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Fractional-order model for biocontrol of the lesser date moth in palm trees and its discretization

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Abstract

In this paper, a fractional-order model of palm trees, the lesser date moth and the predator is presented. Existence conditions of the local asymptotic stability of the equilibrium points of the fractional system are analyzed. We prove that the positive equilibrium point is globally stable also. The numerical simulations come to illustrate the dynamical behaviors of the model such as bifurcation and chaos phenomenon, and the numerical simulations confirm the validity of our theoretical results.

Keywords: natural enemy; fractional-order model; discrete time; stability; numerical method

1 Introduction

Date palm is a unisexual fruit tree native to the hot arid regions of the world, mainly grown in the Middle East and North Africa. Since ancient times this majestic plant has been recognized as the tree of life because of its integration into human settlement, well-being, and food security in hot regions of the world, where only a few plant species can flourish [1].

It is well known that approximately one third of the world food production is lost due to pests. Pesticides have a great role in destroying pests and increasing crop yield. But the excessive use of pesticides exerts harmful effects on human health. The extensive use of chemical pesticides has had many well documented adverse consequences. So, the present trend in pest control is to minimize the use of pesticides with optimum reduction in the pest population. Such a situation can be achieved when the balance between the pest and its natural enemies is least disturbed by selective use of pesticides [2].

The lesser date moth, Batrachedra amydraula (Lepidoptera: Batrachedridae), is a serious pest of date palms. Its distribution is from Bangladesh to the entire Middle East, as well as most of North America. It is one of the most important pests on date palms that may cause more than 50% loss of the crop [3].

In recent decades, the fractional calculus and fractional differential equations have attracted much attention and increasing interest due to their potential applications in science and engineering [4–17]. In this paper, we consider a fractional-order model consisting of palm trees, the lesser date moth and the predator. Sufficient conditions for the existence of the solutions of the fractional-order model are investigated. The equilibrium points and their asymptotic stability are discussed. Also, the conditions for the existence of a flip bifurcation are considered. The necessary conditions for this system to exhibit chaotic dynamics are also derived.

2 Fractional calculus

A great deal of research has been conducted on prey-predator models based on fractional-order differential equations. A property of these fractional models is their nonlocal property which is not present in integer-order differential equations. Nonlocal property means that the next state of a model depends not only upon its current state, but also upon all of its historical states as the case in epidemics. Fractional-order differential equations can be used to model phenomena which cannot be adequately modeled by integer-order differential equations [10, 13, 18–21]. There are several definitions of fractional derivatives. One of the most common definitions is the Caputo concept. This definition is often used in real applications.

Definition 1 The fractional integral of order $\beta \in R^+$ of the function $f(t)$ is defined by

$$I^\beta f(t) = \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau)\, d\tau,$$

provided the integral on the right-hand side is defined for almost every $t > 0$. This occurs, for example, if $f \in L^1(0,+\infty)$. The fractional derivative of order $\alpha \in (n-1,n)$, $n \in N$ is defined as follows for a function f such that the n-order derivative exists and, for example, $f^{(n)} \in L^1(0,+\infty)$:

$$D_t^\alpha f(t) = I^{n-\alpha} f^{(n)}(t), \quad \alpha > 0.$$

3 Fractional-order lesser date moth model and its discretization

Following [22, 23], the model of biocontrol of the lesser date moth in palm trees can be written as a set of three coupled nonlinear ordinary differential equations as follows:

$$\begin{aligned} \frac{dP}{d\tau} &= rP\left(1-\frac{P}{K}\right) - \frac{bPL}{a+P}, \\ \frac{dL}{d\tau} &= -dL + \frac{mPL}{a+P} - pLN, \\ \frac{dN}{d\tau} &= -\mu N + qLN. \end{aligned} \tag{1}$$

The model consists of three populations. The palm tree whose population density at time t is denoted by P, the pest (lesser date moth) whose population density is denoted by L and the predator whose population density is denoted by N. In the absence of predators, the prey population density grows according to a logistic curve with carrying capacity K and with an intrinsic growth rate constant r.

The maximal growth rate of the pest is denoted by b. The half saturation a is constant, d denotes the death rate of the pests, m is the conversion rate of the pests, p is the quantity that represents decrease in the growth rate of the pests due to predator attack, q is the rate of increase in the predator population, and μ denotes the intrinsic mortality rate of the predators. Here all the parameters r, K, b, a, d, m , p, μ, and q are positive.

One can reduce the number of parameters in system (1) by using the following transformations:

$$P = Kx, \qquad L = \frac{Kr}{b}y, \qquad N = \frac{r}{p}z, \qquad \tau = \frac{t}{r},$$

then we have the following dimensionless system:

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{xy}{\beta + x}, \\ \frac{dy}{dt} &= -\delta y + \frac{\gamma xy}{\beta + x} - yz, \\ \frac{dz}{dt} &= -\eta z + \sigma yz, \end{aligned} \tag{2}$$

where

$$\beta = \frac{a}{K}, \qquad \delta = \frac{d}{r}, \qquad \gamma = \frac{m}{r}, \qquad \eta = \frac{\mu}{r} \quad \text{and} \quad \sigma = \frac{qk}{b}.$$

Fractional-order models are more accurate than integer-order models as they allow more degrees of freedom. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes [10, 13]. Now we introduce fractional order into the ODE model (2). The new system is described by the following set of fractional-order differential equations:

$$\begin{aligned} D^\alpha x &= x(1-x) - \frac{xy}{\beta + x}, \\ D^\alpha y &= -\delta y + \frac{\gamma xy}{\beta + x} - yz, \\ D^\alpha z &= -\eta z + \sigma yz, \end{aligned} \tag{3}$$

where D^α is the Caputo fractional derivative. Following [21, 24, 25], we discretize the fractional lesser date moth and predator model (3). After the discretization with piecewise constant arguments, the system is reduced to

$$\begin{aligned} x_{n+1} &= x_n + \frac{h^\alpha}{\Gamma(1+\alpha)}\left[x_n(1-x_n) - \frac{x_n y_n}{\beta + x_n}\right], \\ y_{n+1} &= y_n + \frac{h^\alpha}{\Gamma(1+\alpha)}\left[-\delta y_n + \frac{\gamma x_n y_n}{\beta + x_n} - y_n z_n\right], \\ z_{n+1} &= z_n + \frac{h^\alpha}{\Gamma(1+\alpha)}[-\eta z_n + \sigma y_n z_n]. \end{aligned} \tag{4}$$

Remark 1 If the fractional order $\alpha \to 1$, then we have the forward Euler discretization of system (2).

