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Some identities involving generalized Gegenbauer polynomials

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Abstract

In this paper, we investigate some interesting identities on the Bernoulli, Euler, Hermite and generalized Gegenbauer polynomials arising from the orthogonality of generalized Gegenbauer polynomials in the generalized inner product

$$\langle p_1(x), p_2(x)\rangle = \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} (\alpha q - p^2x^2)^{\lambda-\frac{1}{2}} p_1(x)p_2(x)\,dx.$$

MSC: 11B83; 11M06

Keywords: generalized Gegenbauer polynomials; orthogonality; generalized inner product space

1 Introduction

In recent years, the orthogonal polynomial has had a very important position in the areas of functional neural network and its properties, operators and identification of distributed parameter systems, intelligent instrument automatic calibration, operator and control theory, and so on. And the Gegenbauer polynomial is very important of orthogonal polynomials. It seems to be interesting and important in the area of mathematical physics. Recently, many authors have studied Gegenbauer polynomials related to mathematical physics (see [1–6]).

The information entropy of Gegenbauer polynomials is relevant since it is related to the angular part of the information entropies of certain quantum mechanical systems such as the harmonic oscillator and the hydrogen atom in D dimensions (see [1, 7, 8]). In this paper we will promote the results in [9, 10] to the generalized Gegenbauer polynomials. It will help solve the above general problems.

The generalized Gegenbauer polynomials are given in terms of generating function by

$$\frac{1}{(\alpha - 2pxt + qt^2)^\lambda} = \sum_{n=0}^{\infty} C^{\lambda}_{\alpha,p,q,n}(x)t^n. \tag{1}$$

By Newton's binomial theorem, we get

$$\frac{1}{(\alpha-2pxt+qt^2)^{\lambda}} = \left(1-\sqrt{\frac{q}{\alpha}}t\right)^{-2\lambda}\left[\alpha+2t\left(1-\sqrt{\frac{q}{\alpha}}t\right)^{-2}(\sqrt{\alpha q}-px)\right]^{-\lambda}$$
$$= \sum_{k=0}^{\infty}\frac{(-2)^k\Gamma(k+\lambda)(\sqrt{\alpha q}-px)^k t^k(1-\sqrt{\frac{q}{\alpha}}t)^{-2k-2\lambda}}{k!\Gamma(\lambda)}\alpha^{-\lambda-k},$$
$$t^k\left(1-\sqrt{\frac{q}{\alpha}}t\right)^{-2k-2\lambda} = \sum_{m=0}^{\infty}\frac{\Gamma(m+2k+2\lambda)}{m!\Gamma(2k+2\lambda)}t^{m+k}\left(\sqrt{\frac{q}{\alpha}}\right)^m.$$

Then

$$\frac{1}{(\alpha-2pxt+qt^2)^{\lambda}} = \sum_{k=0}^{\infty}\frac{(-2)^k\Gamma(k+\lambda)(\sqrt{\alpha q}-px)^k}{k!\Gamma(\lambda)}\sum_{m=0}^{\infty}\frac{\Gamma(m+2k+2\lambda)}{m!\Gamma(2k+2\lambda)}t^{m+k}\alpha^{-\lambda-k-\frac{m}{2}}q^{\frac{m}{2}}.$$

When $|t|>0$ is small enough, the double series is absolutely convergent, and the coefficient of t^k is known by the diagonal summation method

$$C^{\lambda}_{\alpha,p,q,n}(x) = \sum_{k=0}^{n}\frac{(-2)^k\Gamma(k+\lambda)\Gamma(n+k+2\lambda)(\sqrt{\alpha q}-px)^k}{k!(n-k)!\Gamma(\lambda)\Gamma(2k+2\lambda)}\alpha^{-\lambda-\frac{k}{2}-\frac{n}{2}}q^{\frac{n}{2}-\frac{k}{2}}.$$

Thus, there are

$$C^{\lambda}_{\alpha,p,q,n}(x) = \binom{n+2\lambda-1}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}(2\lambda+n)_k}{(\lambda+\frac{1}{2})_k}\left(\frac{px-\sqrt{\alpha q}}{2}\right)^k\alpha^{-\lambda-\frac{k}{2}-\frac{n}{2}}q^{\frac{n}{2}-\frac{k}{2}},$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$.

From [11], we have the recursion formula about the generalized Gegenbauer polynomials

$$C^{\lambda}_{\alpha,p,q,0}(x) = \alpha^{-\lambda}, \qquad C^{\lambda}_{\alpha,p,q,1}(x) = 2\lambda px\alpha^{-\lambda-1},$$
$$\frac{n+1}{2\lambda}C^{\lambda}_{\alpha,p,q,n+1}(x) = pxC^{\lambda+1}_{\alpha,p,q,n}(x) - qC^{\lambda+1}_{\alpha,p,q,n-1}(x).$$

By [11], we get

$$C^{\lambda}_{\alpha,p,q,n}(x) = \frac{1}{\Gamma(\lambda)}\sum_{l=0}^{[\frac{n}{2}]}\frac{(-1)^l\Gamma(\lambda+n-l)}{l!(n-2l)!}(2xp)^{n-2l}q^l\alpha^{-\lambda-n+l}.$$

