Shen et al. *Advances in Difference Equations* 2013, 2013:295
http://www.advancesindifferenceequations.com/content/2013/1/295

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Solvability of fractional boundary value problem with p-Laplacian operator at resonance

Tengfei Shen, Wenbin Liu* and Xiaohui Shen

*Correspondence: wblium@163.com
College of Sciences, China University of Mining and Technology, Xuzhou, 221116, P.R. China

Abstract
In this paper, a class of multi-point boundary value problems for nonlinear fractional differential equations at resonance with p-Laplacian operator is considered. By using the extension of Mawhin's continuation theorem due to Ge, the existence of solutions is obtained, which enriches previous results.

MSC: 34A08; 34B15

Keywords: fractional differential equation; boundary value problem; p-Laplacian operator; coincidence degree theory; resonance

1 Introduction

In the recent years, fractional differential equations played an important role in many fields such as physics, electrical circuits, biology, control theory, *etc.* (see [1–7]). Thus, many scholars have paid more attention to fractional differential equations and gained some achievements (see [8–22]). For example, Wang [21] considered a class of fractional multi-point boundary value problems at resonance by Mawhin's continuation theorem (see [23]):

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, u(t), D_{0^+}^{\alpha-1}u(t)), \quad t \in (0,1), \text{ a.e. } t \in (0,1), \\ u(0) = 0, \qquad D_{0^+}^{\alpha-1}u(1) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-1}u(\xi_i), \\ D_{0^+}^{\alpha-2}u(1) = \sum_{i=1}^{m} b_i D_{0^+}^{\alpha-2}u(\eta_i), \end{cases} \tag{1.1}$$

where $1 < \alpha \le 2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1$, $\sum_{i=1}^{m} a_i = 1$, $\sum_{i=1}^{n} b_i = 1$, $\sum_{i=1}^{n} b_i\eta_i = 1$, $D_{0^+}^{\alpha}$ is the standard fractional derivative, $f : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ satisfies the Carathéodory condition.

But Mawhin's continuation theorem is not suitable for quasi-linear operators. In [24], Ge and Ren had extended Mawhin's continuation theorem, which was used to deal with more general abstract operator equations. In [25], Pang *et al.* considered a higher order nonlinear differential equation with a p-Laplacian operator at resonance:

$$\begin{cases} (\varphi_p(u^{(n-1)}(t)))' = f(t, u(t), \ldots, u^{(n-1)}(t)) + e(t), \quad t \in (0,1), \\ u^{(i)}(0) = 0, \quad i = 1, 2, \ldots, n-1, \\ u(1) = \int_0^1 u(s)\,dg(s), \end{cases} \tag{1.2}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $f : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ and $e : [0,1] \to \mathbf{R}$ are continuous, $n \geq 2$ is an integer. $g : [0,1] \to \mathbf{R}$ is a nondecreasing function with $\int_0^1 dg(s) = 1$, the integral in the second part of (1.2) is meant in the Riemann-Stieltjes sense.

However, there are few articles which consider the fractional multi-point boundary value problem at resonance with p-Laplacian operator and $\dim \operatorname{Ker} M = 2$, because p-Laplacian operator is a nonlinear operator, and it is hard to construct suitable continuous projectors. In this paper, we will improve and generalize some known results.

Motivated by the work above, our article is to investigate the multi-point boundary value problem at resonance for a class of Riemanne-Liouville fractional differential equations with p-Laplacian operator and $\dim \operatorname{Ker} M = 2$ by constructing suitable continuous projectors and using the extension of Mawhin's continuation theorem:

$$\begin{cases} D_{0^+}^{\beta}\varphi_p(D_{0^+}^{\alpha}u(t)) = f(t, u(t), D_{0^+}^{\alpha-2}u(t), D_{0^+}^{\alpha-1}u(t), D_{0^+}^{\alpha}u(t)), \quad t \in (0,1), \\ u(0) = D_{0^+}^{\alpha}u(0) = 0, \qquad u(1) = \sum_{i=1}^{m} a_i u(\xi_i), \\ D_{0^+}^{\alpha-1}u(1) = \sum_{i=1}^{m} b_i D_{0^+}^{\alpha-1}u(\eta_i), \end{cases} \tag{1.3}$$

where $2 < \alpha \leq 3$, $0 < \beta \leq 1$, $3 < \alpha + \beta \leq 4$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, $a_i \in \mathbf{R}$, $b_i \in \mathbf{R}$, $1 < m$, $m \in N$, $\sum_{i=1}^{m} a_i\xi_i^{\alpha-1} = 1$, $\sum_{i=1}^{m} a_i\xi_i^{\alpha-2} = 1$, $\sum_{i=1}^{m} b_i = 1$, $\varphi_p(s) = |s|^{p-2}s$, $\varphi_p(0) = 0$, $1 < p$, $1/p + 1/q = 1$, φ_p is invertible and its inverse operator is φ_q, $D_{0^+}^{\alpha}$ is Riemann-Liouville standard fractional derivative, $f : [0,1] \times \mathbf{R}^4 \to \mathbf{R}$ is continuous.

In order to investigate the problem, we need to suppose that the following conditions hold:

$$\Lambda = \Lambda_1\Lambda_4 - \Lambda_2\Lambda_3 \neq 0,$$

where

$$\begin{aligned} \Lambda_1 &= \frac{\Gamma(\alpha)^q\Gamma(\alpha q + \beta q - q - \alpha - \beta + 2)}{\Gamma(\alpha + \beta)^{q-1}\Gamma(\alpha q + \beta q - q - \beta + 2)}\left(1 - \sum_{i=1}^{m} a_i\xi_i^{\alpha q+\beta q-q-\beta+1}\right), \\ \Lambda_2 &= \frac{\Gamma(\alpha - 1)^{q-1}\Gamma(\alpha)\Gamma(\alpha q + \beta q - 2q - \alpha - \beta + 3)}{\Gamma(\alpha + \beta - 1)^{q-1}\Gamma(\alpha q + \beta q - 2q - \beta + 3)}\left(1 - \sum_{i=1}^{m} a_i\xi_i^{\alpha q+\beta q-2q-\beta+2}\right), \\ \Lambda_3 &= \frac{\Gamma(\alpha)^{q-1}}{\Gamma(\alpha + \beta)^{q-1}(\alpha q + \beta q - q - \alpha - \beta + 2)}\left(1 - \sum_{i=1}^{m} b_i\eta_i^{\alpha q+\beta q-q-\alpha-\beta+2}\right), \\ \Lambda_4 &= \frac{\Gamma(\alpha - 1)^{q-1}}{\Gamma(\alpha + \beta - 1)^{q-1}(\alpha q + \beta q - 2q - \alpha - \beta + 3)}\left(1 - \sum_{i=1}^{m} b_i\eta_i^{\alpha q+\beta q-2q-\alpha-\beta+3}\right). \end{aligned}$$

The rest of this article is organized as follows: In Section 2, we give some notations, definitions and lemmas. In Section 3, based on the extension of Mawhin's continuation theorem due to Ge, we establish a theorem on existence of solutions for BVP (1.3).

