}$. *Every constant function* $f : \mathbb{T} \to \mathbb{R}$, $f(t) \equiv c$, is *Stieltjes* Δ*-integrable with respect to g on I and*

$$\int_a^b c\Delta g(t) = c(g(b) - g(a)).$$

Theorem 2.5. [26]*Let* $t \in \mathbb{T}$ *and* $f : \mathbb{T} \to \mathbb{R}$. *If f is Riemann-Stieltjes* Δ*-integrable with respect to g from t to* $\sigma(t)$, *then*

$$\int_t^{\sigma(t)} f(\tau)\Delta g(\tau) = f(t)(g^{\sigma}(t) - g(t)),$$

where $g^{\sigma} = g \circ \sigma$. *Moreover, if g is* Δ*-differentiable at t, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta g(\tau) = \mu(t)f(t)g^{\Delta}(t).$$

Theorem 2.6. [26]*Let* $a, b, c \in \mathbb{T}$ *with* $a < b < c$. *If f is bounded on* $[a, c]_{\mathbb{T}}$ *and g is monotonically increasing on* $[a, c]_{\mathbb{T}}$, *then*

$$\int_a^c f\Delta g = \int_a^b f\Delta g + \int_b^c f\Delta g.$$

Lemma 2.3. [26]*Let* $I = [a, b]_{\mathbb{T}}$, $a, b \in \mathbb{T}$. *Suppose that g is an increasing function such that* g^{Δ} *is continuous on* $(a, b)_{\mathbb{T}}$ *and* f^{σ} *is a real-bounded function on I. Then,* $f^{\sigma} \in \mathcal{R}_{\Delta}(g, I)$ *if and only if* $f^{\sigma}g^{\Delta} \in \mathcal{R}_{\Delta}(g, I)$. *Moreover,*

$$\int_a^b f^{\sigma}(t)\Delta g(t) = \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t.$$

Lemma 2.4. (Delta integration by parts) *Let* $I = [a, b]_{\mathbb{T}}$, $a, b \in \mathbb{T}$. *Suppose that g is an increasing function such that* g^{Δ} *is continuous on* $(a, b)_{\mathbb{T}}$ *and* f^{σ} *is a real-bounded function on I. Then*

$$\int_a^b f^{\sigma}\Delta g = [fg]_a^b - \int_a^b g\Delta f.$$

Proof. Lemma 2.3 imply that

$$\int_a^b f^{\sigma}(t)\Delta g(t) = \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t;$$

furthermore,

$$\int_a^b f^\sigma(t)g^\Delta(t)\Delta t = [fg]_a^b - \int_a^b f^\Delta(t)g(t)\Delta t.$$

Hence,

$$\int_a^b f^\sigma \Delta g = [fg]_a^b - \int_a^b g\Delta f.$$

The proof of this lemma is complete.

Lemma 2.5. *Let* $I = [0, T]_{\mathbb{T}}$, $0, T \in \mathbb{T}$. *Assume that* f^σ *is a real-bounded function on* I *and*

$$g(s) = \sum_{i=1}^{k} a_i \chi(s - t_i),$$

where $k \geq 1$ *is an integer,* $a_i \in [0, \infty)$, $i = 1, ..., k$, $\{t_i\}_{i=1}^k$ *is a finite increasing sequence of distinct points in* $[0, T]_{\mathbb{T}}$ *and* $\chi(s)$ *is the characteristic function, that is,*

