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On certain inequalities and their applications in the oscillation theory

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Abstract

In the paper, we offer a set of inequalities involving delayed argument and offer their application for higher-order differential equations of the form

$$x^{(n)}(t) + q(t)x(\tau(t)) = 0$$

to be oscillatory. The conditions obtained essentially improve many other known results.

MSC: 34K11; 34C10

Keywords: higher order differential equations; delay argument; oscillation

1 Introduction

The paper is organized as follows. In the first part we consider only properties of functions and their derivatives, and later we connect the estimate obtained with properties of solutions of differential equations. We shall investigate the properties of a couple of functions $\tau(t) \in C(I)$ and $x(t) \in C^{\ell}(I)$, $I = [t_0, \infty)$.

Lemma 1 *Assume that ℓ is a positive integer such that*

$$x(t) > 0, \qquad x'(t) > 0, \qquad \ldots, \qquad x^{(\ell)}(t) > 0, \qquad x^{(\ell+1)}(t) < 0, \tag{$\bar{C}_\ell$}$$

eventually. Then for any constant $\lambda \in (0,1)$ and for every $i = 1, 2, \ldots, \ell$,

$$\frac{t^i x^{(i)}(t)}{i!} < \frac{1}{\lambda}\binom{\ell}{i} x(t), \tag{1.1}$$

eventually.

Proof Assume that $(\bar{C}_\ell)$ holds for $t \ge t_0$. Using the monotonicity of $x^{(\ell)}(t)$, it is easy to see that for any $k \in (0,1)$,

$$x^{(\ell-1)}(t) > \int_{t_0}^{t} x^{(\ell)}(s)\,ds \ge (t - t_0)x^{(\ell)}(t) \ge ktx^{(\ell)}(t), \tag{1.2}$$

eventually, let us say, for $t \geq t_1 \geq t_0$. We define a sequence of functions $\{\rho_i(t)\}_1^\ell$ as follows:

$$\begin{aligned}
\rho_1(t) &= x^{(\ell-1)}(t) - ktx^{(\ell)}(t),\\
\rho_2(t) &= 2x^{(\ell-2)}(t) - ktx^{(\ell-1)}(t),\\
&\vdots\\
\rho_\ell(t) &= \ell x(t) - ktx'(t).
\end{aligned}$$

It follows from (1.2) that $\rho_1(t) > 0$. An integration of this from t_1 to t yields

$$x^{(\ell-2)}(t)[1+k] - ktx^{(\ell-1)}(t) \geq c = k\bigl(x^{(\ell-2)}(t_1) - t_1x^{(\ell-1)}(t_1)\bigr).$$

On the other hand, since $x^{(\ell-2)}(t) \to \infty$ as $t \to \infty$, we see that

$$x^{(\ell-2)}(t)[1-k] > c.$$

Combining the last two inequalities, we conclude that

$$\rho_2(t) > 0.$$

Proceeding as above, we verify that $\rho_i(t) > 0$, eventually, for all $i = 1, 2, \dots, \ell - 1$. Therefore,

$$\begin{aligned}
&\ell x(t) > ktx'(t),\\
&(\ell-1)x'(t) > ktx''(t),\\
&\vdots\\
&x^{(\ell-1)}(t) > ktx^{(\ell)}(t),
\end{aligned}$$

or in other words

$$\begin{aligned}
&tx'(t) < \frac{1}{k}\ell x(t),\\
&t^2x''(t) < \frac{1}{k^2}\ell(\ell-1)x(t),\\
&\vdots\\
&t^\ell x^{(\ell)}(t) < \frac{1}{k^\ell}\ell! x(t).
\end{aligned}$$

Setting $\lambda = k^\ell$, the last inequalities imply (1.1) and the proof is complete. □

Lemma 2 *Assume that $\tau(t) \leq t$ and that ℓ is a positive integer such that $(\bar{C}_\ell)$ holds. Then, for any constant $\lambda \in (0,1)$,*

$$x\bigl(\tau(t)\bigr) \geq \lambda\left(\frac{\tau(t)}{t}\right)^\ell x(t), \tag{1.3}$$

eventually.

