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Approximate perfect differential equations of second order

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Abstract

In this paper we prove the Hyers-Ulam stability of the perfect linear differential equation $f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) = Q(t)$, where $f, y \in C^2[a,b]$, $Q \in C[a,b]$, $f_2(t) = f_1'(t) - f''(t)$ and $-\infty < a < b < +\infty$.

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Keywords: Hyers-Ulam stability; differential equation

1 Introduction

The question concerning the stability of group homomorphisms was posed by Ulam [1]. Hyers [2] solved the case of approximately additive mappings in Banach spaces and T.M. Rassias generalized the result of Hyers [3].

Definition 1.1 Let X be a normed space over a scalar field $\mathbb{K}$ and let I be an open interval. Assume that $a_0, a_1, \ldots, a_n, h : I \to \mathbb{K}$ are continuous functions. We say that the differential equation

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + h(t) = 0 \tag{1.1}$$

has the Hyers-Ulam stability if, for any function $f : I \to X$ satisfying the differential inequality

$$\left\| a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0 y(t) + h(t) \right\| \le \varepsilon$$

for all $t \in I$ and some $\varepsilon \ge 0$, there exists a solution $g : I \to X$ of (1.1) such that $\|f(t) - g(t)\| \le K(\varepsilon)$ for all $t \in I$, where $K(\varepsilon)$ is a function depending only on ε.

Obłoza [4, 5] was the first author who investigated the Hyers-Ulam stability of differential equations (also see [6]).

Jung [7] solved the inhomogeneous differential equation of the form $y'' + 2xy' - 2ny = \sum_{m=0}^{\infty} a_m x^m$, where n is a positive integer, and he used this result to prove the Hyers-Ulam stability of the differential equation $y'' + 2xy' - 2ny = 0$ in a special class of analytic functions.

Li and Shen [8] proved that if the characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$ has two different positive roots, then the linear differential equation of second order with constant coefficients $y''(x) + \alpha y'(x) + \beta y(x) = f(x)$ has the Hyers-Ulam stability where $y \in C^2[a,b]$, $f \in C[a,b]$ and $-\infty < a < b < +\infty$ (see also [9, 10]). Abdollahpour and Najati [11] proved that the third-order differential equation $y^{(3)}(t) + \alpha y''(t) + \beta y'(t) + \gamma y(t) = f(t)$ has the Hyers-Ulam stability. Ghaemi *et al.* [12] proved the Hyers-Ulam stability of the exact second-order linear differential equation

$$p_0(x)y'' + p_1(x)y' + p_2(x)y + f(x) = 0$$

with $p_0''(x) - p_1'(x) + p_2(x) = 0$. Here p_0, p_1, p_2, $f : (a,b) \to \mathbb{R}$ are continuous functions. For more results about the Hyers-Ulam stability of differential equations, we can refer to [13–21].

Definition 1.2 We say that the differential equation

$$f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) = Q(t), \tag{1.2}$$

is perfect if it can be written as $\frac{d}{dt}[f(t)y'(t) + (f_1(t) - f'(t))y(t)] = Q(t)$.

It is clear that the differential equation (1.2) is perfect if and only if $f_2(t) = f_1'(t) - f''(t)$. The aim of this paper is to investigate the Hyers-Ulam stability of the perfect differential equation (1.2), where $f, y \in C^2[a,b]$, $Q \in C[a,b]$, $f_1 \in C^1[a,b]$, $f_2(t) = f_1'(t) - f''(t)$ and $-\infty < a < b < +\infty$. More precisely, we prove that the equation (1.2) has the Hyers-Ulam stability.

2 Hyers-Ulam stability of the perfect differential equation $f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) = Q(t)$

In the following theorem, we prove the Hyers-Ulam stability of the differential equation (1.2).

Throughout this section, a and b are real numbers with $-\infty < a < b < +\infty$.

Theorem 2.1 *The perfect differential equation*

$$f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) = Q(t)$$

has the Hyers-Ulam stability, where $f, y \in C^2[a,b]$, $f_1 \in C^1[a,b]$, $Q \in C[a,b]$ *and* $f(t) \neq 0$ *for all* $t \in [a,b]$.