In the following, we will study the dynamics of system (4).

4 Dynamical behaviors of the discretized fractional-order lesser date moth and predator model

4.1 Stability of the fixed points of the system

In this subsection, we study the asymptotic stability of the fixed points of system (4) which has the same fixed points of system (4). First, we need the following two definitions.

Definition 2 ([26] (Local stability when all eigenvalues are real)) Consider the discrete, nonlinear dynamical system in (4) with a steady-state equilibrium $\bar{x}$. The linearized system is given by (4). The associated Jacobian matrix has three real eigenvalues λ_i ($i = 1, 2, 3$).

Lemma 1

(i) *The steady-state equilibrium* $\bar{x}$ *is called a stable node if* $|\lambda_i| < 1$ *for all* $i = 1, 2, 3$.

(ii) *The steady-state equilibrium* $\bar{x}$ *is called a two-dimensional saddle if one* $|\lambda_i| > 1$.

(iii) *The steady-state equilibrium* $\bar{x}$ *is called a one-dimensional saddle if one* $|\lambda_i| < 1$.

(iv) *The steady-state equilibrium* $\bar{x}$ *is called an unstable node if* $|\lambda_i| > 1$ *for all* $i = 1, 2, 3$.

(v) *The steady-state equilibrium* $\bar{x}$ *is called hyperbolic if one* $|\lambda_i| = 1$.

Definition 3 ([26] (Local stability when complex eigenvalues)) Consider the discrete, nonlinear dynamical system in (4) with a steady-state equilibrium $\bar{x}$. The linearized system is given by (4). The associated Jacobian matrix has a pair of complex eigenvalues $\lambda_{1,2} = \rho + \omega i$ and one real eigenvalue λ_3.

(i) The steady-state equilibrium $\bar{x}$ is called a sink if $|\lambda_i| < 1$ for all $i = 1, 2, 3$.

(ii) The steady-state equilibrium $\bar{x}$ is called a two-dimensional saddle if one $|\lambda_3| > 1$.

(iii) The steady-state equilibrium $\bar{x}$ is called a one-dimensional saddle if one $|\lambda_3| < 1$.

(iv) The steady-state equilibrium $\bar{x}$ is called a source if $|\lambda_i| > 1$ for all $i = 1, 2, 3$.

(v) The steady-state equilibrium $\bar{x}$ is called hyperbolic if one $|\lambda_i| = 1$.

Now, the Jacobian matrix $J(E_0)$ for system given in (4) evaluated at $E_0(0, 0, 0)$ is as follows:

$$J(E_0) = \begin{pmatrix} 1 + \frac{h^\alpha}{\Gamma(1+\alpha)} & 0 & 0 \\ 0 & 1 - \frac{\delta h^\alpha}{\Gamma(1+\alpha)} & 0 \\ 0 & 0 & 1 - \frac{\eta h^\alpha}{\Gamma(1+\alpha)} \end{pmatrix}. \tag{5}$$

Theorem 1 *The trivial-equilibrium point* E_0 *has at least three different topological types for its all values of parameters as follows*:

(i) E_0 *is a source if* $h > \max\{\sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\delta}}, \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\eta}}\}$,

(ii) E_0 *is a two-dimensional saddle if* $0 < h < \min\{\sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\delta}}, \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\eta}}\}$,

(iii) E_0 *is a one-dimensional saddle if* $\sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\delta}} < h < \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\eta}}$ *or* $\sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\eta}} < h < \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\delta}}$.

Proof The eigenvalues corresponding to the equilibrium point E_0 are $\lambda_{01} = 1 + \frac{h^\alpha}{\Gamma(1+\alpha)} > 0$, $\lambda_{02} = 1 - \frac{\delta h^\alpha}{\Gamma(1+\alpha)}$, and $\lambda_{03} = 1 - \frac{\eta h^\alpha}{\Gamma(1+\alpha)}$, where $\alpha \in (0, 1]$ and h, $\frac{h^\alpha}{\Gamma(1+\alpha)} > 0$. Hence, applying the stability conditions using Definition 2, one can obtain the results (i)-(iii). □

The Jacobian matrix $J(E_1)$ for system (4), evaluated at $E_1 = (1,0,0)$, is given by

$$J(E_1) = \begin{pmatrix} 1 - \frac{h^\alpha}{\Gamma(1+\alpha)} & \frac{-h^\alpha}{(1+\beta)\Gamma(1+\alpha)} & 0 \\ 0 & 1 + \frac{\delta(R_0-1)h^\alpha}{\Gamma(1+\alpha)} & 0 \\ 0 & 0 & 1 - \frac{\eta h^\alpha}{\Gamma(1+\alpha)} \end{pmatrix}. \tag{6}$$

Theorem 2 *If the semi-trivial equilibrium point E_1 exists, then it has at least three different topological types for its all values of parameters.*

(i) *E_1 is a source if* $h > \max\{\sqrt[\alpha]{2\Gamma(1+\alpha)}, \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\mu}}\}$,

(ii) *E_1 is a two-dimensional saddle if* $0 < h < \min\{\sqrt[\alpha]{2\Gamma(1+\alpha)}, \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\eta}}\}$,

(iii) *E_1 is a one-dimensional saddle if* $\sqrt[\alpha]{2\Gamma(1+\alpha)} < h < \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\eta}}$ *or* $\sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\eta}} < h < \sqrt[\alpha]{2\Gamma(1+\alpha)}$.

Proof The eigenvalues corresponding to the equilibrium point E_0 are $\lambda_{11} = 1 - \frac{h^\alpha}{\Gamma(1+\alpha)}$, $\lambda_{12} = 1 + \frac{\delta(R_0-1)h^\alpha}{\Gamma(1+\alpha)} > 0$, and $\lambda_{13} = 1 - \frac{\eta h^\alpha}{\Gamma(1+\alpha)}$. Hence, applying the stability conditions using Definition 2, one can obtain the results (i)-(iii). □

For investigating the stability of $E_2 = (\frac{\beta\delta}{\gamma-\delta}, \frac{\gamma(\beta+1)(R_0-1)x_2^2}{\beta\delta}, 0)$, let $J(E_2)$ be the Jacobian matrix for the system given in (4) evaluated at E_2, then

$$J(E_2) = \begin{pmatrix} 1 - \frac{h^\alpha}{\Gamma(1+\alpha)}[x_2 - \frac{x_2 y_2}{(\beta+x_2)^2}] & \frac{-x_2 h^\alpha}{(\beta+x_2)\Gamma(1+\alpha)} & 0 \\ \frac{\beta\gamma y_2 h^\alpha}{(\beta+x_2)^2\Gamma(1+\alpha)} & 1 & \frac{-y_2 h^\alpha}{\Gamma(1+\alpha)} \\ 0 & 0 & 1 - \frac{h^\alpha}{\Gamma(1+\alpha)}(\eta - \sigma y_2) \end{pmatrix}. \tag{7}$$