Proof Taylor's theorem implies that

$$\begin{aligned} x(t) &\le x\bigl(\tau(t)\bigr) + x'\bigl(\tau(t)\bigr)\bigl(t-\tau(t)\bigr) + \cdots + x^{(\ell-2)}\bigl(\tau(t)\bigr)\frac{(t-\tau(t))^{\ell-1}}{(\ell-1)!} \\ &\quad + x^{(\ell)}\bigl(\tau(t)\bigr)\frac{(t-\tau(t))^{\ell}}{\ell!}. \end{aligned}$$

Employing (1.1), we have

$$\begin{aligned} x(t) &\le \frac{x(\tau(t))}{\lambda}\left\{1 + \binom{\ell}{1}\left(\frac{t}{\tau(t)} - 1\right) + \binom{\ell}{2}\left(\frac{t}{\tau(t)} - 1\right)^2 + \cdots + \binom{\ell}{\ell}\left(\frac{t}{\tau(t)} - 1\right)^\ell\right\} \\ &= \frac{x(\tau(t))}{\lambda}\left[1 + \left(\frac{t}{\tau(t)} - 1\right)\right]^\ell = \frac{1}{\lambda}\left(\frac{t}{\tau(t)}\right)^\ell x\bigl(\tau(t)\bigr). \end{aligned}$$

The proof is complete. □

The obtained estimates can be used, *e.g.*, in the theory of functional equations. In the paper, we present their application in discussing oscillatory and asymptotic properties of higher-order delay differential equations.

2 Main results

We consider higher-order delay differential equation

$$x^{(n)}(t) + q(t)x\bigl(\tau(t)\bigr) = 0, \tag{E}$$

where

(H_1) $q(t) > 0$, $\tau(t) \le t$.

Denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (E). It follows from the classical lemma of Kiguradze [1] that the set $\mathcal{N}$ has the following decomposition:

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{n-1}, \quad n\text{-odd}, \\ \mathcal{N} &= \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{n-1}, \quad n\text{-even}, \end{aligned}$$

where the nonoscillatory solution $x(t)$, let us say positive, satisfies

$$x(t) \in \mathcal{N}_\ell \quad \Leftrightarrow \quad \begin{cases} x^{(i)}(t) > 0; & i = 0, 1, \ldots, \ell, \\ (-1)^{i-\ell}x^{(i)}(t) > 0; & i = \ell, \ldots, n-1. \end{cases}$$

A nonoscillatory solution $x(t)$ of (E) is said to be of degree ℓ if $x(t) \in \mathcal{N}_\ell$. Following Kondratiev and Kiguradze, we say that (E) has property (A) provided that

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_0, \quad n\text{-odd}, \\ \mathcal{N} &= \emptyset, \quad n\text{-even}. \end{aligned}$$

The investigation of oscillatory properties of the second- and higher-order linear differential equations started with the Sturm comparison theorem [2]. Later Mahfoud [3] essentially contributed to the subject and presented a very useful comparison technique for studying the properties of a delay differential equation from those of a differential equation without delay. A new impetus to investigation in this direction was given by papers of Chanturia and Kiguradze [4], Kusano and Naito [5] and Koplatadze *et al.* [6, 7]. See also [1–20]. In the paper, we employ Lemma 2 to establish new criteria for oscillation of (E).

It is interesting to note that the condition

$$\int^{\infty} \tau^{n-1}(t)q(t)\,\mathrm{d}t = \infty \tag{2.1}$$

is necessary for property (A) of (E). This fact has been observed in [6] and [7].

Theorem 1 *Assume that* (E) *has a solution of degree* $\ell > 0$, *then for any* $\lambda \in (0,1)$ *so does the ordinary equation*

$$x^{(n)}(t) + \lambda\left(\frac{\tau(t)}{t}\right)^{\ell} q(t)x(t) = 0. \tag{E_ℓ}$$

Proof Assume that (E) possesses a nonoscillatory solution $x(t) \in \mathcal{N}_\ell$. We may assume that $x(t)$ is positive. Then condition (1.3) of Lemma 1 implies that $x(t)$ is a positive solution of the differential inequality

$$x^{(n)}(t) + \lambda\left(\frac{\tau(t)}{t}\right)^{\ell} q(t)x(t) \le 0.$$

On the other hand, it follows from Theorem 2 of [5] that the corresponding equation (E_ℓ) has also a solution of degree ℓ. The proof is complete. □

So, if we eliminate solutions of degree ℓ of equations (E_ℓ), we get property (A) of studied equation (E). To do it, we recall the following comparison result which is due to Chanturia [4].