Meanwhile, by the recursion formula about generalized Gegenbauer polynomials and mathematical induction, we get

$$C^{\lambda}_{\alpha,p,q,n}(x) = \frac{\binom{n+2\lambda-1}{n}}{\binom{n+\lambda-\frac{1}{2}}{n}}\sum_{k=0}^{n}\frac{\binom{n+\lambda-\frac{1}{2}}{n-k}\binom{n+\lambda-\frac{1}{2}}{k}(\frac{px-\sqrt{\alpha q}}{2})^k}{(\frac{px+\sqrt{\alpha q}}{2})^{k-n}}\alpha^{-\lambda-n}. \tag{2}$$

It is not difficult to show that $C^{\lambda}_{\alpha,p,q,n}(x)$ is a solution of the following Gegenbauer differential equation:

$$\frac{1}{p}\left(\alpha q - p^2x^2\right)y^{(2)} - p(2\lambda + 1)xy' + pn(n + 2\lambda)y = 0.$$

From the above equation and mathematical induction, we acquire Rodrigues' formula for the generalized Gegenbauer polynomials

$$\left(\alpha q - p^2x^2\right)^{\lambda-\frac{1}{2}} C^{\lambda}_{\alpha,p,q,n}(x) = \frac{1}{\alpha^{\lambda+n}p^n}\frac{(-2)^n(\lambda)_n}{n!(n+2\lambda)_n}\left(\frac{d}{dx}\right)^n\left(\alpha q - p^2x^2\right)^{n+\lambda-\frac{1}{2}}. \tag{3}$$

Applying (1) and (2), we get

$$\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}}\left(\alpha q - p^2x^2\right)^{\lambda-\frac{1}{2}} C^{\lambda}_{\alpha,p,q,n}(x)C^{\lambda}_{\alpha,p,q,m}(x)\,dx = \frac{q^{n+\lambda}}{p\alpha^{n+\lambda}}\frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)(\Gamma(\lambda))^2}\delta_{m,n}, \tag{4}$$

where $\delta_{m,n}$ is the Kronecker symbol and it holds for each fixed $\lambda \in \mathbf{R}$ with $\lambda > -\frac{1}{2}$ and $\lambda \neq 0$.

Equation (5) implies the orthogonality of $C^{\lambda}_{\alpha,p,q,n}(x)$, and equation (5) is important in deriving our results in this paper. From (1), we can derive the following derivative of generalized Gegenbauer polynomials $C^{\lambda}_{\alpha,p,q,n}(x)$: for $k \geq 1$,

$$\left(\frac{d}{dx}\right)^k C^{\lambda}_{p,q,n}(x) = 2^k(\lambda)_k p^k C^{\lambda+k}_{\alpha,p,q,n-k}(x). \tag{5}$$

The so-called Euler polynomials $E_n(x)$ are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{6}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0, E_0 = E_n$ are called the nth Euler number.

The Bernoulli polynomials are also defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{7}$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$. In the special case, $x = 0, B_0 = B_n$ are called the nth Bernoulli number. From the above equation, we note that

$$\begin{aligned} B_n(x) &= \sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k, \\ E_n(x) &= \sum_{k=0}^{n}\binom{n}{k}E_{n-k}x^k. \end{aligned} \tag{8}$$

For $n \in \mathbf{Z}_+$, we have

$$\frac{dB_n(x)}{dx} = nB_{n-1}(x),$$
$$\frac{dE_n(x)}{dx} = nE_{n-1}(x). \tag{9}$$

By the definition of Bernoulli and Euler polynomials, we get

$$B_0 = 1, \qquad B_n(1) - B_n = \delta_{1,n},$$
$$E_0 = 1, \qquad E_n(1) + E_n = 2\delta_{0,n}.$$

The Hermite polynomials are defined by the generating function to be

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

For $n \in \mathbf{Z}_+$, $k \in \mathbf{Z}_+$, we have

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x), \qquad \left(\frac{d}{dx}\right)^k H_n(x) = 2^k \frac{n!}{(n-k)!} H_{n-k}(x), \tag{10}$$

$$H_n(x) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(2x)^k, \tag{11}$$

where H_n is the nth Hermite number.

For each fixed $\lambda \in \mathbf{R}$ with $\lambda > -\frac{1}{2}$ and $\lambda \neq 0$, let $\mathbf{P}_n = \{p(x) \in \mathbf{R}[x] \mid \deg p(x) \leq n\}$ be an inner product space with respect to the generalized inner product

$$\langle p_1(x), p_2(x)\rangle = \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\alpha q - p^2x^2\right)^{\lambda-\frac{1}{2}} p_1(x)p_2(x)\,dx. \tag{12}$$

In this paper, we derive some interesting identities involving Gegenbauer polynomials arising from the orthogonality of those for the generalized inner product space $\mathbf{P}_n$ with respect to the weighted inner product. Our methods used in this paper are useful in finding some new identities and relations on the Bernoulli, Euler and Hermite polynomials involving generalized Gegenbauer polynomials.