The characteristic equation of the Jacobian matrix (7) is

$$\bigl[\lambda_{21} - \bigl(1 - H[\eta - \sigma y_2]\bigr)\bigr]\bigl[\lambda^2 - Tr_2\lambda + Det_2\bigr] = 0, \tag{8}$$

where $H = \frac{h^\alpha}{\Gamma(1+\alpha)}$, $Tr_2 = 2 - B_2 H$, $B_2 = \frac{(1+\beta)x_2^2}{(\beta+x_2)}[1 - R_0(1-\beta)]$, $Det_2 = A_2 H^2 - B_2 H + 1$, and $A_2 = \frac{\beta\gamma x_2 y_2}{(\beta+x_2)^3}$.

Theorem 3 *If* $1 < R_0 < \frac{\beta\delta\eta}{\gamma\sigma(1+\beta)x_2^2} + 1$, *then the semi-trivial equilibrium point E_2 has at least four different topological types for its all values of parameters*

(i) *E_2 is asymptotically stable (sink) if one of the following conditions holds:*
  - (i.1) $\Delta \geq 0$ *and* $0 < h < \min\{h_1, h_2\}$,
  - (i.2) $\Delta < 0$ *and* $0 < h < h_2$;

(ii) *E_2 is unstable (source) if one of the following conditions holds:*
  - (ii.1) $\Delta \geq 0$ *and* $h > \max\{h_2, h_3\}$,
  - (ii.2) $\Delta < 0$ *and* $h > h_2$;

(iii) *E_2 is a two-dimensional saddle if the following case is satisfied:*
  $\Delta \geq 0$ *and* $h < \min\{h_1, h_2\}$;

(iv) *E_2 is a one-dimensional saddle if one of the following cases is satisfied:*
  - (iv.1) $\Delta \geq 0$ *and* $h_3 < h < h_2$ *or* $\max\{h_1, h_2\} < h < h_3$,
  - (iv.2) $\Delta < 0$ *and* $0 < h < h_2$;

(v) *E_2 is non-hyperbolic if one of the following conditions holds:*

(v.1) $\Delta \geq 0$ *and* $h = h_1$ *or* h_3,

(v.2) $\Delta < 0$ *and* $h = h_2$,

where

$$h_1 = \sqrt[\alpha]{\frac{4\Gamma(1+\alpha)}{B_2 + \sqrt{\Delta}}}, \qquad h_3 = \sqrt[\alpha]{\frac{4\Gamma(1+\alpha)}{B_2 - \sqrt{\Delta}}},$$

$$\Delta = B_2^2 - 4A_2, \qquad h_2 = \sqrt[\alpha]{\frac{2\Gamma(1+\alpha)}{\eta - \sigma y_2}}.$$

Proof The eigenvalues corresponding to the equilibrium point E_2 are the roots of the characteristic equation (8), which is $\lambda_{21} = 1 - H[\eta - \sigma y_2]$ and $\lambda_{22,23} = \frac{1}{2}(Tr_2 \pm \sqrt{Tr_2^2 - 4Det_2}) = 1 - \frac{H}{2}(B_2 \pm \sqrt{B_2^2 - 4A_2})$. Hence, applying the stability conditions using Lemma 1, one can obtain the results (i)-(v). □

The fourth and fifth equilibrium points are $E_j = (x_j, y_j, z_j)$, $j = 3, 4$, where

$$x_3 = \frac{1}{2}\left[1 - \beta - \sqrt{(1-\beta)^2 - 4\left(\frac{\eta}{\sigma} - \beta\right)}\right],$$

$$x_4 = \frac{1}{2}\left[1 - \beta + \sqrt{(1-\beta)^2 - 4\left(\frac{\eta}{\sigma} - \beta\right)}\right],$$

$$y_j = \frac{\eta}{\sigma},$$

$$z_j = \frac{\beta\gamma\sigma}{\eta}(x_j - 1) + \gamma - \delta.$$

For the dynamical properties of the interior (positive) equilibrium point E_j ($j = 3, 4$) we need to state these lemmas.

Lemma 2 ([27]) *Let the equation* $x^3 + bx^2 + cx + d = 0$, *where* $b, c, d \in R$. *Let further* $A = b^2 - 3c$, $B = bc - 9d$, $C = c^2 - 3bd$, *and* $\Delta = B^2 - 4AC$. *Then*

(1) *The equation has three real roots if and only if* $\Delta \leq 0$.

(2) *The equation has one real root* x_1 *and a pair of conjugate complex roots if and only if* $\Delta > 0$. *Furthermore, the conjugate complex roots* $x_{2,3}$ *are*

$$x_{2,3} = \frac{1}{6}\left[\sqrt[3]{y_1} + \sqrt[3]{y_2} - 2b \pm \sqrt{3}i\left(\sqrt[3]{y_1} - \sqrt[3]{y_2}\right)\right],$$

where

$$y_{1,2} = bA + \frac{3}{2}\left(-B \pm \sqrt{B^2 - 4AC}\right).$$

Lemma 3 ([28–31]) *Let* $F(\lambda) = \lambda^2 - Tr\lambda + Det$. *Suppose that* $F(1) > 0$, λ_1 *and* λ_2 *are the two roots of* $F(\lambda) = 0$. *Then*

(i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $Det < 1$,

(ii) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ (*or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$) *if and only if* $F(-1) < 0$,

(iii) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $Det > 1$,

(iv) $\lambda_1 = -1$ *and* $\lambda_2 \neq 1$ *if and only if* $F(-1) = 0$ *and* $Tr \neq 0, 2$,
(v) λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2|$ *if and only if* $Tr^2 - 4Det < 0$ *and* $Det = 1$.

The necessary and sufficient conditions ensuring that $|\lambda_1| < 1$ and $|\lambda_2| < 1$ are as follows [28]:

$$\begin{aligned}
&\text{(i)} \quad 1 - TrJ + \det J > 0, \\
&\text{(ii)} \quad 1 + TrJ + \det J > 0, \\
&\text{(iii)} \quad \det J < 1.
\end{aligned} \tag{9}$$

If one of conditions (9) is not satisfied, then we have one of the following cases [25].

