

POSITIVE SOLUTIONS OF FUNCTIONAL DIFFERENCE EQUATIONS WITH p -LAPLACIAN OPERATOR

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The author studies the boundary value problems with p -Laplacian functional difference equation $\Delta\phi_p(\Delta x(t)) + r(t)f(x_t) = 0$, $t \in [0, N]$, $x_0 = \psi \in C^+$, $x(0) - B_0(\Delta x(0)) = 0$, $\Delta x(N + 1) = 0$. By using a fixed point theorem in cones, sufficient conditions are established for the existence of twin positive solutions.

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1. Introduction

For notation, given $a < b$ in \mathbb{Z} , we employ intervals to denote discrete sets such as $[a, b] = \{a, a + 1, \dots, b\}$, $[a, b) = \{a, a + 1, \dots, b - 1\}$, $[a, \infty) = \{a, a + 1, \dots\}$, and so forth. Let $\tau, N \in \mathbb{Z}$ and let $0 \leq \tau \leq N$. In this paper, we are concerned with the p -Laplacian functional difference equation

$$\begin{aligned} \Delta\phi_p(\Delta x(t)) + r(t)f(x_t) &= 0, \quad t \in [0, N], \\ x_0 &= \psi \in C^+, \quad x(0) - B_0(\Delta x(0)) = 0, \quad \Delta x(N + 1) = 0, \end{aligned} \tag{1.1}$$

where $\phi_p(u)$ is the p -Laplacian operator, that is, $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $(\phi_p)^{-1}(u) = \phi_q(u)$, $1/p + 1/q = 1$. For all $t \in \mathbb{Z}$, let $x_t = x_t(k) = x(t + k)$, $k \in [-\tau, -1]$; then $x_t \in C$, where $C = C([-\tau, -1], \mathbb{R})$ is a Banach space with the norm $\|\varphi\|_C = \max_{k \in [-\tau, -1]} |\varphi(k)|$. Let $C^+ = \{\varphi \in C : \varphi(k) \geq 0, k \in [-\tau, -1]\}$ and let $d = \max_{k \in [-\tau, -1]} \psi(k)$, $\psi \in C^+$. As usual, Δ denotes the forward difference operator defined by $\Delta x(t) = x(t + 1) - x(t)$.

We will assume that

(H₁) $f(\varphi)$ is a nonnegative continuous functional defined on C^+ ;

(H₂) $r(t)$ is a nonnegative function defined on $[0, N]$;

(H₃) $B_0 : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies that there are $\beta \geq \alpha \geq 0$ such that $\alpha s \leq B_0(s) \leq \beta s$ for $s \in \mathbb{R}^+$, where \mathbb{R}^+ denotes the set of nonnegative real numbers.

2 Positive solutions of difference equations

Recently, the existence of positive solutions of finite difference equations with different boundary value conditions is investigated in [1–5] and references therein. In this paper, we consider the functional difference equation (1.1) and apply the twin fixed point theorem to obtain at least two positive solutions of the boundary value problem (BVP) (1.1) when growth conditions are imposed on f . Finally, we present two corollaries that show that under the assumptions that f is superlinear or sublinear, BVP (1.1) has at least two positive solutions. An example to illustrate our results in this paper is included.

We note that $x(t)$ is a solution of (1.1) if and only if

$$x(t) = \begin{cases} B_0 \left(\phi_q \left(\sum_{n=0}^N r(n) f(x_n) \right) \right) + \sum_{m=0}^{t-1} \phi_q \left(\sum_{n=m}^N r(n) f(x_n) \right), & t \in [0, N+2], \\ \psi, & t \in [-\tau, -1]. \end{cases} \quad (1.2)$$

We assume that $\bar{x}(t)$ is the solution of BVP (1.1) with $f \equiv 0$. Clearly, it can be expressed as

$$\bar{x}(t) = \begin{cases} 0, & t \in [0, N+2], \\ \psi, & t \in [-\tau, -1]. \end{cases} \quad (1.3)$$

It is obvious that $\bar{x}_n \equiv 0$ for $n \in [\tau, N]$.

Let $x(t)$ be a solution of BVP (1.1) and $y(t) = x(t) - \bar{x}(t)$. Noting that $y(t) = x(t)$ for $t \in [0, N+2]$, then we have from (1.2) that

$$y(t) = \begin{cases} B_0 \left(\phi_q \left(\sum_{n=0}^N r(n) f(y_n + \bar{x}_n) \right) \right) + \sum_{m=0}^{t-1} \phi_q \left(\sum_{n=m}^N r(n) f(y_n + \bar{x}_n) \right), & t \in [0, N+2], \\ 0, & t \in [-\tau, -1]. \end{cases} \quad (1.4)$$

Let $E = \{y : [-\tau, N+2] \rightarrow \mathbb{R}\}$ with norm $\|y\| = \max_{t \in [-\tau, N+2]} |y(t)|$, then $(E, \|\cdot\|)$ is a Banach space.