2 Some identities involving generalized Gegenbauer polynomials

Lemma 1 *For $p(x) \in \mathbf{P}_n$, let*

$$p(x) = \sum_{k=0}^{n} d_k C^{\lambda}_{\alpha,p,q,k}(x) \quad (d_k \in \mathbf{R}). \tag{13}$$

Then

$$d_k = \frac{p^{1-k}}{q^{k+\lambda}} \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})} \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\frac{d^k}{dx^k}\left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}}\right) p(x)\,dx. \tag{14}$$

Proof Let us take $p(x) = \sum_{k=0}^{n} d_k C_{\alpha,p,q,k}^{\lambda}(x) \in \mathbf{P}_n$, $d_k \in \mathbf{R}$. Then by (4) and (12), we get

$$\begin{aligned}
\langle p(x), C_{\alpha,p,q,k}^{\lambda}(x)\rangle &= d_k \langle C_{\alpha,p,q,k}^{\lambda}(x), C_{\alpha,p,q,k}^{\lambda}(x)\rangle \\
&= d_k \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\alpha q - p^2x^2\right)^{\lambda-\frac{1}{2}} C_{\alpha,p,q,k}^{\lambda}(x) C_{\alpha,p,q,k}^{\lambda}(x)\, dx \\
&= d_k \frac{q^{k+\lambda}}{p\alpha^{\lambda+k}} \frac{\pi 2^{1-2\lambda}\Gamma(k+2\lambda)}{k!(k+\lambda)(\Gamma(\lambda))^2}.
\end{aligned}$$

Thus, by the above equation, we get

$$d_k = \frac{p\alpha^{\lambda+k}}{q^{k+\lambda}} \frac{k!(k+\lambda)(\Gamma(\lambda))^2}{\pi 2^{1-2\lambda}\Gamma(k+2\lambda)} \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\alpha q - p^2x^2\right)^{\lambda-\frac{1}{2}} p(x) C_{\alpha,p,q,k}^{\lambda}(x)\, dx.$$

From the above equation and (3), we have

$$\begin{aligned}
d_k &= \frac{p\alpha^{\lambda+k}}{q^{k+\lambda}} \frac{k!(k+\lambda)(\Gamma(\lambda))^2}{\pi 2^{1-2\lambda}\Gamma(k+2\lambda)} \frac{\alpha^{-\lambda-k}}{p^k} \frac{(-2)^k(\lambda)_k}{k!(k+2\lambda)_k} \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\frac{d^k}{dx^k}\left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}}\right) p(x)\, dx \\
&= \frac{p^{1-k}}{q^{k+\lambda}} \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})} \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\frac{d^k}{dx^k}\left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}}\right) p(x)\, dx.
\end{aligned}$$

This proves Lemma 1. □

Theorem 1 *For $n \in \mathbf{Z}_+$, we have*

$$x^n = \sum_{0\le k\le n, n-k\equiv 0 (\bmod 2)} \frac{\sqrt{\alpha}^{n+k+2\lambda}\sqrt{q}^{n-k}}{p^n} \frac{(k+\lambda)n!\Gamma(\lambda)}{2^n(\frac{n-k}{2})!\Gamma(\frac{n+k+2\lambda+2}{2})} C_{\alpha,p,q,k}^{\lambda}(x).$$

Proof Let $p(x) = x^n \in \mathbf{P}_n$, from (14) we have

$$\begin{aligned}
d_k &= \frac{p^{1-k}}{q^{k+\lambda}} \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})} \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\frac{d^k}{dx^k}\left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}}\right) x^n\, dx \\
&= \frac{p^{1-k}}{q^{k+\lambda}} \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})} \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} x^n\, d\left(\frac{d^{k-1}}{dx^{k-1}}\left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}}\right) \\
&= (-n)\frac{p^{1-k}}{q^{k+\lambda}} \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})} \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} x^{n-1}\left(\frac{d^{k-1}}{dx^{k-1}}\left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}}\right) dx \\
&= \cdots \\
&= \frac{p^{1-k}}{q^{k+\lambda}} \frac{(k+\lambda)\Gamma(\lambda)\frac{n!}{(n-k)!}}{(2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})} \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} x^{n-k}\left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}}\, dx \\
&= \left(1+(-1)^{n-k}\right)\frac{p^{1-k}}{q^{k+\lambda}} \frac{(k+\lambda)\Gamma(\lambda)\frac{n!}{(n-k)!}}{2^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})} \int_{0}^{\frac{\sqrt{\alpha q}}{p}} x^{n-k}\left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}}\, dx \\
&= \left(1+(-1)^{n-k}\right)\frac{\sqrt{\alpha}^{n+k+2\lambda}\sqrt{q}^{n-k}}{p^n} \frac{(k+\lambda)\Gamma(\lambda)\frac{n!}{(n-k)!}}{2^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})} \int_{0}^{1} y^{n-k}\left(1-y^2\right)^{k+\lambda-\frac{1}{2}}\, dy.
\end{aligned}$$