1. A saddle-node (often called fold bifurcation in maps), transcritical or pitchfork bifurcation if one of the eigenvalues = 1 and other eigenvalues (real) $\neq 1$. This local bifurcation leads to the stability switching between two different steady states;
2. A flip bifurcation if one of the eigenvalues = –1, other eigenvalues (real) $\neq -1$. This local bifurcation entails the birth of a period 2-cycle;
3. A Neimark-Sacker (secondary Hopf) bifurcation; in this case we have two conjugate eigenvalues and the modulus of each of them = 1.

This local bifurcation implies the birth of an invariant curve in the phase plane. The Neimark-Sacker bifurcation is considered to be an equivalent to the Hopf bifurcation in continuous time and in fact the major instrument to prove the existence of quasi-periodic orbits for the map.

Note You can get any local bifurcation (fold, flip and Neimark-Sacker) by taking specific parameter value such that one of the conditions of each bifurcation is satisfied.

The Jacobian matrix $J(E_j)$ for system (4) evaluated at the interior equilibrium point E_j is as follows:

$$J(E_j) = \begin{pmatrix} 1 - H[x_j - \frac{x_j y_j}{(\beta + x_j)^2}] & \frac{-x_j H}{\beta + x_j} & 0 \\ \frac{\beta \gamma y_j H}{(\beta + x_j)^2} & 1 & -y_j H \\ 0 & \sigma z_j H & 1 \end{pmatrix}, \tag{10}$$

then the characteristic equation of the Jacobian matrix (10) is

$$F(\lambda) = \lambda^3 + b_j \lambda^2 + c_j \lambda + d_j = 0, \tag{11}$$

where

$$b_j = 3 - \varepsilon_j H, \quad \varepsilon_j = \frac{x_j}{\beta + x_j}[2x_j + \beta - 1],$$

$$c_j = \psi_j H^2 - 2\varepsilon_j H + 3, \quad \psi_j = \frac{\beta \gamma x_j y_j}{(\beta + x_j)^3} + \sigma y_j z_j,$$

and

$$d_j = \nu_j H^3 - \psi_j H^2 + \varepsilon_j H - 1, \quad \nu_j = \sigma \varepsilon_j y_j z_j.$$

By some computation, we have

$$A = b_j^2 - 3c_j = (\varepsilon_j^2 - 3\psi_j)H^2,$$
$$B = b_jc_j - 9d_j = (\varepsilon_j\psi_j - 9\nu_j)H^3 + (6\psi_j - 2\varepsilon_j^2)H^2,$$
$$C = c_j^2 - 3b_jd_j = (\psi_j^2 - 3\varepsilon_j\nu_j)H^4 + (9\nu_j - \varepsilon_j\psi_j)H^3 + (\varepsilon_j^2 - 3\psi_j)H^2,$$

and

$$\Delta_j = \bar{\Delta}_j H^6,$$

where

$$\bar{\Delta}_j = 3(27\nu_j^2 + 4\varepsilon_j^3\nu_j + 4\psi_j^3 - \varepsilon_j^2\psi_j^2 - 18\varepsilon_j\psi_j\nu_j).$$

It is clear that equation $F'(\lambda) = 0$ has the following two roots:

$$\lambda_{1,2}^* = \frac{1}{3}\left(-b_j \pm \sqrt{b_j^2 - 3c_j}\right) = 1 - \frac{H}{3}\left(\varepsilon_j \pm \sqrt{\varepsilon_j^2 - 3\psi_j}\right).$$

If $\bar{\Delta}_j \leq 0$, then $\Delta_j \leq 0$; by Lemma 1, equation (10) has three real roots λ_i, $i = 1, 2, 3$ (let $\lambda_1 \leq \lambda_2 \leq \lambda_3$). From this, we note that two roots $\lambda_{1,2}^*$ (let $\lambda_1^* \leq \lambda_2^*$) of equation $F'(\lambda) = 0$ also are real.

When $\bar{\Delta}_j > 0$, namely, $\Delta_j > 0$, by Lemma 2, we have that equation (11) has one real root λ_1 and a pair of conjugate complex roots $\lambda_{2,3}$. The conjugate complex roots are as follows:

$$\lambda_{2,3} = \frac{1}{6}\left[\sqrt[3]{y_1} + \sqrt[3]{y_2} - 2b \pm \sqrt{3}i\left(\sqrt[3]{y_1} - \sqrt[3]{y_2}\right)\right],$$

where

$$y_{1,2} = \frac{H^3}{2}\left(2\varepsilon_j^3 - 9\varepsilon_j\psi_j - 27\nu_j \pm 3\sqrt{\bar{\Delta}_j}\right),$$

and

$$F(1) = \nu_jH^3 \quad \text{and} \quad F(-1) = -8 + 4\varepsilon_jH - 2\psi_jH^2 + \nu_jH^3.$$

Now, we will introduce the stability of E_j, we have the following theorem.

Theorem 4 *If the positive equilibrium point E_j exists, then it has the following topological types of its all values of parameters*:

(1) *E_j is a sink if one of the following conditions holds*:
   (1.i) $\bar{\Delta}_j \leq 0$, $F(1) > 0$, $F(-1) < 0$ *and* $-1 < \lambda_{1,2}^* < 1$,
   (1.ii) $\bar{\Delta}_j > 0$, $F(1) > 0$, $F(-1) < 0$ *and* $|\lambda_{2,3}| < 1$.

(2) *E_j is a source if one of the following conditions holds*:
   (2.i) $\bar{\Delta}_j \leq 0$ *and one of the following conditions holds*:
      (2.i.a) $F(1) > 0$, $F(-1) > 0$ *and* $\lambda_2^* < -1$ *or* $\lambda_2^* > 1$,
      (2.i.b) $F(1) < 0$, $F(-1) < 0$ *and* $\lambda_2^* < -1$ *or* $\lambda_2^* > 1$,

(2.ii) $\bar{\Delta}_j > 0$ *and one of the following conditions holds*:
(2.ii.a) $F(1) < 0$ *and* $|\lambda_{2,3}| > 1$,
(2.ii.b) $F(-1) > 0$ *and* $|\lambda_{2,3}| > 1$.