2 Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions for fractional calculus theory that can be found in the recent literature (see [1, 3, 14, 21, 24, 25]).

Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|u\|_Y$, respectively. A continuous operator

$$M|_{\operatorname{dom} M\cap X} : X \cap \operatorname{dom} M \to Y$$

is said to be quasi-linear if

(i) $\operatorname{Im} M := M(X \cap \operatorname{dom} M)$ is a closed subset of Y,

(ii) $\operatorname{Ker} M := \{u \in X \cap \operatorname{dom} M : Mu = 0\}$ is linearly homeomorphic to R^n, $n < \infty$.

Let $X_1 = \operatorname{Ker} M$ and X_2 be the complement space of X_1 in X, then $X = X_1 \oplus X_2$. On the other hand, suppose that Y_1 is a subspace of Y, and Y_2 is the complement space of Y_1 in Y, so that $Y = Y_1 \oplus Y_2$. Let $P : X \to X_1$ be a projector and $Q : Y \to Y_1$ a semi-projector, and $\Omega \subset X$ an open and bounded set with origin $\theta \in \Omega$. θ is the origin of a linear space.

Suppose that $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is a continuous operator. Denote N_1 by N. Let $\Sigma_\lambda = \{u \in \overline{\Omega} : Mu = N_\lambda u\}$. N_λ is said to be M-compact in $\overline{\Omega}$ if there is an $Y_1 \subset Y$ with $\dim Y_1 = \dim X_1$ and an operator $R : \overline{\Omega} \times [0,1] \to X$ continuous and compact such that for $\lambda \in [0,1]$,

$$(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Y, \tag{2.1}$$

$$QN_\lambda x = \theta, \quad \lambda \in (0,1) \quad \Leftrightarrow \quad QNx = \theta, \tag{2.2}$$

$$R(\cdot,\lambda)|_{\Sigma_\lambda} = (I-P)|_{\Sigma_\lambda} \tag{2.3}$$

and $R(\cdot,0)$ is the zero operator,

$$M\big[P + R(\cdot,\lambda)\big] = (I-Q)N_\lambda. \tag{2.4}$$

Lemma 2.1 (Ge-Mawhin's continuation theorem [24]) *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces, and $\Omega \subset X$ an open and bounded nonempty set. Suppose that $M : X \cap \operatorname{dom} M \to Y$ is a quasi-linear operator $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is M-compact in $\overline{\Omega}$. In addition, if*

(i) *$Lu \neq N_\lambda u$, $\forall (u,\lambda) \in (\operatorname{dom} M \cap \partial\Omega) \times (0,1)$,*

(ii) *$\deg(JQN, \operatorname{Ker} M \cap \Omega, 0) \neq 0$,*

where $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism with $J(\theta) = \theta$ and $N = N_1$, then the equation $Mu = Nu$ has at least one solution in $\operatorname{dom} M \cap \overline{\Omega}$.

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function u is given by

$$I_{0^+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,ds,$$

provided the right-hand side integral is pointwise almost everywhere defined on $(0,+\infty)$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function u is given by

$$D_{0^+}^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

provided the right-hand side integral is pointwise everywhere defined on $(0,+\infty)$, where $n = [\alpha] + 1$.

Definition 2.3 Let X be a Banach space, and $X_1 \subset X$ is a subspace. A mapping $Q : X \to X_1$ is a semi-projector if Q satisfies

(i) $Q^2x = Qx$, $\forall x \in X$,
(ii) $Q(\mu x) = \mu Qx$, $\forall x \in X$, $\mu \in \mathbf{R}$.

Lemma 2.2 *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_Nt^{\alpha-N}$$

for some $c_i \in \mathbf{R}$, $i = 1,2,\ldots,N$, where $N = [\alpha] + 1$.

Lemma 2.3 *Assume that $u(t) \in C[0,1]$, $0 \le p \le q$, then*

$$D^{q}_{0^+}I^{p}_{0^+}u(t) = I^{p-q}_{0^+}u(t).$$

Lemma 2.4 *Assume that $\alpha \ge 0$, then*:

(i) *If $\lambda > -1$, $\lambda \neq \alpha - i$, $i = 1,2,\ldots,[\alpha] + 1$, we have that*

$$D^{\alpha}_{0^+}t^{\lambda} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \alpha + 1)}t^{\lambda-\alpha}.$$

(ii) $D^{\alpha}_{0^+}t^{\alpha-i} = 0$, $i = 1,2,\ldots,[\alpha] + 1$.

In this paper, we take $X = \{u \mid u, D^{\alpha-2}_{0^+}u, D^{\alpha-1}_{0^+}u, D^{\alpha}_{0^+}u \in C[0,1]\}$ with the norm $\|u\|_X = \max\{\|u\|_\infty, \|D^{\alpha-2}_{0^+}u\|_\infty\|D^{\alpha-1}_{0^+}u\|_\infty, \|D^{\alpha}_{0^+}u\|_\infty\}$, where $\|u\|_\infty = \max_{t\in[0,1]}|u(t)|$, and $Y = C[0,1]$ with the norm $\|y\|_Y = \|y\|_\infty$. By means of the linear functional analysis theory, it is easy to prove that X and Y are Banach spaces, so we omit it.