Let us assume that $n-k\equiv 0(\bmod 2)$ and $y=\sqrt{x}$. Then we get

$$\begin{aligned} d_k &= \frac{\sqrt{\alpha}^{n+k+2\lambda}\sqrt{q}^{n-k}}{p^n}\frac{(k+\lambda)\Gamma(\lambda)\frac{n!}{(n-k)!}}{2^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\int_0^1 x^{\frac{n-k-1}{2}}(1-x)^{k+\lambda-\frac{1}{2}}\,dx \\ &= \frac{\sqrt{\alpha}^{n+k+2\lambda}\sqrt{q}^{n-k}}{p^n}\frac{(k+\lambda)\Gamma(\lambda)\frac{n!}{(n-k)!}}{2^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}B\left(k+\lambda+\frac{1}{2},\frac{n-k+1}{2}\right) \\ &= \frac{\sqrt{\alpha}^{n+k+2\lambda}\sqrt{q}^{n-k}}{p^n}\frac{(k+\lambda)\Gamma(\lambda)\frac{n!}{(n-k)!}}{2^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\frac{\Gamma(\frac{n-k+1}{2})\Gamma(k+\lambda+\frac{1}{2})}{\Gamma(\frac{n+k+2\lambda+2}{2})}, \end{aligned}$$

where $B(\alpha,\beta)$ is the beta function which is defined by $B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

It is not difficult to show that

$$\Gamma\left(\frac{n-k+1}{2}\right)=\frac{(n-k)!\sqrt{\pi}}{2^{n-k}(\frac{n-k}{2})!}.$$

Therefore, from (13), we get

$$x^n=\sum_{0\le k\le n,n-k\equiv 0(\bmod 2)}\frac{\sqrt{\alpha}^{n+k+2\lambda}\sqrt{q}^{n-k}}{p^n}\frac{(k+\lambda)n!\Gamma(\lambda)}{2^n(\frac{n-k}{2})!\Gamma(\frac{n+k+2\lambda+2}{2})}C^{\lambda}_{\alpha,p,q,k}(x).$$

□

Theorem 2 *For $n\in\mathbf{Z}_+$, we have the identities*

$$\begin{aligned} \frac{B_n(x)}{n!} &= \Gamma(\lambda)\sum_{k=0}^{n}\left(\frac{(k+\lambda)}{2^k(n-k)!}\sum_{0\le l\le n-k,l\equiv 0(\bmod 2)}\frac{\frac{\sqrt{\alpha}^{2k+2\lambda+l}\sqrt{q}^l}{p^{l+k}}\binom{n-k}{l}B_{n-k-l}l!}{2^l(\frac{l}{2})!\Gamma(\frac{2k+2\lambda+l+2}{2})}\right) \\ &\quad\times C^{\lambda}_{\alpha,p,q,k}(x), \\ \frac{E_n(x)}{n!} &= \Gamma(\lambda)\sum_{k=0}^{n}\left(\frac{(k+\lambda)}{2^k(n-k)!}\sum_{0\le l\le n-k,l\equiv 0(\bmod 2)}\frac{\frac{\sqrt{\alpha}^{2k+2\lambda+l}\sqrt{q}^l}{p^{l+k}}\binom{n-k}{l}E_{n-k-l}l!}{2^l(\frac{l}{2})!\Gamma(\frac{2k+2\lambda+l+2}{2})}\right) \\ &\quad\times C^{\lambda}_{\alpha,p,q,k}(x). \end{aligned}$$

Proof Let us take $p(x)=B_n(x)\in\mathbf{P}_n$. Then, applying (9) and (14), we get

$$\begin{aligned} d_k &= \frac{p^{1-k}}{q^{k+\lambda}}\frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}}\left(\frac{d^k}{dx^k}(\alpha q-p^2x^2)^{k+\lambda-\frac{1}{2}}\right)B_n(x)\,dx \\ &= (-n)\frac{p^{1-k}}{q^{k+\lambda}}\frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})} \\ &\quad\times\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}}\left(\frac{d^{k-1}}{dx^{k-1}}(\alpha q-p^2x^2)^{k+\lambda-\frac{1}{2}}\right)B_{n-1}(x)\,dx \\ &= \cdots \\ &= \frac{p^{1-k}}{q^{k+\lambda}}\frac{(k+\lambda)\Gamma(\lambda)\frac{n!}{(n-k)!}}{(2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}}(\alpha q-p^2x^2)^{k+\lambda-\frac{1}{2}}B_{n-k}(x)\,dx. \end{aligned}$$

From (8), we have

$$\begin{aligned}
&\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}} B_{n-k}(x)\,dx \\
&= \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l} \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}} x^l\,dx \\
&= \sum_{0\le l\le n-k, l=0(\bmod 2)} \binom{n-k}{l} B_{n-k-l} \frac{(\alpha q)^{k+\lambda+\frac{l}{2}}}{p^{l+1}} \int_0^1 (1-x)^{k+\lambda-\frac{1}{2}} x^{\frac{l-1}{2}}\,dx \\
&= \sum_{0\le l\le n-k, l=0(\bmod 2)} \binom{n-k}{l} B_{n-k-l} \frac{(\alpha q)^{k+\lambda+\frac{l}{2}}}{p^{l+1}} \frac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(\frac{l+1}{2})}{\Gamma(\frac{2k+2\lambda+l+2}{2})}.
\end{aligned}$$