(3) E_j *is a one-dimensional saddle if one of the following conditions holds*:
(3.i) $\bar{\Delta}_j \leq 0$ *and one of the following conditions holds*:
(3.i.a) $F(1) > 0, F(-1) < 0$ *and* $\lambda_1^* < -1$ *or* $\lambda_2^* > 1$,
(3.i.b) $F(1) < 0, F(-1) > 0$.
(3.ii) $\bar{\Delta}_j > 0$ *and one of the following conditions holds*:
(3.ii.a) $F(1) > 0, F(-1) < 0$ *and* $|\lambda_{2,3}| > 1$,
(3.ii.b) $F(1) < 0$ *and* $|\lambda_{2,3}| < 1$,
(3.ii.c) $F(-1) > 0$ *and* $|\lambda_{2,3}| < 1$.

(4) E_j *is a two-dimensional saddle if one of the following conditions holds*:
(4.i) $\bar{\Delta}_j \leq 0$ *and one of the following conditions holds*:
(4.i.a) $F(1) > 0, F(-1) > 0$ *and* $-1 < \lambda_2^* < 1$,
(4.i.b) $F(1) < 0, F(-1) < 0$ *and* $-1 < \lambda_1^* < 1$,
(4.ii) $\bar{\Delta}_j > 0$ *and one of the following conditions holds*:
(4.ii.a) $F(-1) < 0$ *and* $|\lambda_{2,3}| < 1$,
(4.ii.b) $F(1) > 0$ *and* $|\lambda_{2,3}| < 1$.

(5) E_j *is non-hyperbolic if one of the following conditions holds*:
(5.i) $\bar{\Delta}_j \leq 0$ *and* $F(1) = 0$ *or* $F(-1) = 0$,
(5.ii) $\bar{\Delta}_j > 0$ *and* $F(1) = 0$ *or* $F(-1) = 0$ *or* $|\lambda_{2,3}| = 1$.

Proof Let $\bar{\Delta}_j \leq 0$. From Lemma 2, equation (11) has three real roots λ_i, $i = 1, 2, 3$. Further, we obtain that equation $F'(\lambda) = 0$ has also two real roots λ_1^* and λ_2^*. From the expression of $F'(\lambda)$, we have $F'(\lambda) > 0$ for all $\lambda \in (-\infty, \lambda_1^*) \cup (\lambda_2^*, \infty)$ and $F'(\lambda) < 0$ for all $\lambda \in (\lambda_1^*, \lambda_2^*)$. Hence, $F(\lambda)$ is increasing for all $\lambda \in (-\infty, \lambda_1^*) \cup (\lambda_2^*, \infty)$ and decreasing for all $\lambda \in (\lambda_1^*, \lambda_2^*)$. Therefore, we finally obtain $F(\lambda_1^*) \geq 0$, $F(\lambda_1^*) \leq 0$, $\lambda_1 \in (-\infty, \lambda_1^*]$, $\lambda_2 \in [\lambda_1^*, \lambda_2^*)$ and $\lambda_3 \in [\lambda_2^*, \infty)$.

If condition (1.i) holds, then we obviously have $\lambda_1 \in (-1, \lambda_1^*]$, $\lambda_2 \in [\lambda_1^*, \lambda_2^*]$ and $\lambda_3 \in [\lambda_2^*, 1)$. Therefore, E_j is a sink.

If condition (2.i.a) holds. When $\lambda_2^* < -1$, we have $\lambda_1 < -1$ and $\lambda_2 < -1$. Since $F(\lambda)$ is increasing for all $\lambda \in [\lambda_2^*, \infty)$ and $F(-1) > 0$, we can obtain $\lambda_3 < -1$. Therefore, E_j is a source. When $\lambda_2^* > -1$, then from $F(-1) > 0$ we have $\lambda_1 < -1$. Hence $F(\lambda) > 0$ for all $\lambda \in (-1, 1)$. Consequently, $\lambda_2 > 1$ and $\lambda_3 > 1$. Therefore, E_j is a source.

By the same way, we prove that when condition (2.i.b) holds, E_j is also a source.

If condition (3.i.a) holds, then, when $\lambda_1^* < -1$ we have $\lambda_1 < -1$ and when $\lambda_2^* > 1$ we have $\lambda_3 > 1$. From $F(1) > 0$ and $F(-1) < 0$ we have $\lambda_2 \in (-1, 1)$. Therefore, E_j is a one-dimensional saddle.

If condition (3.i.b) holds, then we clearly have $\lambda_1 < -1$, $\lambda_2 \in (-1, 1)$ and $\lambda_3 > 1$. Therefore, E_j is a one-dimensional saddle too.

If condition (4.i.a) holds, we have $\lambda_{2,3} \in (-1, 1)$ and $\lambda_1^* \in (-\infty, 1)$, $F(\lambda)$ is increasing for all $\lambda \in (-\infty, \lambda_1^*]$, we obtain $\lambda_1 \in (-\infty, -1)$. Therefore, E_j is a two-dimensional saddle.

By the same way, we can prove that when condition (4.i.b) holds, E_j is also a two-dimensional saddle.

If condition (5.i) holds, then we can easily prove that E_j is non-hyperbolic.

Now, we let $\bar{\Delta}_j > 0$. From Lemma 2, equation (11) has one real root λ_1 and a pair of conjugate complex roots $\lambda_{2,3}$. If condition (1.ii) holds, then from $F(1) > 0$ and $F(-1) < 0$ we have that a real root $\lambda_1 \in (-1, 1)$. Therefore, from $|\lambda_{2,3}| < 1$ we obtain that E_j is a sink.

If condition (2.ii.a) holds, then from $F(-1) < 0$ we have a real root $\lambda_1 > 1$. Therefore, from $|\lambda_{2,3}| > 1$ we obtain that E_j is a source.

By the same way, we can prove that if condition (2.ii.b) holds, then E_j is also a source.

If condition (3.ii.a) holds, then we have a real root $\lambda_1 \in (-1, 1)$. Therefore, from $|\lambda_{2,3}| > 1$ we have that E_j is a one-dimensional saddle.

By the same way, we can prove that when conditions (3.ii.b) and (3.ii.c) hold, then E_j is also a one-dimensional saddle.

If condition (4.ii.a) holds, then from $F(-1) < 0$ we have a real root $\lambda_1 > 1$. Therefore, from $|\lambda_{2,3}| > 1$ we have that E_j is a two-dimensional saddle.

By the same way, we can prove that if condition (4.ii.b) holds, then E_j is also a two-dimensional saddle.