Define the operator $M : \operatorname{dom} M \to Y$ by

$$Mu = D^{\beta}_{0^+}\varphi_p\bigl(D^{\alpha}_{0^+}u(t)\bigr), \tag{2.5}$$

$$\operatorname{dom} M = \left\{ u \in X \,\middle|\, D^{\beta}_{0^+}\varphi_p\bigl(D^{\alpha}_{0^+}u\bigr) \in Y, u(0) = D^{\alpha}_{0^+}u(0) = 0, \right.$$
$$\left. u(1) = \sum_{i=1}^{m} a_iu(\xi_i), D^{\alpha-1}_{0^+}u(1) = \sum_{i=1}^{m} b_iD^{\alpha-1}_{0^+}u(\eta_i) \right\}. \tag{2.6}$$

Based on the definition of $\operatorname{dom} M$, it is easy to find that $\operatorname{dom} M \neq \varnothing$ such as $u(t) = ct^{\alpha-1} \in \operatorname{dom} M$, $c \in \mathbf{R}$.

Define the operator $N_\lambda : X \to Y$, $\lambda \in [0,1]$,

$$N_\lambda u(t) = \lambda f\bigl(t, u(t), D^{\alpha-2}_{0^+}u(t), D^{\alpha-1}_{0^+}u(t), D^{\alpha}_{0^+}u(t)\bigr), \quad t \in [0,1].$$

Then BVP (1.3) is equivalent to the operator equation $Mu = Nu$, where $N = N_1$.

3 Main result

In this section, a theorem on existence of solutions for BVP (1.3) will be given.

Define operators $T_j : Y \to Y$, $j = 1, 2$ as follows:

$$T_1 y = \int_0^1 (1-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}y(\tau)\,d\tau\right)ds - \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}y(\tau)\,d\tau\right)ds,$$

$$T_2 y = \int_0^1 \varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}y(\tau)\,d\tau\right)ds - \sum_{i=1}^m b_i \int_0^{\eta_i} \varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}y(\tau)\,d\tau\right)ds.$$

Let us make some assumptions, which will be used in the sequel.

(H1) There exist nonnegative functions $r, d, e, h, k \in Y$ such that for all $t \in [0,1]$, $(u, v, w, z) \in \mathbf{R}^4$,

$$\left|f(t,u,v,w,z)\right| \le r(t) + d(t)|u|^{p-1} + e(t)|v|^{p-1} + h(t)|w|^{p-1} + k(t)|z|^{p-1}.$$

(H2) There exists a constant $A > 0$ such that for $u \in \operatorname{dom} M$, if $|D_{0^+}^{\alpha-1}u(t)| > A$ for all $t \in [0,1]$, then

$$\operatorname{sgn}\left\{D_{0^+}^{\alpha-1}u(t)\right\}\frac{1}{\Lambda}\left(\Lambda_4 T_1 Nu(t) - \Lambda_3 T_2 Nu(t)\right) > 0$$

or

$$\operatorname{sgn}\left\{D_{0^+}^{\alpha-1}u(t)\right\}\frac{1}{\Lambda}\left(\Lambda_4 T_1 Nu(t) - \Lambda_3 T_2 Nu(t)\right) < 0.$$

(H3) There exists a constant $B > 0$ such that for $u \in \operatorname{dom} M$, if $|D_{0^+}^{\alpha-2}u(t)| > B$ for all $t \in [0,1]$, then

$$\operatorname{sgn}\left\{D_{0^+}^{\alpha-2}u(t)\right\}\frac{1}{\Lambda}\left(-\Lambda_2 T_1 Nu(t) + \Lambda_1 T_2 Nu(t)\right) > 0$$

or

$$\operatorname{sgn}\left\{D_{0^+}^{\alpha-2}u(t)\right\}\frac{1}{\Lambda}\left(-\Lambda_2 T_1 Nu(t) + \Lambda_1 T_2 Nu(t)\right) < 0.$$

Theorem 3.1 *Let $f : [0,1] \times \mathbf{R}^4 \to \mathbf{R}$ be continuous and condition* (H1)-(H3) *hold, then BVP* (1.3) *has at least one solution, provided that*

$$\frac{1}{\Gamma(\beta+1)}\left(A_1\|d\|_\infty + \|e\|_\infty + \|h\|_\infty + \|k\|_\infty\right) < 1, \tag{3.1}$$

where $A_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{2}{\Gamma(\alpha)} + \frac{7}{2\Gamma(\alpha-1)}$.

In order to prove Theorem 3.1, we need to prove some lemmas below.

Lemma 3.1 *The operator $M : \operatorname{dom} M \cap X \to Y$ is quasi-linear.*

$$\operatorname{Ker} M = \{u \in X \mid u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, c_1, c_2 \in \mathbf{R}\}, \tag{3.2}$$

$$\operatorname{Im} M = \{y \in Y \mid T_j y = 0, j = 1, 2\}. \tag{3.3}$$

Proof Suppose that $u(t) \in \operatorname{dom} M$, by $D_{0^+}^{\beta} \varphi_p(D_{0^+}^{\alpha} u(t)) = 0$, we have

$$D_{0^+}^{\alpha} u(t) = \varphi_q(c_0 t^{\beta-1}).$$

Based on $D_{0^+}^{\alpha} u(0) = 0$, one has

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3},$$

which together with $u(0) = 0$ yields that

$$\operatorname{Ker} M = \{u \in X \mid u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, c_1, c_2 \in \mathbf{R}\}.$$

It is clear that $\dim \operatorname{Ker} M = 2$. So, $\operatorname{Ker} M$ is linearly homeomorphic to $\mathbf{R}^2$.

If $y \in \operatorname{Im} M$, then there exists a function $u \in \operatorname{dom} M$ such that $y(t) = D_{0^+}^{\beta} \varphi_p(D_{0^+}^{\alpha} u(t))$. Based on Lemmas 2.2 and 2.3, we have

$$u(t) = I_{0^+}^{\alpha} \varphi_q(I_{0^+}^{\beta} y(s)) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2},$$
$$D_{0^+}^{\alpha-1} u(t) = D_{0^+}^{\alpha-1} I_{0^+}^{\alpha} \varphi_q(I_{0^+}^{\beta} y(s)) + c_1 \Gamma(\alpha),$$

which together with $\sum_{i=1}^{m} a_i \xi_i^{\alpha-1} = 1$, $\sum_{i=1}^{m} a_i \xi_i^{\alpha-2} = 1$ and $\sum_{i=1}^{m} b_i = 1$ yields that $T_j y(t) = 0$, $j = 1, 2$.