It is easy to show that

$$\Gamma\left(\frac{l+1}{2}\right) = \left(\frac{l-1}{2}\right)\left(\frac{l-3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{(\frac{l}{2})^l l!\Gamma(\frac{1}{2})}{(\frac{l}{2})!} = \frac{l!\sqrt{\pi}}{2^l(\frac{l}{2})!}.$$

So, we get

$$d_k = \frac{n!(k+\lambda)\Gamma(\lambda)}{2^k(n-k)!} \sum_{0\le l\le n-k, l=0(\bmod 2)} \frac{\sqrt{\alpha}^{2k+2\lambda+l}\sqrt{q}^l}{p^{l+k}} \frac{\binom{n-k}{l} B_{n-k-l} l!}{2^l(\frac{l}{2})!\Gamma(\frac{2k+2\lambda+l+2}{2})}.$$

By the same method, we have

$$d_k = \frac{n!(k+\lambda)\Gamma(\lambda)}{2^k(n-k)!} \sum_{0\le l\le n-k, l=0(\bmod 2)} \frac{\sqrt{\alpha}^{2k+2\lambda+l}\sqrt{q}^l}{p^{l+k}} \frac{\binom{n-k}{l} E_{n-k-l} l!}{2^l(\frac{l}{2})!\Gamma(\frac{2k+2\lambda+l+2}{2})}.$$

Now Theorem 2 follows from (13). □

Theorem 3 *For $n, k \in \mathbf{Z}_+$ with $n \ge k$, we have*

$$\begin{aligned}
C^{\lambda}_{\alpha,p,q,n-k}(x) C^{\lambda}_{\alpha,p,q,k}(x) &= 2^{\lambda+1} \binom{n-k+2\lambda-1}{n-k} \binom{k+2\lambda-1}{k} \\
&\quad \times \sum_{r=0}^{n} \sum_{i=r}^{n} \sum_{m=0}^{i} \Bigg\{ (-1)^{i+r} \times (r+\lambda) \\
&\quad \times \frac{\binom{n-k}{i-m}\binom{k}{m}(2\lambda+k)_m(2\lambda+n-k)_{i-m} i! (\lambda+\frac{1}{2})_i}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_{i-m}(i-r)!(2\lambda)_{r+i+1}} \Bigg\} \\
&\quad \times \frac{\sqrt{\alpha}^{-2\lambda-n+r}\sqrt{q}^{n-r}}{p^r} C^{\lambda}_{\alpha,p,q,r}(x).
\end{aligned}$$

Proof From the front expression, we get

$$C^{\lambda}_{\alpha,p,q,n-k}(x) C^{\lambda}_{\alpha,p,q,k}(x) = \binom{n-k+2\lambda-1}{n-k} \binom{k+2\lambda-1}{k}$$

$$\times \sum_{i=0}^{n} \left(\sum_{m=0}^{i} \frac{\binom{n-k}{i-m}\binom{k}{m}(2\lambda+k)_m(2\lambda+n-k)_{i-m}}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_{i-m}} \right) \times \left(\frac{px-\sqrt{\alpha q}}{2} \right)^i q^{\frac{n-i}{2}} \alpha^{-2\lambda-\frac{n}{2}-\frac{i}{2}},$$

$p(x) = C^{\lambda}_{\alpha,p,q,n-k}(x)C^{\lambda}_{\alpha,p,q,k}(x) \in \mathbf{P}_n$, $p(x) = \sum_{r=0}^{n} d_r C^{\lambda}_{\alpha,p,q,r}(x)$, by (14)

$$\begin{aligned}
d_r &= \frac{p^{1-r}}{q^{r+\lambda}} \frac{(r+\lambda)\Gamma(\lambda)}{(-2)^r\sqrt{\pi}\Gamma(r+\lambda+\frac{1}{2})} \\
&\quad \times \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\frac{d^r}{dx^r} (\alpha q - p^2x^2)^{r+\lambda-\frac{1}{2}} \right) C^{\lambda}_{\alpha,p,q,n-k}(x) C^{\lambda}_{\alpha,p,q,k}(x)\, dx \\
&= \frac{p^{1-r}}{q^{r+\lambda}} \frac{(r+\lambda)\Gamma(\lambda)}{(-2)^r\sqrt{\pi}\Gamma(r+\lambda+\frac{1}{2})} \binom{n-k+2\lambda-1}{n-k} \binom{k+2\lambda-1}{k} \\
&\quad \times \sum_{i=0}^{n} \left(\sum_{m=0}^{i} \frac{\binom{n-k}{i-m}\binom{k}{m}(2\lambda+k)_m(2\lambda+n-k)_{i-m}}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_{i-m}} \right) q^{\frac{n-i}{2}} \alpha^{-2\lambda-\frac{n}{2}-\frac{i}{2}} \\
&\quad \times \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\frac{d^r}{dx^r} (\alpha q - p^2x^2)^{r+\lambda-\frac{1}{2}} \right) \left(\frac{px-\sqrt{\alpha q}}{2} \right)^i dx \\
&= \frac{p^{1-r}}{q^{r+\lambda}} \frac{(r+\lambda)\Gamma(\lambda)}{(-2)^r\sqrt{\pi}\Gamma(r+\lambda+\frac{1}{2})} \binom{n-k+2\lambda-1}{n-k} \binom{k+2\lambda-1}{k} \\
&\quad \times \sum_{i=r}^{n} \left(\sum_{m=0}^{i} \frac{\binom{n-k}{i-m}\binom{k}{m}(2\lambda+k)_m(2\lambda+n-k)_{i-m}}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_{i-m}} \right) q^{\frac{n-i}{2}} \alpha^{-2\lambda-\frac{n}{2}-\frac{i}{2}} \\
&\quad \times \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\frac{d^r}{dx^r} (\alpha q - p^2x^2)^{r+\lambda-\frac{1}{2}} \right) \left(\frac{px-\sqrt{\alpha q}}{2} \right)^i dx.
\end{aligned}$$