$$\chi(s) = \begin{cases} 1, & s > 0, \\ 0, & s \leq 0. \end{cases}$$

Then

$$f(T) = \int_0^T f^\sigma(s)\Delta g(s) = \sum_{i=1}^{k} a_i f(t_i),$$

where t_i, $i = 1, 2, ..., k$ *can be determined.*

Proof. By Lemma 2.4, it leads to

$$\begin{aligned}
f(T) &= \int_0^T f^\sigma(s)\Delta g(s) \\
&= \left(\int_0^{t_1} + \int_{t_1}^{t_2} + \cdots + \int_{t_k}^{T} \right) f^\sigma(s)\Delta g(s) \\
&= \left([fg]_0^{t_1} - \int_0^{t_1} g(s)\Delta f(s) \right) + \cdots + \left([fg]_{t_k}^{T} - \int_{t_k}^{T} g(s)\Delta f(s) \right) \\
&= f(T)g(T) - \left(\int_0^{t_1} 0\Delta f(s) + \int_{t_1}^{t_2} a_1 \Delta f(s) + \cdots + \int_{t_k}^{T} (a_1 + a_2 + \cdots + a_k)\Delta f(s) \right) \\
&= (a_1 + a_2 + \cdots + a_k)f(T) - \left(-\sum_{i=1}^{k} a_i f(t_i) + (a_1 + a_2 + \cdots + a_k)f(T) \right) \\
&= \sum_{i=1}^{k} a_i f(t_i).
\end{aligned}$$

This completes the proof.

3 Main results

In this section, first we provide some background materials from Banach spaces and preliminary results, and then we illustrate and prove some important lemmas and theorems.

Definition 3.1. *Let × and Y be Banach spaces. A linear operator* $L : \mathrm{Dom}\, L \subset X \to Y$ *is called a Fredholm operator if the following two conditions hold*

(i) Ker$L$ *has a finite dimension;*
(ii) Im L *is closed and has a finite codimension.*

L is a Fredholm operator, and its Fredholm index is the integer Ind L = dimKer L - codimIm L. In this paper, we are interested in a Fredholm operator of index zero, i.e., dimKer L = codimIm L.

From Definition 3.1, we know that there exist continuous projector $P : X \to X$ and $Q : Y \to Y$ such that Im P = Ker L, Ker Q = Im L, X = Ker $L \oplus$ Ker P, Y = Im $L \oplus$ ImQ, and the operator $L|_{\mathrm{Dom}\ L \cap \mathrm{Ker} P} : \mathrm{Dom}\ L \cap \mathrm{Ker}\ P \to \mathrm{Im}\ L$ is invertible; we denote the inverse of $L|_{\mathrm{Dom}\ L \cap \mathrm{Ker} P}$ by $K_P : \mathrm{Im}\ L \to \mathrm{Dom}\ L \cap \mathrm{Ker}\ P$. The generalized inverse of L denoted by $K_{P,Q} : Y \to \mathrm{Dom}\ L \cap \mathrm{Ker}\ P$ is defined by $K_{P,Q} = K_P\,(I - Q)$.

Now, we state the coincidence degree theorem of Mawhin [27].

Theorem 3.1. *Let* $\Omega \subset X$ *be open-bounded set, L be a Fredholm operator of index zero and N be L-compact on* $\bar{\Omega}$ *. Assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in (\mathrm{Dom}\ L \backslash \mathrm{Ker}\ L) \cap \partial\Omega \times [0, T]_{\mathbb{T}}$;
(ii) $Nx \notin \mathrm{Im}\ L$ *for every* $x \in \mathrm{Ker}\ L \cap \partial\Omega$;
(iii) $\deg(QN|_{\mathrm{Ker}\ L \cap \partial\Omega}, \Omega \cap \mathrm{Ker}\ L, 0) \neq 0$ *with* $Q : Y \to Y$ *a continuous projector such that* Ker Q = Im L.

Then, the equation Lu = Nu admits at least one nontrivial solution in Dom $L \cap \bar{\Omega}$.

Definition 3.2. *A mapping* $f : [0, T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies the Carathéodory conditions with respect to* $L^{\Delta}[0, T]_{\mathbb{T}}$, *where* $L^{\Delta}[0, T]_{\mathbb{T}}$ *denotes that all Lebesgue Δ-integrable functions on* $[0, T]_{\mathbb{T}}$, *if the following conditions are satisfied:*

(i) *for each* $(x_1, x_2) \in \mathbb{R}^2$, *the mapping* $t \to f(t, x_1, x_2)$ *is Lebesgue measurable on* $[0, T]_{\mathbb{T}}$;
(ii) *for a.e.* $t \in [0, T]_{\mathbb{T}}$, *the mapping* $(x_1, x_2) \to f(t, x_1, x_2)$ *is continuous on* $\mathbb{R}^2$;
(iii) *for each* $r > 0$, *there exists* $\alpha_r \in L^{\Delta}([0, T]_{\mathbb{T}}, \mathbb{R})$ *such that for a.e.* $t \in [0, T]_{\mathbb{T}}$ *and every* x_1 *such that* $|x_1| \leq r$, $|f(t, x_1, x_2)| \leq \alpha_r$.