Define a cone P by

$$P = \{y \in E : y(t) = 0 \text{ for } t \in [-\tau, -1]; y(t) \geq 0 \text{ for } t \in [0, N+2], \\ \text{and } \Delta^2 y(t) \leq 0, \Delta y(t) \geq 0 \text{ for } t \in [0, N+2], \Delta y(N+1) = 0\}. \quad (1.5)$$

Clearly, $\|y\| = \|y\|_{[0, N+2]} = y(N+2)$ for $y(t) \in P$, where $\|y\|_{[0, N+2]} = \max_{t \in [0, N+2]} |y(t)|$. Define $T : P \rightarrow E$ by

$$Ty(t) = \begin{cases} B_0 \left(\phi_q \left(\sum_{n=0}^N r(n) f(y_n + \bar{x}_n) \right) \right) + \sum_{m=0}^{t-1} \phi_q \left(\sum_{n=m}^N r(n) f(y_n + \bar{x}_n) \right), & t \in [0, N+2], \\ 0, & t \in [-\tau, -1]. \end{cases} \quad (1.6)$$

The following lemma will play an important role in the proof of our results and can be found in [2]. Let

$$\begin{aligned} P(\delta, e) &= \{x \in P : \delta(x) < e\}, \\ \partial P(\delta, e) &= \{x \in P : \delta(x) = e\}, \\ \overline{P(\delta, e)} &= \{x \in P : \delta(x) \leq e\}. \end{aligned} \tag{1.7}$$

LEMMA 1.1. *Let X be a real Banach space, P a cone of X , γ and α two nonnegative increasing continuous maps, θ a nonnegative continuous map, and $\theta(0) = 0$. There are two positive numbers c and M such that*

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \quad \|x\| \leq M\gamma(x) \quad \text{for } x \in \overline{P(\gamma, c)}. \tag{1.8}$$

In addition, assume that $T : \overline{P(\gamma, c)} \rightarrow P$ is completely continuous. There are positive numbers $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda \theta(x) \quad \forall \lambda \in [0, 1], x \in \partial P(\theta, b), \tag{1.9}$$

and

- (i) $\gamma(Tx) > c$ for $x \in \partial P(\gamma, c)$;
- (ii) $\theta(Tx) < b$ for $x \in \partial P(\theta, b)$;
- (iii) $\alpha(Tx) > a$ and $P(\alpha, a) \neq \emptyset$ for $x \in \partial P(\alpha, a)$.

Then T has at least two fixed points x_1 and $x_2 \in \overline{P(\gamma, c)}$ satisfying

$$a < \alpha(x_1), \quad \theta(x_1) < b, \quad b < \theta(x_2), \quad \gamma(x_2) < c. \tag{1.10}$$

The following lemma is similar to Lemma 1.1; the proof is omitted.

LEMMA 1.2. *Let X be a real Banach space, P a cone of X , γ and α two nonnegative increasing continuous maps, θ a nonnegative continuous map, and $\theta(0) = 0$. There are two positive numbers c and M such that*

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \quad \|x\| \leq M\gamma(x) \quad \text{for } x \in \overline{P(\gamma, c)}. \tag{1.11}$$

In addition, assume that $T : \overline{P(\gamma, c)} \rightarrow P$ is completely continuous. There are positive numbers $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda \theta(x) \quad \forall \lambda \in [0, 1], x \in \partial P(\theta, b), \tag{1.12}$$

and

- (i) $\gamma(Tx) < c$ for $x \in \partial P(\gamma, c)$;
- (ii) $\theta(Tx) > b$ for $x \in \partial P(\theta, b)$;
- (iii) $\alpha(Tx) < a$ and $P(\alpha, a) \neq \emptyset$ for $x \in \partial P(\alpha, a)$.

Then T has at least two fixed points x_1 and $x_2 \in \overline{P(\gamma, c)}$ satisfying

$$a < \alpha(x_1), \quad \theta(x_1) < b, \quad b < \theta(x_2), \quad \gamma(x_2) < c. \tag{1.13}$$

4 Positive solutions of difference equations

2. Main results

Choose $h = [(N + 2)/2]$, where $[x]$ is the greatest integer not greater than x .

LEMMA 2.1. *Let T be defined by (1.4). If $y \in P$, then*

- (i) $T(P) \subset P$;
- (ii) $T : P \rightarrow P$ is completely continuous;
- (iii) finding positive solutions of BVP (1.1) is equivalent to finding fixed points of the operator T on P ;
- (iv) if $y \in P$, then

$$y(t) \geq \frac{1}{2} \|y\| = \frac{1}{2} y(N + 2), \quad t \in [h, N + 2]. \quad (2.1)$$

The proof is simple and is omitted.