Theorem 2 *Assume that*

$$\int_t^{\infty} q(s)\,\mathrm{d}s \ge \int_t^{\infty} p(s)\,\mathrm{d}s. \tag{2.2}$$

If the differential equation

$$x^{(n)}(t) + p(t)x(t) = 0$$

has no solution of degree ℓ, *neither does the equation*

$$x^{(n)}(t) + q(t)x(t) = 0.$$

In view of Theorem 1, we apply this comparison theorem to equations (E_ℓ) and the Euler equation

$$x^{(n)}(t) + \frac{a}{t^n}x(t) = 0 \tag{2.3}$$

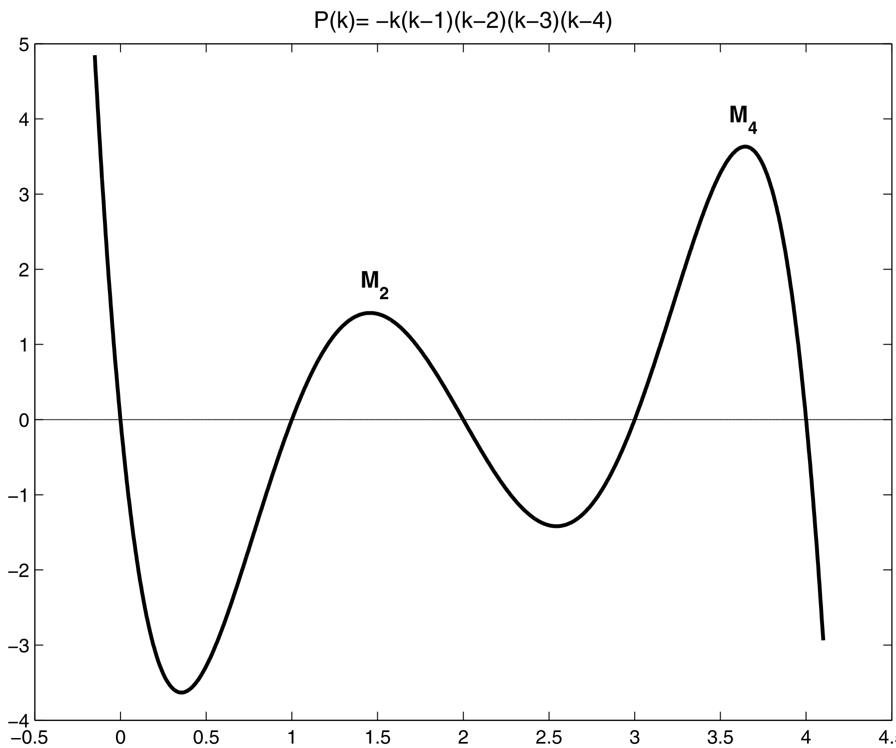

Figure 1 Graph of function $P(k)$.

to obtain new criteria for property (A) of (E). Properties of (2.3) are connected with properties of the polynomial $P_n(k) = -k(k-1)\cdots(k-n+1)$. Let us denote

$$M_j = \max_{k\in(j-1;j)}\{P_n(k)\},$$

where $j = 2,4,\dots,n-1$ for n odd, while $j = 1,3,\dots,n-1$ for n even. In other words, M_j represents all local maxima of the polynomial $P_n(k)$ (see Figure 1). Then it is easy to verify (see also [15]) that the following criterion for the $\mathcal{N}_\ell$ to be empty holds true.

Lemma 3 *Let $\ell > 0$. If*

$$a > M_\ell, \tag{2.4}$$

where $\ell = 1,3,\dots,n-1$ for n odd and $\ell = 1,3,\dots,n-1$ for n even, then (2.3) *has no solution of degree ℓ.*

Employing Theorem 2 to (2.3) and (E_ℓ), in view of Theorem 1, one gets the following theorem.

Theorem 3 *Assume that*

$$\liminf_{t\to\infty} t^{n-1}\int_t^\infty \left(\frac{\tau(s)}{s}\right)^{n-1} q(s)\,\mathrm{d}s > \frac{M_{n-1}}{(n-1)}. \tag{P}$$

Then (E) *has property* (A).

Proof Assume that n is odd. Observing that $\tau(t) \le t$ and $M_{n-1} > M_\ell$ for every $\ell = 2, 4, \dots, n-3$, it follows from (P) that for every $\ell = 2, 4, \dots, n-1$,

$$\liminf_{t\to\infty} t^{n-1} \int_t^\infty \left(\frac{\tau(s)}{s}\right)^\ell q(s)\,\mathrm{d}s > \frac{M_\ell}{(n-1)}. \tag{P_ℓ}$$

On the other hand, (P_ℓ) implies that there exists a couple of constants $\lambda \in (0,1)$ and $a > M_\ell$ such that

$$\lambda \int_t^\infty \left(\frac{\tau(s)}{s}\right)^\ell q(s)\,\mathrm{d}s > \frac{a}{(n-1)t^{n-1}}. \tag{2.5}$$

Since $a > M_\ell$, Euler equation (2.3) has no solution of degree ℓ. On the other hand, taking (2.5) into account, Theorem 2 ensures that (E_ℓ) has no solution of degree ℓ. Finally, Theorem 1 guarantees that (E) has property (A). The proof is complete. □

For $\tau(t) = \alpha t$, $0 < \alpha \le 1$, the previous result simplifies to the following.