Let the Banach space $X = C^{\Delta}[0, T]_{\mathbb{T}}$ with the norm $||x|| = \max\{||x||_\infty, ||x^{\Delta}||_\infty\}$, where $||x||_\infty = \sup_{t \in [0,T]_{\mathbb{T}}} |x(t)|$. Let

$$L^{\Delta}_{1\mathrm{oc}}[0, T]_{\mathbb{T}} = \{x : x|_{[s,t]_{\mathbb{T}}} \in L^{\Delta}[0, T]_{\mathbb{T}} \text{ for each } [s, t]_{\mathbb{T}} \subset [0, T]_{\mathbb{T}}\},$$

set $Y = L^{\Delta}_{1oc}[0, T]_{\mathbb{T}}$ with the norm $||x||_L = \int_0^T |x(t)|\Delta t$. We use the space $W^{2,1}[0, T]_{\mathbb{T}}$ defined by

$$\{x : [0, T]_{\mathbb{T}} \to \mathbb{R} | x(t), x^{\Delta}(t) \text{ is absolutely continuous on } [0, T]_{\mathbb{T}} \text{with } x^{\Delta\Delta} \in L^{\Delta}_{1oc}[0, T]_{\mathbb{T}}\}.$$

Define the linear operator L and the nonlinear operator N by

$$\begin{aligned} &L : X \cap \mathrm{Dom}\, L \to Y, \quad Lx(t) = x^{\Delta\Delta}(t), \quad \text{for} \quad x \in X \cap \mathrm{Dom}\, L, \\ &N : X \to Y, \quad Nx(t) = f(t, x(t), x^{\Delta}(t)) + e(t), \ \text{for } x \in X, \end{aligned}$$

respectively, where

$$\mathrm{Dom}\, L = \left\{ x \in W^{2,1}[0, T]_{\mathbb{T}}, x^{\Delta}(0) = 0, x(T) = \int_0^T x^{\sigma}(s)\Delta g(s) \right\}.$$

Lemma 3.1. *L : Dom $L \subset X \to Y$ is a Fredholm mapping of index zero. Furthermore, the continuous linear project operator $Q : Y \to Y$ can be defined by*

$$Qy = \frac{1}{\Lambda} \int_0^T \int_{\sigma(s)}^T \int_0^t y(\tau)\Delta\tau\Delta t\Delta g(s), \quad \text{for } y \in Y,$$

where $\Lambda = \int_0^T \int_{\sigma(s)}^T \int_0^t \Delta\tau\Delta t\Delta g(s) \neq 0$. *Linear mapping K_P can be written by*

$$K_P y(t) = \int_0^t (t - \sigma(s)) y(s)\Delta s, \quad \text{for } y \in \mathrm{Im}\, L.$$

Proof. It is clear that Ker $L = \{x(t) \equiv c, c \in \mathbb{R}\} = \mathbb{R}$, i.e., dimKer $L = 1$. Moreover, we have

$$\mathrm{Im}\, L = \left\{ y \in Y, \int_0^T \int_{\sigma(s)}^T \int_0^t y(\tau)\Delta\tau\Delta t\Delta g(s) = 0 \right\}. \tag{3.1}$$

If $y \in$ Im L, then there exists $x \in$ Dom L such that $x^{\Delta\Delta}(t) = y(t)$. Integrating it from 0 to t, we have

$$x^{\Delta}(t) = \int_0^t y(\tau)\Delta\tau.$$

Integrating the above equation from s to T, we get

$$x(s) = x(T) - \int_s^T \int_0^t y(\tau)\Delta\tau\Delta t. \tag{3.2}$$

Substituting the boundary condition $x(T) = \int_0^T x^\sigma(s)\Delta g(s)$ into (3.2), and by the condition $\int_0^T \Delta g(s) = 1$, we have

$$\int_0^T \int_{\sigma(s)}^T \int_0^t y(\tau)\Delta\tau\Delta t\Delta g(s) = 0. \tag{3.3}$$

On the other hand, $y \in Y$ satisfies (3.3), we take $x \in \text{Dom } L \subset X$ as given by (3.2), then $x^{\Delta\Delta}(t) = y(t)$ and

$$x^\Delta(0) = 0, \quad x(T) = \int_0^T x^\sigma(s)\Delta g(s).$$

Therefore, (3.1) holds.