On the other hand, suppose that $y \in Y$ and satisfies (3.3), and let $u(t) = I_{0^+}^{\alpha} \varphi_q(I_{0^+}^{\beta} y(t))$, then $u \in \operatorname{dom} M$ and $Mu(t) = D_{0^+}^{\beta} \varphi_p(D_{0^+}^{\alpha} u(t)) = y$, so $y \in \operatorname{Im} M$ and $\operatorname{Im} M := M(\operatorname{dom} M)$ is a closed subset of Y. Thus, M is a quasi-linear operator. □

Lemma 3.2 *Let $\Omega \subset X$ be an open and bounded set, then N_λ is M-compact in $\overline{\Omega}$.*

Proof Define the continuous projector $P : X \to X_1$ by

$$Pu(t) = \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} u(0) t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)} D_{0^+}^{\alpha-2} u(0) t^{\alpha-2}, \quad t \in [0,1].$$

Define the continuous projector $Q : Y \to Y_1$, by

$$Qy(t) = (Q_1 y(t)) t^{\alpha-1} + (Q_2 y(t)) t^{\alpha-2}, \quad t \in [0,1],$$

where

$$Q_1 y(t) = \varphi_p\left(\frac{1}{\Lambda}(\Lambda_4 T_1 y(t) - \Lambda_3 T_2 y(t))\right),$$
$$Q_2 y(t) = \varphi_p\left(\frac{1}{\Lambda}(-\Lambda_2 T_1 y(t) + \Lambda_1 T_2 y(t))\right).$$

Obviously, $X_1 = \operatorname{Ker} M = \operatorname{Im} P$ and $Y_1 = \operatorname{Im} Q$. Thus, we have $\dim Y_1 = \dim X_1 = 2$. For any $y \in Y$, we have

$$\begin{aligned} Q_1\bigl(Q_1 y(t)t^{\alpha-1}\bigr) &= \varphi_p\biggl(\frac{1}{\Lambda}\bigl(\Lambda_4 T_1\bigl(Q_1 y(t)t^{\alpha-1}\bigr) - \Lambda_3 T_2\bigl(Q_1 y(t)t^{\alpha-1}\bigr)\bigr)\biggr) \\ &= Q_1 y(t)\varphi_p\biggl(\frac{1}{\Lambda}(\Lambda_4\Lambda_1 - \Lambda_3\Lambda_2)\biggr) = Q_1 y(t). \end{aligned}$$

Similarly, we can get

$$Q_1\bigl(Q_2 y(t)t^{\alpha-2}\bigr) = 0, \qquad Q_2\bigl(Q_1 y(t)t^{\alpha-1}\bigr) = 0, \qquad Q_2\bigl(Q_2 y(t)t^{\alpha-2}\bigr) = Q_2 y(t).$$

Hence, the map Q is idempotent. Similarly, we can get $Q(\mu y) = \mu Qy$, for all $y \in Y$, $\mu \in \mathbf{R}$. Thus, Q is a semi-projector. For any $y \in \operatorname{Im} M$, we can get that $Qy = 0$ and $y \in \operatorname{Ker} Q$, conversely, if $y \in \operatorname{Ker} Q$, we can obtain that $Qy = 0$, that is to say, $y \in \operatorname{Im} M$. Thus, $\operatorname{Ker} Q = \operatorname{Im} M$. Let $\Omega \subset X$ be an open and bounded set with $\theta \in \Omega$, for each $u \in \overline{\Omega}$, we can get $Q[(I-Q)N_\lambda(u)] = 0$. Thus, $(I-Q)N_\lambda(u) \in \operatorname{Im} M = \operatorname{Ker} Q$. Take any $y \in \operatorname{Im} M$ in the type $y = (y - Qy) + Qy$, since $Qy = 0$, we can get $y \in (I-Q)Y$. So, (2.1) holds. It is easy to verify (2.2).

Define $R : \overline{\Omega} \times [0,1] \to X_2$ by

$$R(u,\lambda)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\biggl(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\bigl((I-Q)N_\lambda u(\tau)\bigr)\,d\tau\biggr)\,ds.$$

By the continuity of f, it is easy to get that $R(u,\lambda)$ is continuous on $\overline{\Omega} \times [0,1]$. Moreover, for all $u \in \overline{\Omega}$, there exists a constant $T > 0$ such that $|I_{0^+}^{\beta}(I-Q)N_\lambda u(\tau)| \le T$, so, we can easily obtain that $R(\overline{\Omega},\lambda)$, $D_{0^+}^{\alpha-2}R(\overline{\Omega},\lambda)$, $D_{0^+}^{\alpha-1}R(\overline{\Omega},\lambda)$ and $D_{0^+}^{\alpha}R(\overline{\Omega},\lambda)$ are uniformly bounded. By Arzela-Ascoli theorem, we just need to prove that $R : \overline{\Omega} \times [0,1] \to X_2$ is equicontinuous.

For $u \in \overline{\Omega}$, $0 < t_1 < t_2 \le 1$, $2 < \alpha \le 3$, $0 < \beta \le 1$, $3 < \alpha + \beta \le 4$, we have

$$\begin{aligned} &\bigl|R(u,\lambda)(t_2) - R(u,\lambda)(t_1)\bigr| \\ &\quad = \frac{1}{\Gamma(\alpha)}\biggl|\int_0^{t_2} (t_2-s)^{\alpha-1}\varphi_q\bigl(I_{0^+}^{\beta}\bigl((I-Q)N_\lambda u(\tau)\bigr)\bigr)\,ds \\ &\qquad - \int_0^{t_1} (t_1-s)^{\alpha-1}\varphi_q\bigl(I_{0^+}^{\beta}\bigl((I-Q)N_\lambda u(\tau)\bigr)\bigr)\,ds\biggr| \\ &\quad \le \frac{\varphi_q(L)}{\Gamma(\alpha)}\biggl(\int_0^{t_1} \bigl((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\bigr)\,ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\,ds\biggr) \\ &\quad = \frac{\varphi_q(T)}{\Gamma(\alpha+1)}\bigl(t_2^{\alpha} - t_1^{\alpha}\bigr), \\ &\bigl|D_{0^+}^{\alpha-2}R(u,\lambda)(t_2) - D_{0^+}^{\alpha-2}R(u,\lambda)(t_1)\bigr| \\ &\quad = \biggl|\int_0^{t_2} (t-s)\varphi_q\bigl(I_{0^+}^{\beta}\bigl((I-Q)N_\lambda u(\tau)\bigr)\bigr)\,ds - \int_0^{t_1} (t-s)\varphi_q\bigl(I_{0^+}^{\beta}\bigl((I-Q)N_\lambda u(\tau)\bigr)\bigr)\,ds\biggr| \\ &\quad \le \varphi_q(T)\biggl(\int_0^{t_1} (t_2-s) - (t_1-s)\,ds + \int_{t_1}^{t_2} (t_2-s)\,ds\biggr) \\ &\quad = \frac{\varphi_q(T)}{2}\bigl(t_2^2 - t_1^2\bigr) \end{aligned}$$