Corollary 1 *Assume that*

$$\liminf_{t\to\infty} t^{n-1} \int_t^\infty q(s)\,\mathrm{d}s > \frac{M_{n-1}}{\alpha^{n-1}(n-1)}. \tag{P^*}$$

Then the delay differential equation

$$x^{(n)}(t) + q(t)x(\alpha t) = 0, \quad 0 < \alpha \le 1, \tag{E^*}$$

has property (A).

Theorem 4 *Let n be odd. Assume that (E) has property* (A). *Then every nonoscillatory solution $x(t)$ of (E) satisfies*

$$\lim_{t\to\infty} x(t) = 0.$$

Proof First note that property (A) of (E) implies (2.1). Moreover, it follows from the definition of property (A) that every nonoscillatory solution $x(t) \in \mathcal{N}_0$, which implies that there exists $\lim_{t\to\infty} x(t) = c \ge 0$. We claim that $c = 0$. If not, then $x(\tau(t)) \ge c > 0$. An integration of (E) from t to ∞ yields

$$x^{(n-1)}(t) \ge \int_t^\infty q(s)x(\tau(s))\,\mathrm{d}s \ge c\int_t^\infty q(s)\,\mathrm{d}s.$$

Having repeated this procedure, we are led to

$$x(t_1) \ge \frac{c}{(n-1)!}\int_t^\infty (s-t_1)^{n-1}q(s)\,\mathrm{d}s,$$

which contradicts (2.1) and we conclude that $c = 0$. □

We support our results with the following illustrative example.

Example 1 Consider the fifth-order delay differential equation

$$x^{(5)}(t) + \frac{a}{t^5}x(\alpha t) = 0, \quad a > 0, 0 < \alpha < 1. \tag{E_{x1}}$$

The graph of the polynomial $P(k) = -k(k-1)(k-2)(k-3)(k-4)$ that corresponds to the fifth-order equation is presented in Figure 1. Employing Matlab, we easily evaluate that

$$M_4 = 3.6314.$$

Consequently, criterion (P^*) for property (A) of (E) reduces for (E_{x1}) to

$$a\alpha^4 > 3.6314.$$

3 Comparison

Theorem 2 essentially improves Chanturia's test [4] that guarantees property (A) of

$$x^{(n)}(t) + q(t)x(t) = 0, \tag{$E1$}$$

provided that

$$\limsup_{t\to\infty} t \int_t^\infty s^{n-2} q(s)\,\mathrm{d}s > (n-1)!.$$

Kiguradze's test [1] that for property (A) of (E) requires

$$\int_{t_0}^\infty s^{n-1-\varepsilon} q(s)\,\mathrm{d}s = \infty \quad \text{for some } \varepsilon > 0,$$

and Koplatadze's test [7] for property (A) of (E) that claims the condition

$$\limsup_{t\to\infty} \left\{ \tau(t) \int_{\tau(t)}^\infty \tau^{n-2}(s) q(s)\,\mathrm{d}s + \int_{\tau(t)}^t \tau^{n-1}(s) q(s)\,\mathrm{d}s + \frac{1}{\tau(t)} \int_{t_2}^{\tau(t)} s\tau^{n-1}(s) q(s)\,\mathrm{d}s \right\} > (n-1)!, \tag{3.1}$$

where $\tau(t)$ is nondecreasing.

We provide details while comparing those criteria with our one.

Example 2 Consider once more the fifth-order delay differential equation (E_{x1}). It is easy to see that Chanturia's test can be applied only when $\alpha = 1$ and requires

$$a > 4!$$

for property (A) of (E_{x1}). Kiguradze's test fails. On the other hand, Koplatadze's test simplifies for $\alpha = 1$ and $\alpha = 0.8$ to

$$a > 6 \quad \text{and} \quad a > 23.9522,$$

respectively, while our criterion needs only

$$a > 3.6314 \quad \text{and} \quad a > 8.8658,$$

respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made the same contribution. All authors read and approved the final manuscript.

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