Set $\Lambda = \int_0^T \int_{\sigma(s)}^T \int_0^t \Delta\tau\Delta t\Delta g(s)$. It is easy to show that $\Lambda \neq 0$, and then we define the mapping $Q : Y \to Y$ by

$$Qy = \frac{1}{\Lambda}\int_0^T \int_{\sigma(s)}^T \int_0^t y(\tau)\Delta\tau\Delta t\Delta g(s), \quad \text{for} \quad y \in Y,$$

and it is easy to see that $Q : Y \to Y$ is a linear continuous projector.

For the mapping L and continuous linear projector Q, it is not difficult to check that Im L = Ker Q. Set $y = (y - Qy) + Qy$; thus, $y - Qy \in$ Ker Q = Im L and $Qz \in$ Im Q, so Y = ImL + Im Q. If $y \in$ Im $L \cap$ Q, then $y(t) = 0$, hence Y = Im $L \oplus$ Im Q. From Ker L = $\mathbb{R}$, we obtain that Ind L = dim Ker L - codim Im L = dim Ker L - dim Im Q = 0, that is, L is a Fredholm mapping of index zero.

Take $P : X \to X$ as follows

$$Px(t) = x(0), \quad \text{for } x \in X.$$

Obviously, Im P = Ker L and X = Ker $L \oplus$ Ker P. Then, the inverse K_P : Im $L \to$ Dom $L \cap$ Ker P is defined by

$$K_P y(t) = \int_0^t (t - \sigma(s))y(s)\Delta s.$$

For $y \in$ Im L, we have

$$(LK_P)y(t) = \left(\int_0^t (t - \sigma(s))y(s)\Delta s\right)^{\Delta\Delta},$$

from Lemma 2.1, we obtain

$$\left(\int_0^t (t-\sigma(s))y(s)\Delta s\right)^{\Delta\Delta} = (\int_0^t y(s)\Delta s)^{\Delta} = y(t),$$

that is

$$(LK_P)y(t) = \left(\int_0^t (t-\sigma(s))y(s)\Delta s\right)^{\Delta\Delta} = y(t). \tag{3.4}$$

On the other hand, for $x \in$ Dom $L \cap$ Ker P,

$$(K_P L)x(t) = \int_0^t (t-\sigma(s))x^{\Delta\Delta}(s)\Delta s,$$

using Lemma 2.4 and the boundary conditions, we get

$$\int_0^t (t-\sigma(s))x^{\Delta\Delta}(s)\Delta s = (t-\sigma(s))x^{\Delta\Delta}(s)\Big|_0^t + \int_0^t x^{\Delta}(s)\Delta s = x(t),$$

i.e.,

$$(K_P L)x(t) = \int_0^t (t-\sigma(s))x^{\Delta\Delta}(s)\Delta s = x(t), \quad t \in [0,T]_{\mathbb{T}}. \tag{3.5}$$

(3.4) and (3.5) yield $K_P = (L|_{\text{Dom } L\cap\text{Ker } P})^{-1}$. The proof is completed.
Furthermore,

$$QNx = \frac{1}{\Lambda}\int_0^T \int_{\sigma(s)}^T \int_0^t (Nx)(\tau)\Delta\tau\Delta t\Delta g(s),$$

$$\begin{aligned}(K_{P,Q}N)x(t) &= \int_0^t (t-\sigma(s))(Nx)(s)\Delta s \\ &\quad - \int_0^t (t-\sigma(s))\left[\frac{1}{\Lambda}\int_0^T \int_{\sigma(s')}^T \int_0^{t'} (Nx)(\tau)\Delta\tau\Delta t'\Delta g(s')\right]\Delta s.\end{aligned}$$

Lemma 3.2. *Let* $f : [0,T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$*satisfy the Carathéodory conditions, then the mapping N is L-completely continuous.*