Lastly, we can easily prove that if condition (5.ii) holds, then E_j is non-hyperbolic. □

Theorem 5 *If the positive equilibrium point E_j exists, then E_j loses its stability via*

(i) *a saddle-node bifurcation, if one of the following conditions holds*:
  (i.1) $\bar{\Delta}_j \leq 0$, $F(1) = 0$ *and* $F(-1) \neq 0$,
  (i.2) $\bar{\Delta}_j > 0$, $F(1) = 0$ *and* $|\lambda_{2,3}| \neq 1$;

(ii) *flip bifurcation, if one of the following conditions holds*:
  (ii.1) $\bar{\Delta}_j \leq 0$, $F(-1) = 0$ *and* $F(1) \neq 0$,
  (ii.2) $\bar{\Delta}_j > 0$, $F(-1) = 0$ *and* $|\lambda_{2,3}| \neq 1$;

(iii) *Hopf bifurcation, if the following condition holds*:
  $\bar{\Delta}_j > 0$, $F(-1) \neq 0$, $F(1) \neq 0$ *and* $|\lambda_{2,3}| = 1$.

Proof ([29]) (i) introduced a thorough study of the main types of bifurcations for 3-D maps. In line with this study, we can see that E_j undergoes a saddle-node bifurcation when a single eigenvalue becomes equal to 1. Therefore E_j can lose its stability through the saddle-node bifurcation when one of $\lambda_i = 1$, $i = 1, 2, 3$. A saddle-node bifurcation of E_j may occur if the parameters vary in the small neighborhood of the following sets:

$$S_1 = \{(h, \alpha, \beta, \gamma, \delta, \eta, \sigma) : \bar{\Delta}_j \leq 0, F(1) = 0 \text{ and } F(-1) \neq 0\},$$

or

$$S_2 = \{(h, \alpha, \beta, \gamma, \delta, \eta, \sigma) : \bar{\Delta}_j > 0, F(1) = 0 \text{ and } |\lambda_{2,3}| \neq 1\}.$$

By the same way, we can prove (ii).

(iii) When the Jacobian has a pair of complex conjugate eigenvalues of modulus 1, we get the Hopf bifurcation, then E_j can lose its stability through the Hopf bifurcation when one of $|\lambda_{2,3}| = 1$, and then it also implies that all the parameters locate and vary in the small neighborhood of the following set:

$$S_3 = \{(h, \alpha, \beta, \gamma, \delta, \eta, \sigma) : \bar{\Delta}_j > 0, F(-1) \neq 0, F(1) \neq 0 \text{ and } |\lambda_{2,3}| = 1\}.$$ □

4.2 Numerical simulations

In this section, we give the phase portraits, the attractor of parameter β and bifurcation diagrams to confirm the above theoretical analysis and to obtain more dynamical behaviors of the palm trees, lesser date moth and predator model. Since most of the fractional-order differential equations do not have exact analytic solutions, approximation and numerical techniques must be used.

From the numerical results, it is clear that the approximate solutions depend on the fractional parameters h, α see Figure 1. The approximate solutions x_n, y_n, and z_n are displayed in the figures below.

We use some documented data for some parameters like $\beta = 0.5$, $\gamma = 3$, $\delta = \eta = 1$, and $\sigma = 3$, then we have $(x_1, y_1, z_1) = (0.7, 0.3, 0.8)$. Other parameters will be (a) $h = 0.05$, $\alpha = 0.95$, (b) $h = 0.07$, $\alpha = 0.95$, (c) $h = 0.09$, $\alpha = 0.95$, and (d) $h = 0.09$, $\alpha = 0.75$.

Figure 1 depicts the phase portraits of model (4) according to the chosen parameter values and for various values of the fractional-order parameters h and α. We can see that, whenever the value of α is fixed and the value of h increases, then E_4 moves from the stabilized to the chaotic band. Figure 1(c) depicts the phase portrait for model (4).

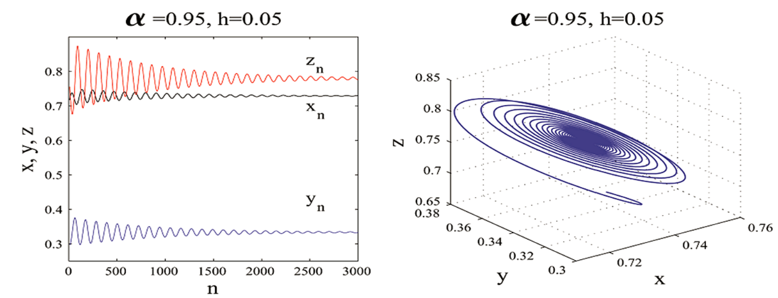
By computing, we have $E_4 \simeq (0.7, 0.33, 0.84)$, and we can get the critical value of flip bifurcation for model (4). In Figure 1(a) we have $\bar{\Delta}_4 = 1{,}153{,}256{,}457 > 0$, $F(1) \simeq 0.00009 > 0$, $F(-1) \simeq -7.872 < 0$, and $|\lambda_{2,3}| \simeq 0.999 < 1$. In this case we get E_4 is a sink according to case (1.ii) in Theorem 4.

In Figure 1(b) we have $\bar{\Delta}_4 = 165{,}217{,}502.2 > 0$, $F(1) \simeq 0.00024 > 0$, $F(-1) \simeq -7.8274 < 0$, and $|\lambda_{2,3}| \simeq 0.999 < 1$. In this case we get E_4 is a sink according to case (1.ii) in Theorem 4.

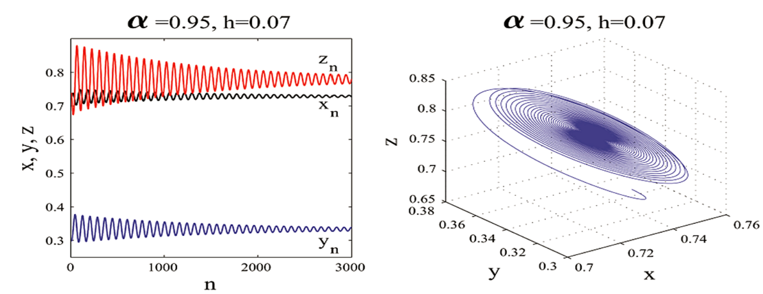
In Figure 1(c) we have $\bar{\Delta}_4 = 38{,}548{,}982.44 > 0$, $F(1) \simeq 0.00049 > 0$, $F(-1) \simeq -7.79 < 0$, and $|\lambda_{2,3}| \simeq 1.0002 > 1$. In this case we get E_4 is a one-dimensional saddle according to case (3.ii.a) in Theorem 4. We see that the fixed point E_4 loses its stability at the Hopf bifurcation parameter value $h \simeq 0.086$. For $h = [0, 0.15]$, there is a cascade of bifurcations. When r increases at certain values, for example, at $h = 0.09$, independent invariant circles appear. When the value of h is increased (Figure 1(c)), the circles break down and some cascades of bifurcations lead to chaos.