We can show that

$$\begin{aligned}
&\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\frac{d^r}{dx^r} (\alpha q - p^2x^2)^{r+\lambda-\frac{1}{2}} \right) \left(\frac{px-\sqrt{\alpha q}}{2} \right)^i dx \\
&= \frac{(-1)^r i!}{2^i(i-r)!} \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} (\alpha q - p^2x^2)^{r+\lambda-\frac{1}{2}} (\sqrt{\alpha q}-px)^{i-r}(-1)^{i-r}\, dx \\
&= \frac{(-1)^i i!}{2^i(i-r)!} \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} (\sqrt{\alpha q}+px)^{r+\lambda-\frac{1}{2}} (\sqrt{\alpha q}-px)^{i+\lambda-\frac{1}{2}}\, dx \\
&= \frac{(-1)^i i!}{2^i(i-r)!} \frac{\sqrt{\alpha q}^{i+r+2\lambda}}{p} \int_0^1 (2x)^{r+\lambda-\frac{1}{2}} (2-2x)^{i+\lambda-\frac{1}{2}} 2\, dx \\
&= \frac{(-1)^i 2^{r+2\lambda} i!}{(i-r)!} \frac{\sqrt{\alpha q}^{i+r+2\lambda}}{p} \frac{\Gamma(i+\lambda+\frac{1}{2})\Gamma(r+\lambda+\frac{1}{2})}{\Gamma(r+i+2\lambda+1)}.
\end{aligned}$$

Applying some identities about gamma function, we get

$$
\begin{aligned}
d_r &= 2^{\lambda+1}(r+\lambda)\binom{n-k+2\lambda-1}{n-k}\binom{k+2\lambda-1}{k}\\
&\quad\times\sum_{i=r}^{n}\left(\sum_{m=0}^{i}(-1)^{r+i}\frac{\binom{n-k}{i-m}\binom{k}{m}(2\lambda+k)_m(2\lambda+n-k)_{i-m}i!(\lambda+\frac{1}{2})_i}{(\lambda+\frac{1}{2})_m(\lambda+\frac{1}{2})_{i-m}(i-r)!(2\lambda)_{r+i+1}}\right)\\
&\quad\times\frac{\sqrt{\alpha}^{-2\lambda-n+r}\sqrt{q}^{n-r}}{p^r}.
\end{aligned}
$$

This proves Theorem 3. □

Theorem 4 *For* $n\in\mathbf{Z}_+$, *we have*

$$
\begin{aligned}
C_{\alpha,p,q,n}(x) &= \sum_{k=0}^{n}\Bigg\{\frac{(\lambda+1)_{k-1}(k+\lambda)2^{2k}\binom{n+k+2\lambda-1}{n-k}}{\binom{n+\lambda-\frac{1}{2}}{n-k}}\sqrt{q}^{n-k}\sqrt{\alpha}^{3k-n}\\
&\quad\times\sum_{l=0}^{n-k}\binom{n+\lambda-\frac{1}{2}}{n-k-l}\binom{n+\lambda-\frac{1}{2}}{l}(-1)^l\frac{\binom{k+\lambda+l-\frac{1}{2}}{l}\binom{\lambda+n-l-\frac{1}{2}}{n-l}}{\binom{n}{l}\binom{k+2\lambda+n}{n+k}\binom{n+k}{k}k!}\Bigg\}C^{\lambda}_{\alpha,p,q,k}(x).
\end{aligned}
$$

Proof By (2), (5) and (14), we get

$$
\begin{aligned}
d_k &= \frac{p^{1-k}}{q^{k+\lambda}}\frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}}\left(\frac{d^k}{dx^k}(\alpha q-p^2x^2)^{k+\lambda-\frac{1}{2}}\right)C^{\lambda}_{\alpha,p,q,n}(x)\,dx\\
&= \frac{(\lambda)_k(k+\lambda)\Gamma(\lambda)}{\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\frac{p}{q^{k+\lambda}}\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}}(\alpha q-p^2x^2)^{k+\lambda-\frac{1}{2}}C^{\lambda+k}_{\alpha,p,q,n-k}(x)\,dx\\
&= \frac{(\lambda)_k(k+\lambda)\Gamma(\lambda)}{\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\frac{p}{q^{k+\lambda}}\frac{\binom{n+k+2\lambda-1}{n-k}}{\binom{n+\lambda-\frac{1}{2}}{n-k}}\sum_{l=0}^{n-k}\binom{n+\lambda-\frac{1}{2}}{n-k-l}\\
&\quad\times\binom{n+\lambda-\frac{1}{2}}{l}(-1)^l\left(\frac{1}{2}\right)^{n-k}\alpha^{-\lambda-n+k}\\
&\quad\times\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}}(\sqrt{\alpha q}-px)^{k+\lambda-\frac{1}{2}+l}(\sqrt{\alpha q}+px)^{\lambda+n-\frac{1}{2}-l}\,dx.
\end{aligned}
$$