and

$$\begin{aligned}
&\left|D_{0^+}^{\alpha-1}R(u,\lambda)(t_2) - D_{0^+}^{\alpha-1}R(u,\lambda)(t_1)\right| \\
&= \left|\int_0^{t_2} \varphi_q\big(I_{0^+}^{\beta}\big((I-Q)N_\lambda u(\tau)\big)\big)\,ds - \int_0^{t_1} \varphi_q\big(I_{0^+}^{\beta}\big((I-Q)N_\lambda u(\tau)\big)\big)\,ds\right| \\
&\le \varphi_q(T)(t_2 - t_1).
\end{aligned}$$

Since t^α is uniformly continuous on $[0,1]$, so, $R(\overline{\Omega},\lambda)$, $D_{0^+}^{\alpha-2}R(\overline{\Omega},\lambda)$ and $D_{0^+}^{\alpha-1}R(\overline{\Omega},\lambda)$ are equicontinuous. Similarly, we can get that $I_{0^+}^{\beta}((I-Q)N_\lambda u(\tau)) \subset C[0,1]$ is equicontinuous. Considering that $\varphi_q(s)$ is uniformly continuous on $[-T,T]$, we have that $D_{0^+}^{\alpha}R(\overline{\Omega},\lambda) = I_{0^+}^{\beta}((I-Q)N_\lambda(\overline{\Omega}))$ is also equicontinuous. So, we can obtain that $R : \overline{\Omega} \times [0,1] \to X_2$ is compact.

For each $u \in \Sigma_\lambda$, we have $D_{0^+}^{\beta}\varphi_p(D_{0^+}^{\alpha}u(t)) = N_\lambda(u(t)) \in \operatorname{Im} M$. Thus,

$$\begin{aligned}
R(u,\lambda)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\big((I-Q)N_\lambda u(\tau)\big)\,d\tau\right)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}D_{0^+}^{\beta}\varphi_p\big(D_{0^+}^{\alpha}u(\tau)\big)\,d\tau\right)ds,
\end{aligned}$$

which together with $D_{0^+}^{\alpha}u(0) = u(0) = 0$ yields that

$$R(u,\lambda)(t) = u(t) - \frac{1}{\Gamma(\alpha)}D_{0^+}^{\alpha-1}u(0)t^{\alpha-1} - \frac{1}{\Gamma(\alpha-1)}D_{0^+}^{\alpha-2}u(0)t^{\alpha-2} = (I-P)u(t).$$

It is easy to verify that $R(u,0)(t)$ is the zero operator. So, (2.3) holds. Besides, for any $u \in \overline{\Omega}$,

$$\begin{aligned}
&M\big[Pu + R(u,\lambda)\big](t) \\
&\quad= M\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}\big((I-Q)N_\lambda u(\tau)\big)\,d\tau\right)ds\right. \\
&\qquad\left. + \frac{1}{\Gamma(\alpha)}D_{0^+}^{\alpha-1}u(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)}D_{0^+}^{\alpha-2}u(0)t^{\alpha-2}\right] \\
&\quad= (I-Q)N_\lambda u(t),
\end{aligned}$$

which implies (2.4). So, N_λ is M-compact in $\overline{\Omega}$. □

Lemma 3.3 *Suppose that* (H1), (H2) *hold, then the set*

$$\Omega_1 = \big\{u \in \operatorname{dom} M \setminus \operatorname{Ker} M \mid Mu = N_\lambda u, \lambda \in (0,1)\big\}$$

is bounded.

Proof By Lemma 2.2, for each $u \in \operatorname{dom} M$, we have

$$u(t) = I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

Combined with $u(0) = 0$, we get $c_3 = 0$. Thus,

$$
\begin{aligned}
&u(t) = I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \\
&D_{0^+}^{\alpha-1} u(t) = I_{0^+}^{1} D_{0^+}^{\alpha} u(t) + c_1 \Gamma(\alpha), \\
&D_{0^+}^{\alpha-2} u(t) = I_{0^+}^{2} D_{0^+}^{\alpha} u(t) + c_1 \Gamma(\alpha) t + c_2 \Gamma(\alpha - 1).
\end{aligned}
$$

By simple calculation, we get

$$
\begin{aligned}
&c_1 = \frac{1}{\Gamma(\alpha)} \left(D_{0^+}^{\alpha-1} u(t) - \int_0^t D_{0^+}^{\alpha} u(s)\, ds \right), \\
&c_2 = \frac{1}{\Gamma(\alpha-1)} \left(D_{0^+}^{\alpha-2} u(t) - \int_0^t (t-s) D_{0^+}^{\alpha-1} u(s)\, ds - \left(D_{0^+}^{\alpha-1} u(t) - \int_0^t D_{0^+}^{\alpha} u(s)\, ds \right) t \right).
\end{aligned}
$$

Take any $u \in \Omega_1$, then $Nu \in \operatorname{Im} M = \operatorname{Ker} Q$ and $QNu = 0$. It follows from (H2) and (H3) that there exist $\varepsilon_1, \varepsilon_2 \in [0,1]$ such that $|D_{0^+}^{\alpha-1} u(\varepsilon_1)| \leq A$, $|D_{0^+}^{\alpha-2} u(\varepsilon_2)| \leq B$. Thus,

$$
\begin{aligned}
&D_{0^+}^{\alpha-1} u(t) = D_{0^+}^{\alpha-1} u(\varepsilon_1) + \int_{\varepsilon_1}^t D_{0^+}^{\alpha} u(t)\, dt, \\
&D_{0^+}^{\alpha-2} u(t) = D_{0^+}^{\alpha-2} u(\varepsilon_2) + \int_{\varepsilon_2}^t D_{0^+}^{\alpha-1} u(t)\, dt, \\
&\left\| D_{0^+}^{\alpha-1} u \right\|_\infty \leq A + \left\| D_{0^+}^{\alpha} u \right\|_\infty, \\
&\left\| D_{0^+}^{\alpha-2} u \right\|_\infty \leq B + \left\| D_{0^+}^{\alpha-1} u \right\|_\infty \leq A + B + \left\| D_{0^+}^{\alpha} u \right\|_\infty.
\end{aligned}
$$