Figures 1(c) and 1(d) explain the effect of the parameter α on the behavior of x, y, and z. Figure 2 demonstrates the sensitivity to initial conditions of system (4). We compute two orbits with initial points (x_1, y_1, z_1) and $(x_1, y_1 + 0.01, z_1)$, respectively. The compositional results are shown in Figure 2. From this figure it is clear that at the beginning the time series are indistinguishable; but after a number of iterations, the difference between them builds up rapidly, which shows that the model has sensitive dependence on the initial conditions of model (4), y-coordinates of the two orbits, plotted against time; the y-coordinates of initial conditions differ by 0.01, and the other coordinates do not change. In Figure 3 we use some documented data for some parameters like $\beta = 0.5$, $\gamma = 3$, $\delta = \eta = 1$, $\sigma = 3$, $\alpha = 0.95$, then we have $(x_1, y_1, z_1) = (0.7, 0.3, 0.8)$, another parameter will be $h = 0.001 : 0.47$. Figure 3 describes the Hopf bifurcation diagram with respect to h, we note that, as h increases, the behavior of this model (4) becomes very complicated, and the changes of parameter h has an effect on the stability of system (4). The Hopf bifurcation diagrams of system (4) in the $(h - x)$, $(h - y)$ and $(h - z)$ planes are given in Fig. 3.

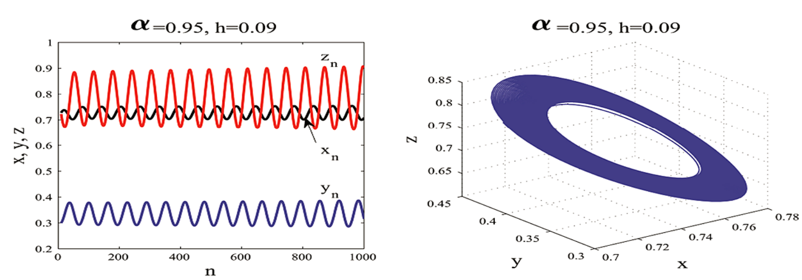
After calculation for the fixed point E_4 of map (4), the Hopf bifurcation emerges from the fixed point $(0.7, 0.33, 0.84)$ at $h = 0.086$ and $(h, \alpha, \beta, \gamma, \delta, \eta, \sigma) \in HB_{E_4}$. From Figure 3, we observe that the fixed point E_4 of map (4) loses its stability through a discrete Hopf



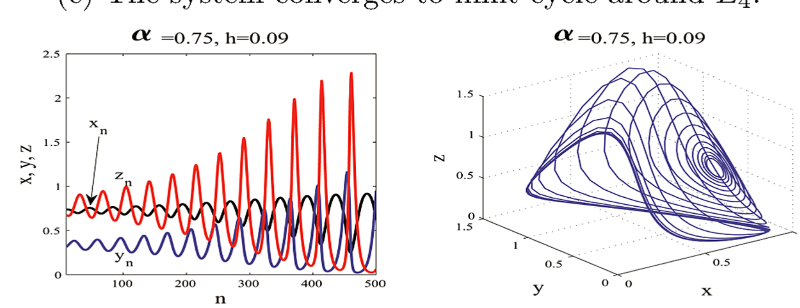
(a) The trajectory of system converges to E_4.



(b) The trajectory of system converges to E_4.



(c) The system converges to limit cycle around E_4.



(d) The trajectory of system (3.4) for $\alpha = 0.75$, $h = 0.09$

Figure 1 Phase portraits of model (4).

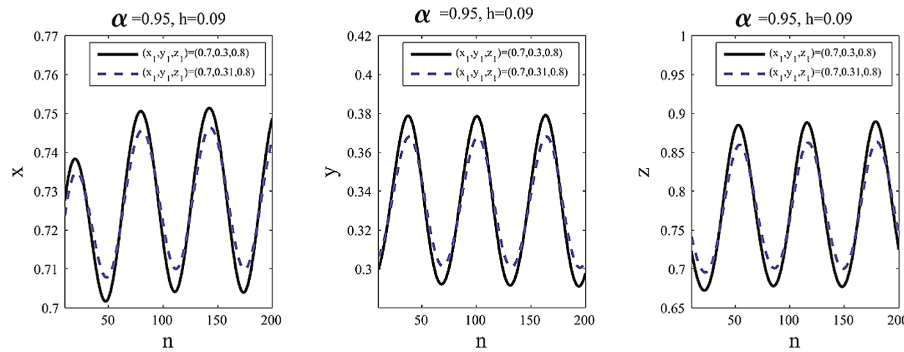


Figure 2 Phase portraits of model (4).

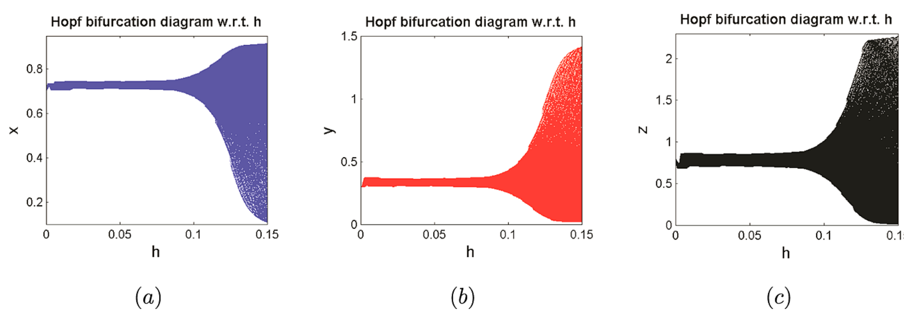


Figure 3 Hopf bifurcation diagram with respect to *h*.

bifurcation for $h = [0, 0.15]$. In Figure 4 we use some documented data for some parameters like $\gamma = 3$, $\delta = \eta = 1$, $\sigma = 3$, $h = 0.85$, $\alpha = 0.95$, other parameters will be (a) $\beta = 1.55$, (b) $\beta = 1.4$, (c) $\beta = 1.2$, and (d) $\beta = 1.15$.

Figure 4(a) describes the stable equilibrium of model (4) according to the values of the parameters set out above. From Figure 4(f), 4(g) we can see that reducing and decreasing β causes disappearance of first-periodic orbits and increase in the chaotic attractors.