Proof Let $\varepsilon > 0$ and $y \in C^2[a,b]$ with

$$\left|f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) - Q(t)\right| \leq \varepsilon.$$

Let $g(t) = f(t)y' + (f_1(t) - f'(t))y$ for all $t \in [a,b]$. It is clear that

$$\left|g'(t) - Q(t)\right| = \left|f(t)y''(t) + f_1(t)y'(t) + f_2(t)y(t) - Q(t)\right| \leq \varepsilon.$$

We define

$$z(x) = g(b) - \int_x^b Q(t)\,dt, \quad x \in [a,b].$$

Then

$$z'(x) = Q(x), \quad x \in [a,b]. \tag{2.1}$$

Also, we have

$$\begin{aligned} |z(x) - g(x)| &= \left| g(b) - g(x) - \int_x^b Q(t)\,dt \right| = \left| \int_x^b g'(t)\,dt - \int_x^b Q(t)\,dt \right| \\ &\le \int_x^b |g'(t) - Q(t)|\,dt \le \varepsilon(b-a) \end{aligned}$$

for all $x \in [a,b]$. Now we define

$$F(x) = \frac{1}{f(x)} \exp\left\{ \int_a^x \frac{f_1(t)}{f(t)}\,dt \right\}, \qquad u(x) = \frac{y(b)F(b)}{F(x)} - \frac{1}{F(x)} \int_x^b \frac{z(t)F(t)}{f(t)}\,dt$$

for all $x \in [a,b]$. It is clear that $u \in C^2[a,b]$ and

$$u'(x)F(x) + u(x)F'(x) = \frac{z(x)F(x)}{f(x)}, \quad F'(x) = \frac{f_1(x) - f'(x)}{f(x)} F(x).$$

Therefore,

$$f(x)u'(x) + \left[f_1(x) - f'(x)\right]u(x) = z(x), \quad x \in [a,b]. \tag{2.2}$$

Hence, (2.1) implies that

$$f(x)u''(x) + f_1(x)u'(x) + f_2(x)u(x) = Q(x), \quad x \in [a,b].$$

Also, we have

$$\begin{aligned} |y(x) - u(x)| &= \left| y(x) - \frac{y(b)F(b)}{F(x)} + \frac{1}{F(x)} \int_x^b \frac{z(t)F(t)}{f(t)}\,dt \right| \\ &= \frac{1}{|F(x)|} \left| y(x)F(x) - y(b)F(b) + \int_x^b \frac{z(t)F(t)}{f(t)}\,dt \right| \\ &= \frac{1}{|F(x)|} \left| \int_x^b \frac{z(t)F(t)}{f(t)}\,dt - \int_x^b \left[y(t)F(t)\right]'\,dt \right| \\ &= \frac{1}{|F(x)|} \left| \int_x^b \left(\frac{z(t)F(t)}{f(t)} - y'(t)F(t) - y(t)F'(t) \right) dt \right| \\ &= \frac{1}{|F(x)|} \left| \int_x^b F(t) \left(\frac{z(t)}{f(t)} - y'(t) - \frac{f_1(t) - f'(t)}{f(t)} y(t) \right) dt \right| \\ &\le \frac{1}{|F(x)|} \int_x^b \left| \frac{F(t)}{f(t)} \right| \left| z(t) - y'(t)f(t) - \left[f_1(t) - f'(t)\right]y(t) \right| dt \\ &= \frac{1}{|F(x)|} \int_x^b \left| \frac{F(t)}{f(t)} \right| \left| z(t) - g(t) \right| dt \\ &\le \varepsilon(b-a) \frac{1}{|F(x)|} \int_x^b \left| \frac{F(t)}{f(t)} \right| dt \end{aligned} \tag{2.3}$$

for all $x \in [a,b]$. Since $\frac{f_1}{f} \in C[a,b]$, there exist constants m' and M' such that $m' \leq \frac{f_1(x)}{f(x)} \leq M'$. Thus

$$\begin{cases} 1 \leq \exp\{\int_a^x \frac{f_1(t)}{f(t)}\,dt\} \leq e^{M'(b-a)} & \text{if } m' \geq 0; \\ e^{m'(b-a)} \leq \exp\{\int_a^x \frac{f_1(t)}{f(t)}\,dt\} \leq e^{M'(b-a)} & \text{if } m' < 0 \leq M'; \\ e^{m'(b-a)} \leq \exp\{\int_a^x \frac{f_1(t)}{f(t)}\,dt\} \leq 1 & \text{if } M' < 0 \end{cases} \tag{2.4}$$

for all $x \in [a,b]$. Since $f \in C[a,b]$ and $|f| > 0$, there exist constants $0 < m \leq M$ such that $m \leq |f(x)| \leq M$ for all $x \in [a,b]$. Hence, (2.4) implies that

$$\frac{1}{M} e^{|m'|(a-b)} \leq \left|F(x)\right| \leq \frac{1}{m} e^{|M'|(b-a)}$$

for all $x \in [a,b]$. It follows from (2.3) that

$$\begin{aligned} \left|y(x) - u(x)\right| &\leq \varepsilon(b-a)\frac{1}{|F(x)|}\int_x^b \left|\frac{F(t)}{f(t)}\right| dt \\ &\leq \varepsilon(b-a)^2 \frac{M}{m^2} e^{(|m'|+|M'|)(b-a)} \end{aligned}$$

for all $x \in [a,b]$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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