Proof. Assume that x_n, $x_0 \in E \subset X$ satisfy $||x_n - x_0|| \to 0$, $(n \to \infty)$; thus, there exists $M > 0$ such that $||x_n|| \le M$ for any $n \ge 1$. One has that

$$||Nx_n - Nx_0||_\infty = \sup_{t\in[0,T]_{\mathbb{T}}} |Nx_n - Nx_0| = \sup_{t\in[0,T]_{\mathbb{T}}} |f(t,x_n(t),x_n^{\Delta}(t)) - f(t,x_0(t),x_0^{\Delta}(t))|.$$

In view of f satisfying the Carathéodory conditions, we can obtain that for a.e. $t \in [0,T]_{\mathbb{T}}$, $||Nx_n - Nx_0||_\infty \to 0$, $(n \to \infty)$. This means that the operator $N : E \to Y$ is continuous. By the definitions of QN and $K_{P,Q}N$, we can obtain that $QN : E \to Y$ and $K_{P,Q} : E \to X$ are continuous.

Let $r = \sup\{||x|| : x \in E\} < \infty$ for a.e. $t \in [0,T]_{\mathbb{T}}$, we have

$$|Nx_n| \leq |f(t, x_n(t), x_n^{\Delta}(t))| + |e(t)| \leq |(\alpha_r(t)| + |e(t)| := \psi(t),$$

$$|QNx_n| \leq \frac{1}{|\Lambda|}\int_0^T \int_{\sigma(s)}^T \int_0^t |(Nx_n)(\tau)|\Delta\tau\Delta t\Delta g(s)$$

$$\leq \frac{1}{|\Lambda|}\int_0^T \int_{\sigma(s)}^T \int_0^t |\psi(\tau)|\Delta\tau\Delta t\Delta g(s),$$

$$|(K_{P,Q}N)x_n(t)| \leq \int_0^t (t-\sigma(s))|(Nx_n)(s)|\Delta s - \int_0^t (t-\sigma(s))|QNx_n|\Delta s.$$

Since functions $\alpha_r(t), e(t) \in L^{\Delta}_{\text{loc}}[0,T]_{\mathbb{T}}$, we get that $\psi(t) \in L^{\Delta}_{\text{loc}}[0,T]_{\mathbb{T}}$. Further

$$||Nx_n||_L \leq \int_0^T |\psi(t)|\Delta t := \chi < \infty.$$

It follows that $(QN)(E)$ and $(K_{P,Q}N)(E)$ are bounded.

It is easy to see that $\{QNx_n\}_{n=1}^{\infty}$ is equicontinuous on a.e. $t \in [0,T]_{\mathbb{T}}$. So, we only show that $\{(K_{P,Q}N)x_n\}_{n=1}^{\infty}$ is equicontinuous on a.e. $t \in [0,T]_{\mathbb{T}}$. For any $t_1, t_2 \in [0,T]_{\mathbb{T}}$ with $t_1 < \rho(t_2)$,

$$\begin{aligned}
&|(K_{P,Q}N)x_n(t_1) - (K_{P,Q}N)x_n(t_2)| \\
&\leq \int_{t_1}^{t_2} |(K_{P,Q}Nx_n)^{\Delta}(s)|\Delta s \\
&\leq \int_{t_1}^{t_2}\int_0^s |(Nx_n)(\tau) - (QNx_n(\tau))|\Delta\tau\Delta s \\
&\leq \int_{t_1}^{t_2}\int_0^s |Nx_n(\tau)|\Delta\tau\Delta s + \int_{t_1}^{t_2}\int_0^s |QNx_n(\tau)|\Delta\tau\Delta s.
\end{aligned} \tag{3.6}$$

Since $|Nx_n| \leq \psi$ with $\psi \in L^{\Delta}_{1\text{oc}}([0,\ T]_{\mathbb{T}})$, (3.6) shows that $\{(K_{P,Q}N)x_n\}_{n=1}^{\infty}$ is equicontinuous on a.e. $t \in [0,T]_{\mathbb{T}}$. Hence, by the Arzelà-Ascoli theorem on time scales, $\{QNx_n\}_{n=1}^{\infty}$ and $\{K_{P,Q}Nx_n\}_{n=1}^{\infty}$ are compact on an arbitrary bounded $E \subset X$, and the mapping $N : X \to Y$ is L-completely continuous. The proof is completed.