In Figure 5, we introduce the 0-1 test for detecting the chaos. Figure 5(a) indicates that, for $\beta = 1.55$, one obtains $K = -0.066 \approx 0$. Then the dynamics is regular. Moreover, Figure 5(b) depicts bounded trajectories in the $(P_c(n), Q_c(n))$ plane. Figure 5(c) indicates that, for $\beta = 1.15$, one obtains $K = 0.9658 \approx 1$. Then the dynamics is chaotic. Moreover, Figure 5(d) depicts Brownian-like (unbounded) trajectories in the $(P_c(n), Q_c(n))$ plane.

In Figure 5(e)-5(f), we show the asymptotic growth rate K as a function of c for regular (chaotic) dynamics. In the case of regular (chaotic) dynamics, most values of c yield $K \approx 0$ ($K \approx 1$) as expected.

Figure 5(e)-5(f) show the two mean square displacements M_c for system (4) with $\beta = 1.55$ ($\beta = 1.15$), which corresponds to regular (chaotic) dynamics.

5 Conclusion

In this paper, we consider the fractional-order model consisting of palm trees, the lesser date moth and the predator. We have got a sufficient condition for the existence and uniqueness of the system solution. We have also studied the local stability of all the equilib-

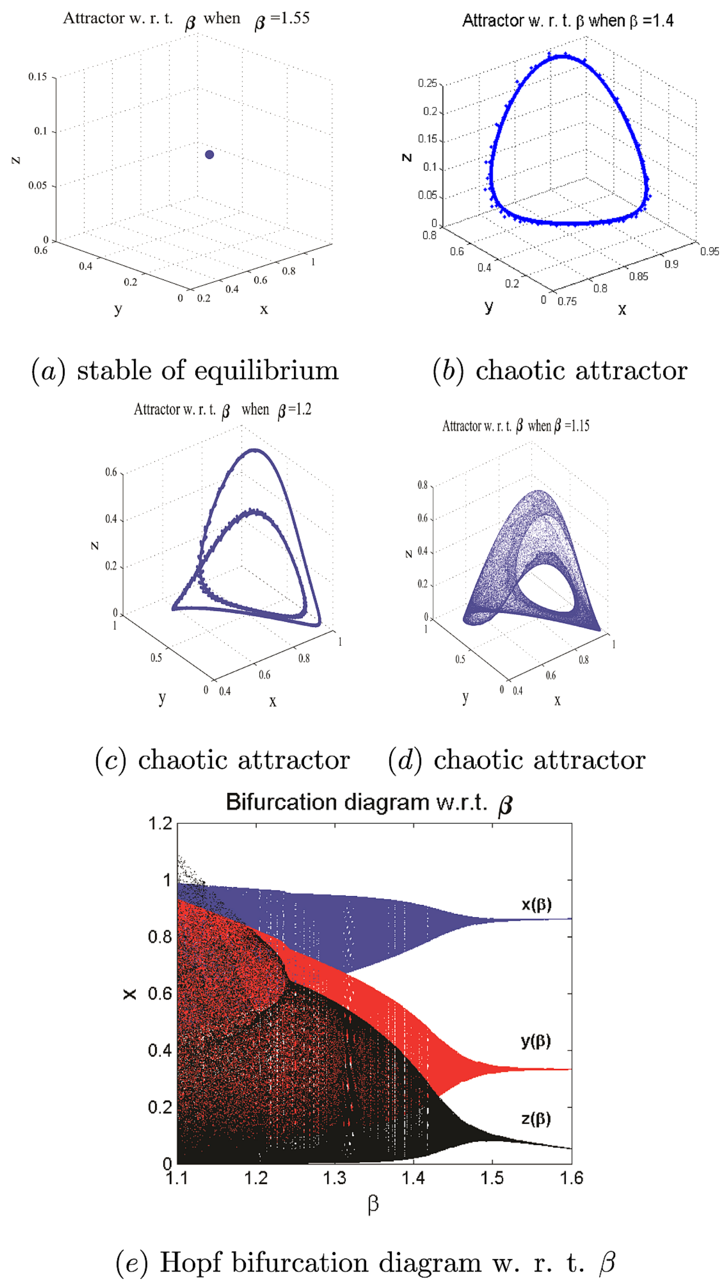
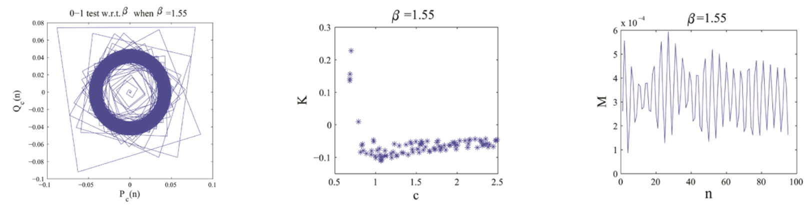


Figure 4 Hopf bifurcation diagram with respect to β.

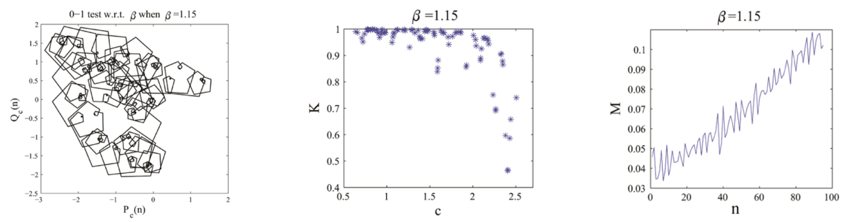
rium states of the discretized fractional-order system. Moreover, it has been found that the fractional parameter α has an effect on the stability of the discretized system. To support our theoretical discussion, we also present numerical simulations. We analyze the bifurcation both by theoretical point of view and by numerical simulations. One also needs to mention that when dealing with real life problems, the order of the system can be determined by using the collected data. The transformation of a classical model into a fractional one makes it very sensitive to the order of differentiation α: a small change in α may result in a big change in the final result. From the numerical results, it is clear that the approximate solutions depend continuously on the fractional derivative α.



(a) Bounded trajectories indicating regular dynamic

(b) Dependence of K on β for a regular dynamic

(c) the mean square displacement versus n corresponding to regular dynamics



(d) Unbounded Brownian-like trajectories indicating chaos

(e) Dependence of K on β for a chaotic dynamics.

(f) the mean square displacement versus n corresponding to chaotic dynamics.

Figure 5 The 0-1 test for detecting chaos.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JJN helped to evaluate, revise and edit the manuscript. ME-S suggested the model, helped in results interpretation and manuscript evaluation. AMA supervised the development of work. IMEA drafted the article.

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