It is not difficult to show that

$$
\begin{aligned}
&\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}}(\sqrt{\alpha q}-px)^{k+\lambda-\frac{1}{2}+l}(\sqrt{\alpha q}+px)^{\lambda+n-\frac{1}{2}-l}\,dx\\
&\quad= \frac{\sqrt{\alpha q}^{k+2\lambda+n}}{p}\int_0^1(2-2x)^{k+\lambda-\frac{1}{2}+l}(2x)^{\lambda+n-\frac{1}{2}-l}\,dx\\
&\quad= \frac{\sqrt{\alpha q}^{k+2\lambda+n}}{p}2^{k+2\lambda+n}\frac{\Gamma(k+\lambda+l+\frac{1}{2})\Gamma(\lambda+n-l+\frac{1}{2})}{\Gamma(k+2\lambda+n+1)}.
\end{aligned}
$$

Applying some identities involving gamma function, we get

$$\Gamma\left(k+\lambda+l+\frac{1}{2}\right)=\binom{k+\lambda+l-\frac{1}{2}}{l}l!\Gamma\left(k+\lambda+\frac{1}{2}\right),$$
$$\Gamma\left(\lambda+n-l+\frac{1}{2}\right)=\binom{\lambda+n-l-\frac{1}{2}}{n-l}(n-l)!\Gamma\left(\lambda+\frac{1}{2}\right),$$

and

$$\Gamma(k+2\lambda+n+1)=\binom{k+2\lambda+n}{n+k}(n+k)!\Gamma(2\lambda+1).$$

As we all know, the duplication formula for the gamma function is given by

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

From this formula, we get

$$\begin{aligned}&\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}}(\sqrt{\alpha q}-px)^{k+\lambda-\frac{1}{2}}(\sqrt{\alpha q}+px)^{\lambda+n-l-\frac{1}{2}}\,dx\\&\quad=\frac{\sqrt{\alpha q}^{k+2\lambda+n}}{p}2^{k+n}\frac{\binom{k+\lambda+l-\frac{1}{2}}{l}\binom{\lambda+n-l-\frac{1}{2}}{n-l}\Gamma(k+\lambda+\frac{1}{2})}{\binom{n}{l}\binom{k+2\lambda+n}{n+k}\binom{n+k}{k}k!2\lambda\Gamma(\lambda)}\sqrt{\pi}.\end{aligned}$$

So, we get

$$\begin{aligned}d_k&=(\lambda+1)_{k-1}(k+\lambda)2^{2k}\frac{\binom{n+k+2\lambda-1}{n-k}}{\binom{n+\lambda-\frac{1}{2}}{n-k}}\sum_{l=0}^{n-k}\binom{n+\lambda-\frac{1}{2}}{n-k-l}\binom{n+\lambda-\frac{1}{2}}{l}\\&\quad\times(-1)^l\sqrt{q}^{n-k}\sqrt{\alpha}^{3k-n}\frac{\binom{k+\lambda+l-\frac{1}{2}}{l}\binom{\lambda+n-l-\frac{1}{2}}{n-l}\binom{n}{l}}{\binom{k+2\lambda+n}{n+k}\binom{n+k}{k}k!}.\end{aligned}$$

This completes the proof of Theorem 4. □

Theorem 5 *For $n\in\mathbf{Z}_+$, we have*

$$\begin{aligned}\frac{B_{n+1}(x+1)-B_{n+1}(x)}{(n+1)!}&=\sum_{0\le k\le n,n-k\equiv0(\bmod 2)}\frac{\sqrt{\alpha}^{2\lambda+k+n}\sqrt{q}^{n-k}}{p^n}\\&\quad\times\frac{(k+\lambda)n!\Gamma(\lambda)}{2^n(\frac{n-k}{2})!\Gamma(\frac{n+k+2\lambda+2}{2})}C^{\lambda}_{\alpha,p,q,k}(x),\\\frac{E_n(x+1)+E_n(x)}{2(n)!}&=\sum_{0\le k\le n,n-k\equiv0(\bmod 2)}\frac{\sqrt{\alpha}^{2\lambda+k+n}\sqrt{q}^{n-k}}{p^n}\\&\quad\times\frac{(k+\lambda)n!\Gamma(\lambda)}{2^n(\frac{n-k}{2})!\Gamma(\frac{n+k+2\lambda+2}{2})}C^{\lambda}_{\alpha,p,q,k}(x).\end{aligned}$$