Now, we are ready to apply the coincidence degree theorem of Mawhin to give the sufficient conditions for the existence of at least one solution to problem (1.1).

Theorem 3.2. *Let* $f : [0,\ T]_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}$*satisfy the Carathéodory conditions, and*

(H_1) *There exist continuous functions* $r : [0,\ T]_{\mathbb{T}} \to \mathbb{R}^+,\ g_i : [0,\ T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}^+,\ i = 1,$ 2, *such that*

$$|f(t,\ x_1,\ x_2)| \leq g_1(t,\ x_1) + g_2(t,\ x_2) + r(t)$$

and

$$\lim_{|x|\to+\infty} \sup_{t\in[0,T]_{\mathbb{T}}} \frac{g_i(t,x)}{|x|} = r_i \in [0,+\infty), i = 1, 2.$$

(H_2) *There is a constant* $M > 0$ *such that for any* $x \lfloor$ Dom $L\backslash$Ker L, *if* $|x(t)| > M$ *for all* $t \in [0,T]_{\mathbb{T}}$; *then,* $\frac{1}{\Lambda}\int_0^T\int_{\sigma(s)}^T\int_0^t [f(\tau,x(\tau),x^{\Delta}(\tau)+e(\tau)]\Delta\tau\Delta t\Delta g(s) \neq 0.$

(H_3) *There is a constant* $M^* > 0$ *such that for any* $c \in \mathbb{R}$, *if* $|c| > M^*$; *then, we have either*

$$\frac{c}{\Lambda}\int_0^T\int_{\sigma(s)}^T\int_0^t [f(\tau,\ c,\ 0)+e(\tau)]\ \Delta\tau\Delta t\Delta g(s) > 0 \tag{3.7}$$

or

$$\frac{c}{\Lambda}\int_0^T\int_{\sigma(s)}^T\int_0^t [f(\tau,\ c,\ 0)+e(\tau)]\ \Delta\tau\Delta t\Delta g(s) < 0. \tag{3.8}$$

Then, problem (1.1) *admits at least one solution provided that*

$$T^2 r_1 + T r_2 < 1.$$

Proof. Let $\Omega_1 = \{x \in$ Dom $L\backslash$Ker $L : Lx = \lambda Nx$ for some $\lambda \in (0, 1)\}$. For $x \in \Omega_1$, we have $x \notin$ Ker L and $Nx \in$ Im L = Ker Q; thus, $QNx = 0$, i.e.,

$$QNx = \frac{1}{\Lambda}\int_0^T\int_{\sigma(s)}^T\int_0^t [f(\tau,\ x(\tau),\ x^{\Delta}(\tau))+e(\tau)]\ \Delta\tau\Delta t\Delta g(s) = 0.$$

Hence by (H_2), we know that there exists $t_0 \in [0,\ T]_{\mathbb{T}}$ such that $|x(t_0)| < M$. Since

$$x(t) = x(t_0) + \int_{t_0}^t x^{\Delta}(s)\Delta s.$$

So we get

$$||x||_\infty \le M + \int_0^T |x^{\Delta}(s)|\Delta s \le M + T||x^{\Delta}||_\infty.$$

Integrating the equation

$$x^{\Delta\Delta}(t) = \lambda[f(t,\ x(t),\ x^{\Delta})(t)+e(t)],\quad t \in [0,\ T]_{\mathbb{T}}$$

from 0 to t, we obtain

$$\begin{aligned}|x^{\Delta}(t)| &= \left|\int_0^t f(s,\ x(s),\ x^{\Delta}(s)) + e(s)\Delta s\right| \\ &\le \int_0^T |f(s,\ x(s),\ x^{\Delta}(s)) + e(s)|\Delta s \\ &\le \int_0^T |e(s)|\Delta s + \int_0^T |r(s)|\Delta s + \int_0^T |g_1(s,\ x(s))|\Delta s + \int_0^T |g_2(s,\ x^{\Delta}(s))|\Delta s.\end{aligned}$$

Let $\varepsilon > 0$ satisfy

$$T[T(r_1 + \varepsilon) + (r_2 + \varepsilon)] < 1.$$

For such ε, there is $\delta > 0$ so that for $i = 1, 2$,

$$|g_i(t,\ x)| < (r_i + \varepsilon)|x|, \text{ uniformly for } t \in [0,\ T]_{\mathbb{T}} \text{ and } |x| > \delta.$$