So, we get

$$
\begin{aligned}
|c_1| &\leq \frac{1}{\Gamma(\alpha)} \left(\left\| D_{0^+}^{\alpha-1} u \right\|_\infty + \left\| D_{0^+}^{\alpha} u \right\|_\infty \right) \leq \frac{1}{\Gamma(\alpha)} \left(A + 2 \left\| D_{0^+}^{\alpha} u \right\|_\infty \right), \\
|c_2| &\leq \frac{1}{\Gamma(\alpha-1)} \left(\left\| D_{0^+}^{\alpha-2} u \right\|_\infty + \frac{3}{2} \left\| D_{0^+}^{\alpha-1} u \right\|_\infty + \left\| D_{0^+}^{\alpha} u \right\|_\infty \right) \\
&\leq \frac{1}{\Gamma(\alpha-1)} \left(\frac{5A}{2} + B + \frac{7}{2} \left\| D_{0^+}^{\alpha} u \right\|_\infty \right), \\
\|u\|_\infty &\leq A_1 \left\| D_{0^+}^{\alpha} u \right\|_\infty + B_1,
\end{aligned}
$$

where $A_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{2}{\Gamma(\alpha)} + \frac{7}{2\Gamma(\alpha-1)}$, $B_1 = \frac{A}{\Gamma(\alpha)} + \frac{5A}{2\Gamma(\alpha-1)} + \frac{B}{\Gamma(\alpha-1)}$.

Based on $D_{0^+}^{\alpha} u(0) = 0$, we have

$$
\varphi_p\left(D_{0^+}^{\alpha} u(t) \right) = \lambda I_{0^+}^{\beta} Nu(t).
$$

From (H1) and $\lambda \in (0,1)$, we have

$$
\begin{aligned}
&\left| \varphi_p\left(D_{0^+}^{\alpha} u(t) \right) \right| \\
&\quad\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left| f\left(s, u(s), D_{0^+}^{\alpha-2} u(t), D_{0^+}^{\alpha-1} u(s), D_{0^+}^{\alpha} u(s) \right) \right| ds \\
&\quad\leq \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(r(s) + d(s) \left| u(s) \right|^{p-1} + e(s) \left| D_{0^+}^{\alpha-2} u(s) \right|^{p-1} \right.
\end{aligned}
$$

$$\begin{aligned}
&\quad + h(s)\left|D_{0^+}^{\alpha-1}u(s)\right|^{p-1} + k(s)\left|D_{0^+}^{\alpha}u(s)\right|^{p-1}\right) ds \\
&\leq \frac{1}{\Gamma(\beta+1)}\left(\|r\|_\infty + \|d\|_\infty\|u\|_\infty^{p-1} + \|e\|_\infty\left\|D_{0^+}^{\alpha-2}u\right\|_\infty^{p-1}\right. \\
&\quad \left. + \|h\|_\infty\left\|D_{0^+}^{\alpha-1}u\right\|_\infty^{p-1} + \|k\|_\infty\left\|D_{0^+}^{\alpha}u\right\|_\infty^{p-1}\right),
\end{aligned}$$

which together with $|\varphi_p(D_{0^+}^{\alpha}u(t))| = |D_{0^+}^{\alpha}u(t)|^{p-1}$, we can get

$$\begin{aligned}
\left\|D_{0^+}^{\alpha}u\right\|_\infty^{p-1} &\leq \frac{1}{\Gamma(\beta+1)}\left(\|r\|_\infty + \|d\|_\infty\|u\|_\infty^{p-1} + \|e\|_\infty\left\|D_{0^+}^{\alpha-2}u\right\|_\infty^{p-1}\right. \\
&\quad \left. + \|h\|_\infty\left\|D_{0^+}^{\alpha-1}u\right\|_\infty^{p-1} + \|k\|_\infty\left\|D_{0^+}^{\alpha}u\right\|_\infty^{p-1}\right) \\
&\leq \frac{1}{\Gamma(\beta+1)}\left(\|r\|_\infty + \|d\|_\infty\left(A_1\left\|D_{0^+}^{\alpha}u\right\|_\infty + B_1\right) + \|e\|_\infty\left(2A + \left\|D_{0^+}^{\alpha}u\right\|_\infty\right)^{p-1}\right. \\
&\quad \left. + \|h\|_\infty\left(A + \left\|D_{0^+}^{\alpha}u\right\|_\infty\right)^{p-1} + \|k\|_\infty\left\|D_{0^+}^{\alpha}u\right\|_\infty^{p-1}\right).
\end{aligned}$$

In view of (3.1), we can obtain that there exists a constant $M_1 > 0$ such that

$$\begin{aligned}
&\left\|D_{0^+}^{\alpha}u\right\|_\infty \leq M_1, \qquad \left\|D_{0^+}^{\alpha-1}u\right\|_\infty \leq A + M_1 := M_2, \\
&\left\|D_{0^+}^{\alpha-2}u\right\|_\infty \leq 2A + M_1 := M_3, \qquad \|u\|_\infty \leq A_1M_1 + B_1 := M_4.
\end{aligned}$$

Thus, we have

$$\|u\|_X = \max\left\{\|u\|_\infty, \left\|D_{0^+}^{\alpha-2}u\right\|_\infty, \left\|D_{0^+}^{\alpha-1}u\right\|_\infty, \left\|D_{0^+}^{\alpha}u\right\|_\infty\right\} \leq \max\{M_1, M_2, M_3, M_4\} := M.$$

So, Ω_1 is bounded. □

Lemma 3.4 *Suppose that* (H2) *holds, then the set*

$$\Omega_2 = \{u \mid u \in \operatorname{Ker} M, Nu \in \operatorname{Im} M\}$$

is bounded.