Proof Applying Theorem 1, we get

$$x^n = \sum_{0\le k\le n, n-k\equiv 0(\bmod 2)} \frac{\sqrt{\alpha}^{2\lambda+k+n}\sqrt{q}^{n-k}}{p^n} \frac{(k+\lambda)n!\Gamma(\lambda)}{2^n(\frac{n-k}{2})!\Gamma(\frac{n+k+2\lambda+2}{2})} C^{\lambda}_{\alpha,p,q,k}(x).$$

From (6) and (7), we have

$$\begin{aligned}
e^{xt} &= \frac{1}{t}\frac{t}{e^t-1}e^{xt}\left(e^t-1\right) = \frac{1}{t}\frac{t}{e^t-1}\left(e^{(x+1)t}-e^{xt}\right) \\
&= \frac{1}{t}\left(\sum_{n=0}^{\infty} B_n(x+1)\frac{t^n}{n!} - \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty} \frac{B_{n+1}(x+1)-B_{n+1}(x)}{n+1}\frac{t^n}{n!}, \\
e^{xt} &= \frac{2}{e^t+1}e^{xt}\left(\frac{e^t+1}{2}\right) = \frac{1}{2}\left(\frac{2}{e^t+1}e^{(x+1)t} + \frac{2}{e^t+1}e^{xt}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} E_n(x+1)\frac{t^n}{n!} + \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}\right).
\end{aligned}$$

So, we get

$$x^n = \frac{B_{n+1}(x+1)-B_{n+1}(x)}{n+1} = \frac{1}{2}\left(E_n(x+1)+E_n(x)\right).$$

We can complete the proof of Theorem 5. □

Theorem 6 *For $n \in \mathbf{Z}_+$, we have*

$$\begin{aligned}
\frac{H_n(x)}{n!} &= \Gamma(\lambda)\sum_{k=0}^{n}\frac{(k+\lambda)}{(n-k)!}\left(\sum_{0\le l\le n-k, l\equiv 0(\bmod 2)} \frac{\sqrt{\alpha}^{2\lambda+2k+l}\sqrt{q}^{l}}{p^{l+k}} \frac{\binom{n-k}{l} l! H_{n-k-l}}{(\frac{l}{2})!\Gamma(\frac{2k+2\lambda+l+2}{2})}\right) \\
&\quad \times C^{\lambda}_{\alpha,p,q,k}(x).
\end{aligned}$$

Proof From (10), (11) and (14), we get

$$\begin{aligned}
d_k &= \frac{p^{1-k}}{q^{k+\lambda}}\frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\frac{d^k}{dx^k}\left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}}\right) H_n(x)\,dx \\
&= \frac{p^{1-k}}{q^{k+\lambda}}\frac{(k+\lambda)\Gamma(\lambda)}{\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\frac{n!}{(n-k)!}\int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}} H_{n-k}(x)\,dx \\
&= \frac{p^{1-k}}{q^{k+\lambda}}\frac{(k+\lambda)\Gamma(\lambda)}{\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\frac{n!}{(n-k)!} \\
&\quad \times \sum_{l=0}^{n-k}\binom{n-k}{l} H_{n-k-l}2^l \int_{-\frac{\sqrt{\alpha q}}{p}}^{\frac{\sqrt{\alpha q}}{p}} \left(\alpha q - p^2x^2\right)^{k+\lambda-\frac{1}{2}} x^l\,dx.
\end{aligned}$$

Let us assume that $l \equiv 0(\bmod 2)$, first let $x = \frac{\sqrt{q}}{p}y$, then $y = \sqrt{x}$, we have

$$\begin{aligned} d_k &= \frac{(k+\lambda)\Gamma(\lambda)}{\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\frac{n!}{(n-k)!} \\ &\quad \times \sum_{l=0}^{n-k}\binom{n-k}{l}H_{n-k-l}2^l\frac{\sqrt{\alpha}^{2\lambda+2k+l}\sqrt{q}^l}{p^{l+k}}\int_0^1(1-x)^{k+\lambda-\frac{1}{2}}x^{\frac{l-1}{2}}\,dx \\ &= \frac{(k+\lambda)\Gamma(\lambda)}{\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\frac{n!}{(n-k)!} \\ &\quad \times \sum_{0\le l\le n-k, l\equiv 0(\bmod 2)}\binom{n-k}{l}H_{n-k-l}2^l\frac{\sqrt{\alpha}^{2\lambda+2k+l}\sqrt{q}^l}{p^{l+k}}\frac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(\frac{l+1}{2})}{\Gamma(\frac{2k+2\lambda+l+2}{2})} \\ &= \frac{(k+\lambda)\Gamma(\lambda)n!}{(n-k)!}\sum_{0\le l\le n-k, l\equiv 0(\bmod 2)}\frac{\binom{n-k}{l}H_{n-k-l}l!}{(\frac{l}{2})!\Gamma(\frac{2k+2\lambda+l+2}{2})}\frac{\sqrt{\alpha}^{2\lambda+2k+l}\sqrt{q}^l}{p^{l+k}}. \end{aligned}$$

□

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The author declares that they have no competing interests.

Authors' contributions
The author has contributed solely to the writing of this paper. He read and approved the manuscript.

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