Let

$$\begin{aligned}&\Gamma_{1,0} = \{t : t \in [0,\ T]_{\mathbb{T}},\ |x(t)| \le \delta\}, \quad \Gamma_{1,1} = \{t : t \in [0,\ T]_{\mathbb{T}},\ |x^{\Delta}(t)| \le \delta\}, \\ &\Gamma_{2,0} = \{t : t \in [0,\ T]_{\mathbb{T}},\ |x(t)| > \delta\}, \quad \Gamma_{2,1} = \{t : t \in [0,\ T]_{\mathbb{T}},\ |x^{\Delta}(t)| > \delta\}, \\ &\bar{g}_i = \max_{t \in [0,T]_{\mathbb{T}}, |x| < \delta} |g_i(t,\ x)|,\ i = 1, 2.\end{aligned}$$

We get

$$\begin{aligned}|x^{\Delta}(t)| &\le \int_0^T |e(s)|\Delta s + \int_0^T |r(s)|\Delta s + \int_0^T |g_1(s, x(s))|\Delta s + \int_0^T |g_2(s, x^{\Delta}(s))|\Delta s \\ &\le \int_0^T |e(s)|\Delta s + \int_0^T |r(s)|\Delta s + \int_{\Gamma_{1,0}} |g_1(s, x(s))|\Delta s + \int_{\Gamma_{2,0}} |g_1(s, x(s))|\Delta s \\ &+ \int_{\Gamma_{1,1}} |g_2(s, x^{\Delta}(s))|\Delta s + \int_{\Gamma_{2,1}} |g_2(s, x^{\Delta}(s))|\Delta s \\ &\le \int_0^T |e(s)|\Delta s + \int_0^T |r(s)|\Delta s + T[(r_1 + \varepsilon)||x||_{\infty} + \bar{g}_1 + (r_2 + \varepsilon)||x^{\Delta}||_{\infty} + \bar{g}_2] \\ &\le \int_0^T |e(s)|\Delta s + \int_0^T |r(s)|\Delta s + T[(r_1 + \varepsilon)M + \bar{g}_1 + \bar{g}_2] \\ &+ T[T(r_1 + \varepsilon) + (r_2 + \varepsilon)]||x^{\Delta}||_{\infty}.\end{aligned}$$

So we get

$$\{1 - T[T(r_1 + \varepsilon) + (r_2 + \varepsilon)]\}||x^{\Delta}||_{\infty} \le \int_0^T |e(s)|\Delta s + \int_0^T |r(s)|\Delta s + T[(r_1 + \varepsilon)M + \bar{g}_1 + \bar{g}_2].$$

It follows from the definition of ε that there is a constant $A > 0$ such that

$$||x^{\Delta}||_{\infty} \leq A := \frac{\int_0^T |e(s)|\Delta s + \int_0^T |r(s)|\Delta s + T[(r_1 + \varepsilon)M + \bar{g}_1 + \bar{g}_2]}{1 - T[T(r_1 + \varepsilon) + (r_2 + \varepsilon)]}.$$

Hence, we have

$$||x|| = \max\{||x||_{\infty}, \ ||x^{\Delta}||_{\infty}\} \leq \max\{M + TA, \ A\},$$

which means that Ω_1 is bounded.

Let $\Omega_2 = \{x \in \text{Ker } L : Nx \in \text{Im } L\}$. For $x \in \Omega_2$, then $x(t) = c$ for some $c \in \mathbb{R}$. $Nx \in \text{Im } L$ implies $QNx = 0$, that is

$$\frac{1}{\Lambda} \int_0^T \int_{\sigma(s)}^T \int_0^t [f(\tau, \ c, \ 0) + e(\tau)] \ \Delta\tau \Delta t \Delta g(s) = 0.$$

From (H_3), we know that $||x|| = |c| \leq M^*$, thus Ω_2 is bounded.