Proof For each $u \in \Omega_2$, we have that $u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2}$, $c_1, c_2 \in \mathbf{R}$ and $QNu = 0$. It follows from (H2) and (H3) that there exists an $\varepsilon_1, \varepsilon_2 \in [0,1]$ such that $|D_{0^+}^{\alpha-1}u(\varepsilon_1)| \leq A$, $|D_{0^+}^{\alpha-2}u(\varepsilon_2)| \leq A$, which implies that $|c_1| \leq \frac{A}{\Gamma(\alpha)}$ and $|c_2| \leq \frac{A+B}{\Gamma(\alpha-1)}$. So, Ω_2 is bounded. □

Define the isomorphism $J^{-1} : \operatorname{Ker} M \to \operatorname{Im} Q$ by $J^{-1}(c_1t^{\alpha-1} + c_2t^{\alpha-2}) = c_1t^{\alpha-1} + c_2t^{\alpha-2}$, $c_1, c_2 \in \mathbf{R}$, $t \in [0,1]$. In fact, for each $c_1, c_2 \in R$, suppose that $(Q_1y(t), Q_2y(t)) = (c_1, c_2)$, we have

$$\begin{cases} \Lambda_4T_1y(t) - \Lambda_3T_2y(t) = \Lambda\varphi_q(c_1) := \widetilde{c_1}, \\ -\Lambda_2T_1y(t) + \Lambda_1T_2y(t) = \Lambda\varphi_q(c_2) := \widetilde{c_2}, \end{cases} \tag{3.4}$$

where $\widetilde{c_1}, \widetilde{c_2} \in \mathbf{R}$, by the condition $\Lambda_4\Lambda_1 - \Lambda_2\Lambda_3 \neq 0$, there exists a unique solution for (3.4), which is $(T_1y(t), T_2y(t)) = (m_1, m_2)$, $m_1, m_2 \in \mathbf{R}$. Now, we will prove that there exists $y \in Y$ such that $(T_1y(t), T_2y(t)) = (m_1, m_2)$. Based on $y(t) \in C[0,1]$, we choose $y(t) = D_{0^+}^{\beta}\overline{y}(t)$, where $\overline{y}(t) = \varphi_p(l_1 t^{(\alpha+\beta-1)(q-1)} + l_2 t^{(\alpha+\beta-2)(q-1)})$, $l_1 = \frac{\Delta_1\Gamma(\alpha)^{q-1}}{\Gamma(\alpha+\beta)^{q-1}}$, $l_2 = \frac{\Delta_2\Gamma(\alpha-1)^{q-1}}{\Gamma(\alpha+\beta-1)^{q-1}}$, $\Delta_1 = \frac{m_1\Lambda_4 - m_2\Lambda_2}{\Lambda}$, $\Delta_2 = \frac{m_2\Lambda_1 - m_1\Lambda_3}{\Lambda}$, $t \in [0,1]$, which together with $2 < \alpha \le 3$, $0 < \beta \le 1$, $3 < \alpha + \beta \le 4$, $q > 1$, we have $(\alpha+\beta-1)(q-1) > 0$ and $(\alpha+\beta-2)(q-1) > 0$. So, we have $\overline{y}(0) = 0$. Thus, we can obtain that

$$I_{0^+}^{\beta} y(t) = I_{0^+}^{\beta} D_{0^+}^{\beta} \overline{y}(t) = \overline{y}(t) + ct^{\beta-1},$$

where $c \in \mathbf{R}$, which together with $\overline{y}(0) = 0$, we have $c = 0$. So, we have

$$\begin{cases} T_1 y(t) = \int_0^1 (1-s)^{\alpha-1}\varphi_q(\overline{y}(s))\,ds - \sum_{i=1}^m a_i \int_0^{\xi_i} (\xi_i - s)^{\alpha-1}\varphi_q(\overline{y}(s))\,ds \\ \qquad\quad = \Delta_1\Lambda_1 + \Delta_2\Lambda_2 = m_1, \\ T_2 y(t) = \int_0^1 \varphi_q(\overline{y}(s))\,ds - \sum_{i=1}^m b_i \int_0^{\eta_i} \varphi_q(\overline{y}(s))\,ds = \Delta_1\Lambda_3 + \Delta_2\Lambda_4 = m_2. \end{cases}$$

Thus, there exists $y \in Y$ such that $(T_1 y(t), T_2 y(t)) = (m_1, m_2)$. So $J^{-1} : \operatorname{Ker} L \to \operatorname{Im} Q$ is well defined.

Lemma 3.5 *Suppose that the first part of* (H3) *holds, then the set*

$$\Omega_3 = \{u \in \operatorname{Ker} M \mid \lambda J^{-1}u + (1-\lambda)QNu = 0, \lambda \in [0,1]\}$$

is bounded.

Proof For each $u \in \Omega_3$, we can get that $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$, $c_1, c_2 \in \mathbf{R}$. By the definition of the set Ω_3, we can obtain that

$$\lambda(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}) + (1-\lambda)\big(Q_1 N(c_1 t^{\alpha-1} + c_2 t^{\alpha-2})t^{\alpha-1} + Q_2 N(c_1 t^{\alpha-1} + c_2 t^{\alpha-2})t^{\alpha-2}\big) = 0.$$

Thus,

$$\lambda c_1 + (1-\lambda)\varphi_p\left(\frac{1}{\Lambda}\big(\Lambda_4 T_1 N(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}) - \Lambda_3 T_2 N(c_1 t^{\alpha-1} + c_2 t^{\alpha-2})\big)\right) = 0, \tag{3.5}$$

$$\lambda c_2 + (1-\lambda)\varphi_p\left(\frac{1}{\Lambda}\big(-\Lambda_2 T_1 N(c_1 t^{\alpha-1} + c_2 t^{\alpha-2}) + \Lambda_1 T_2 N(c_1 t^{\alpha-1} + c_2 t^{\alpha-2})\big)\right) = 0. \tag{3.6}$$

If $\lambda = 0$, then Ω_3 is bounded because of the first part of (H2) and (H3). If $\lambda = 1$, we get $c_1 = c_2 = 0$, obviously, Ω_3 is bounded. If $\lambda \in (0,1)$, by the first part of (H2) and (3.5), we can obtain that $|c_1| \le \frac{A}{\Gamma(\alpha)}$, by the first part of (H3) and (3.6), we have $|c_2| \le \frac{A+B}{\Gamma(\alpha-1)}$. So, Ω_3 is bounded. □

Remark 3.1 If the second part of (H3) holds, then the set

$$\Omega_3' = \{u \in \operatorname{Ker} M \mid -\lambda J^{-1}u + (1-\lambda)QNu = 0, \lambda \in [0,1]\}$$

is bounded.

Proof of Theorem 3.1 Assume that Ω is a bounded open set of X with $\bigcup_{i=1}^{3}\overline{\Omega}_i \cup \Omega_3' \subset \Omega$. By Lemmas 3.1 and 3.2, we can obtain that $M : \operatorname{dom} M \cap X \to Y$ is quasi-linear, and N_λ is M-compact on $\overline{\Omega}$. By the definition of Ω , we have

$$\begin{aligned}
&Lu \neq N_\lambda u, \quad \forall (u,\lambda) \in (\operatorname{dom} M \cap \partial\Omega) \times (0,1),\\
&H(u,\lambda) = \pm\lambda J^{-1}(u) + (1-\lambda)QN(u) \neq 0, \quad (\partial\Omega \cap \operatorname{Ker} M) \times [0,1].
\end{aligned}$$

Thus, by the homotopic property of degree, we can get

$$\begin{aligned}
\deg(JQN, \Omega \cap \operatorname{Ker} M, 0) &= \deg\bigl(H(\cdot,0), \Omega \cap \operatorname{Ker} M, 0\bigr)\\
&= \deg\bigl(H(\cdot,1), \Omega \cap \operatorname{Ker} M, 0\bigr)\\
&= \deg(\pm I, \Omega \cap \operatorname{Ker} M, 0) \neq 0.
\end{aligned}$$

So Lemma 2.1 is satisfied, and $Mu = Nu$ has at least one solution in $\operatorname{dom} M \cap \overline{\Omega}$. Namely, BVP (1.3) have at least one solution in the space X. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in this article. All authors read and approved the final manuscript.

Acknowledgements
The authors are grateful to those who gave useful suggestions about the original manuscript. This research is supported by the National Natural Science Foundation of China (No. 11271364) and the Fundamental Research Funds for the Central Universities (2010LKSX09).

Received: 1 January 2013 Accepted: 2 September 2013 Published: 07 Nov 2013

References
1. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
2. Sabatier, J, Agrawal, OP, Machado, JAT: Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht (2007)
3. Samko, SG, Kilbas, AA, Marichev, OI: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, New York (1993)
4. Magin, R: Fractional calculus models of complex dynamics in biological tissues. Comput. Math. Appl. **59**, 1586-1593 (2010)
5. Metzler, R, Klafter, J: Boundary value problems for fractional diffusion equations. Physica A **278**, 107-125 (2000)
6. Mainardi, F: Fractional diffusive waves in viscoelastic solids. In: Wegner, JL, Norwood, FR (eds.) Nonlinear Waves in Solids, pp. 93-97. ASME, Fairfield (1995)
7. Bai, J, Feng, X: Fractional-order anisotropic diffusion for image denoising. IEEE Trans. Image Process. **16**, 2492-2502 (2007)
8. Agarwal, RP, O'Regan, D, Stanek, S: Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations. J. Math. Anal. Appl. **371**, 57-68 (2010)
9. Kosmatov, N: A boundary value problem of fractional order at resonance. Electron. J. Differ. Equ. **2010**, 135 (2010)
10. Jafari, H, Gejji, VD: Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method. Appl. Math. Comput. **180**, 700-706 (2006)
11. Bai, Z: Solvability for a class of fractional *m*-point boundary value problem at resonance. Comput. Math. Appl. **62**, 1292-1302 (2011)
12. Bai, Z: On solutions of some fractional *m*-point boundary value problems at resonance. Electron. J. Qual. Theory Differ. Equ. **2010**, 37 (2010)
13. Bai, Z, Zhang, Y: Solvability of fractional three-point boundary value problems with nonlinear growth. Appl. Math. Comput. **218**, 1719-1725 (2011)
14. Zhang, Y, Bai, Z: Existence of positive solutions for nonlinear fractional three-point boundary value problem at resonance. Appl. Math. Comput. **36**, 417-440 (2011)
15. Hu, Z, Liu, W: Solvability for fractional order boundary value problem at resonance. Bound. Value Probl. **2011**, 20 (2011)
16. Rui, W: Existence of solutions of nonlinear fractional differential equations at resonance. Electron. J. Qual. Theory Differ. Equ. **2012**, 66 (2012)

17. Wang, G, Liu, W, Yang, J, Zhu, S, Zheng, T: The existence of solutions for a fractional $2m$-point boundary value problems. J. Appl. Math. (2012). doi:10.1155/2012/841349
18. Wang, W: Solvability for a coupled system of fractional differential equations at resonance. Nonlinear Anal. **13**, 2285-2292 (2012)
19. Zhou, H, Kou, C, Xie, F: Existence of solutions for fractional differential equations with multi-point boundary conditions at resonance on a half-line. Electron. J. Qual. Theory Differ. Equ. **2011**, 27 (2011)
20. Chen, Y, Tang, X: Solvability of sequential fractional order multi-point boundary value problems at resonance. Appl. Math. Comput. **218**, 7638-7648 (2012)
21. Wang, J: The existence of solutions to boundary value problems of fractional differential equations at resonance. Nonlinear Anal. **74**, 1987-1994 (2011)
22. Chen, T, Liu, W, Hu, Z: A boundary value problem for fractional differential equation with p-Laplacian operator at resonance. Nonlinear Anal. **75**, 3210-3217 (2012)
23. Mawhin, J: Topological Degree and Boundary Value Problems for Nonlinear Differential Equations in Topological Methods for Ordinary Differential Equations. Lecture Notes in Math., vol. 1537, pp. 74-142 (1993)
24. Ge, W, Ren, J: An extension of Mawhin's continuation theorem and its application to boundary value problems with a p-Laplacian. Nonlinear Anal. **58**, 477-488 (2004)
25. Pang, H, Ge, W, Tian, M: Solvability of nonlocal boundary value problems for ordinary differential equation of higher order with a p-Laplacian. Comput. Math. Appl. **56**, 127-142 (2008)

10.1186/1687-1847-2013-295
Cite this article as: Shen et al.: Solvability of fractional boundary value problem with p-Laplacian operator at resonance. *Advances in Difference Equations* 2013, 2013:295