If (3.7) holds, then let

$$\Omega_3 = \{x \in \text{Ker } L : -\lambda Jx + (1 - \lambda)QNx = 0, \ \lambda \in [0, 1]\},$$

where $J : \text{Ker } L \to \text{Im } Q$ is a linear isomorphism given by $J(k) = k$ for any $k \in \mathbb{R}$. Since $x(t) = k$ thus

$$\lambda k = (1 - \lambda)QNk = \frac{1 - \lambda}{\Lambda} \int_0^T \int_{\sigma(s)}^T \int_0^t [f(\tau, \ k, \ 0) + e(\tau)] \ \Delta\tau \Delta t \Delta g(s).$$

If $\lambda = 1$, then $k = 0$, and in the case $\lambda \in [0, 1)$, if $|k| > M^*$, we have

$$\lambda k^2 = \frac{k(1 - \lambda)}{\Lambda} \int_0^T \int_{\sigma(s)}^T \int_0^t [f(\tau, \ k, \ 0) + e(\tau)] \ \Delta\tau \Delta t \Delta g(s) < 0,$$

which is a contradiction. Again, if (3.8) holds, then let

$$\Omega_3 = \{x \in \text{Ker } L : -\lambda Jx + (1 - \lambda)QNx = 0, \ \lambda \in [0, 1]\},$$

where J as in above, similar to the above argument. Thus, in either case, $||x|| = |k| \leq M^*$ for any $x \in \Omega_3$, that is, Ω_3 is bounded.

Let Ω be a bounded open subset of X such that $\cup_{i=1}^3 \Omega_i \subset \Omega$. By Lemma 3.2, we can check that $K_P(I - Q)N : \bar{\Omega} \to X$ is compact; thus, N is L-compact on $\overline{\Omega}$.

Finally, we verify that the condition (iii) of Theorem 3.1 is fulfilled. Define a homotopy

$$H(x, \ \lambda) = \pm\lambda Jx + (1 - \lambda)QNx.$$

According to the above argument, we have

$$H(x, \ \lambda) \neq 0, \text{for } x \in \partial\Omega \cap \text{Ker } L;$$

thus, by the degree property of homotopy invariance, we obtain

$$\begin{aligned}\deg(QN_{\mathrm{Ker}\,L}, \Omega \cap \mathrm{Ker}\,L, 0) &= \deg(H(\cdot, 0), \Omega \cap \mathrm{Ker}\,L, 0)\\ &= \deg(H(\cdot, 1), \Omega \cap \mathrm{Ker}\,L, 0)\\ &= \deg(\pm J, \Omega \cap \mathrm{Ker}\,L, 0) \neq 0.\end{aligned}$$

Thus, the conditions of Theorem 2.4 are satisfied, that is, the operator equation $Lx = Nx$ admits at least one solution in Dom $L \cap \overline{\Omega}$. Therefore, BVPs (1.1) has at least one solution in $C^{\Delta}[0, T]_{\mathbb{T}}$.

4 An example

In this section, we present an easy example to illustrate our main results.

Example 4.1. Let $\mathbb{T} = \{0\} \bigcup \{\frac{1}{2^{n+1}}\} \bigcup [\frac{1}{2}, 1]$, $n = 1, 2, ..., \infty$. Consider the boundary value

Problem

$$\begin{cases} x^{\Delta\Delta}(t) = \frac{1}{2}tx^2(t) + \frac{1}{3}t^2x^{\Delta}(t) + \sqrt{t}, \text{ a.e. } t \in \mathbb{T}, \\ x^{\Delta}(0) = 0, \qquad x(1) = x(\frac{1}{2}). \end{cases} \tag{4.1}$$

Let

$$g(t) = \begin{cases} 0, \text{ for } 0 \le t \le \frac{1}{2}, \\ 1, \text{ for } \frac{1}{2} \le t \le 1, \end{cases}$$

then $x(1) = \int_0^1 x^{\sigma}(s)\Delta g(s)$. Let

$$g_1(t, x) = \frac{1}{2}x^2(t), \quad g_2(t, x^{\Delta}) = \frac{1}{3}(x^{\Delta}(t))^2, \quad r(t) = \frac{t}{2}.$$

We can get that $r_1 + r_2 = \frac{5}{6} < 1$. It is easy to check other conditions of Theorem 3.1 are satisfied. Hence, boundary value problem (4.1) has at least one solution.

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Authors' contributions

All authors contributed equally to the manuscript and typed, read and approved the final manuscript.

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The authors declare that they have no competing interests.

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