Define the nonnegative, increasing, continuous functionals γ, θ , and α on P by

$$\begin{aligned} \gamma(y) &= y(h), \\ \theta(y) &= \max_{t \in [0, h]} y(t) = y(h), \\ \alpha(y) &= \max_{t \in [0, h]} y(t) = y(h). \end{aligned} \quad (2.2)$$

We have

$$\begin{aligned} \gamma(y) &= \theta(y) = \alpha(y), \quad y \in P, \\ \theta(y) &= \gamma(y) = y(h) \geq \left(\frac{1}{2}\right) y(N + 2) = \left(\frac{1}{2}\right) \|y\| \text{ for each } y \in P. \end{aligned} \quad (2.3)$$

Then

$$\begin{aligned} \|y\| &\leq 2\gamma(y), \quad \text{for each } y \in P, \\ \theta(\lambda y) &= \lambda\theta(y), \quad \forall \lambda \in [0, 1], y \in \partial P(\theta, b). \end{aligned} \quad (2.4)$$

For the notational convenience, we denote σ and ρ by

$$\begin{aligned} \sigma &= (\alpha + 1)\phi_q \left(\sum_{n=h+\tau}^N r(n) \right); \\ \rho &= (\beta + h)\phi_q \left(\sum_{n=0}^N r(n) \right). \end{aligned} \quad (2.5)$$

Throughout the paper, we assume that $h + \tau \leq N$ and $\sum_{n=h+\tau}^N r(n) > 0$.

THEOREM 2.2. *Suppose that there are positive numbers $a < b < c$ such that*

$$0 < a < \frac{\sigma}{\rho} b < \frac{\sigma}{2\rho} (c - d). \quad (2.6)$$

Assume that $f(\varphi)$ satisfies the following conditions:

- (A) $f(\varphi) > \phi_p(c/\sigma)$ for $c \leq \|\varphi\|_C \leq 2c$,
- (B) $f(\varphi) < \phi_p(b/\rho)$ for $0 \leq \|\varphi\|_C \leq 2b + d$,
- (C) $f(\varphi) > \phi_p(a/\sigma)$ for $a \leq \|\varphi\|_C \leq 2a$.

Then BVP (1.1) has at least two positive solutions x_1 and x_2 such that

$$a < \max_{t \in [0, h]} x_1(t) < b < \max_{t \in [0, h]} x_2(t) < c. \quad (2.7)$$

Proof. Firstly, we verify that $y \in \partial P(y, c)$ implies that $\gamma(Ty) > c$.

Since $\gamma(y) = c = y(h)$, one gets $y(t) \geq c$ for $t \in [h, N + 2]$.

Recalling that $\|y\| \leq 2\gamma(y) = 2c$, we know that $c \leq \|y_n\|_C \leq 2c$ for $n \in [h + \tau, N]$.

Then, we get

$$\begin{aligned} \gamma(Ty) &= B_0 \left(\phi_q \left(\sum_{n=0}^N r(n) f(y_n + \bar{x}_n) \right) \right) + \sum_{m=0}^{h-1} \phi_q \left(\sum_{n=m}^N r(n) f(y_n + \bar{x}_n) \right) \\ &\geq \alpha \phi_q \left(\sum_{n=0}^N r(n) f(y_n + \bar{x}_n) \right) + \sum_{m=0}^{h-1} \phi_q \left(\sum_{n=m}^N r(n) f(y_n + \bar{x}_n) \right) \\ &\geq \alpha \phi_q \left(\sum_{n=h+\tau}^N r(n) f(y_n) \right) + \phi_q \left(\sum_{n=h+\tau}^N r(n) f(y_n) \right) \\ &= (\alpha + 1) \phi_q \left(\sum_{n=h+\tau}^N r(n) f(y_n) \right) > (\alpha + 1) \phi_q \left(\sum_{n=h+\tau}^N r(n) \phi_p \left(\frac{c}{\sigma} \right) \right) \\ &= \frac{c}{\sigma} (\alpha + 1) \phi_q \left(\sum_{n=h+\tau}^N r(n) \right) = c. \end{aligned} \quad (2.8)$$

Secondly, we prove that $y \in \partial P(\theta, b)$ implies that $\theta(Ty) < b$.

Since $\theta(y) = b$ implies that $y(h) = b$, it follows that $0 \leq y(t) \leq b$ for $t \in [0, h]$ and

$$b \leq y(t) \leq \|y\| \leq 2\theta(y) = 2b, \quad \text{for } t \in [h + 1, N], \quad y \in P. \quad (2.9)$$

So

$$\|y_n + \bar{x}_n\|_C \leq \|y_n\|_C + \|\bar{x}_n\|_C \leq 2b + d. \quad (2.10)$$

Then, we have

$$\begin{aligned} \theta(Ty) &= B_0 \left(\phi_q \left(\sum_{n=0}^N r(n) f(y_n + \bar{x}_n) \right) \right) + \sum_{m=0}^{h-1} \phi_q \left(\sum_{n=m}^N r(n) f(y_n + \bar{x}_n) \right) \\ &< \beta \phi_q \left(\sum_{n=0}^N r(n) f(y_n + \bar{x}_n) \right) + \sum_{m=0}^{h-1} \phi_q \left(\sum_{n=0}^N r(n) f(y_n + \bar{x}_n) \right) \\ &= \frac{b}{\rho} (\beta + h) \phi_q \left(\sum_{n=0}^N r(n) \right) = b. \end{aligned} \quad (2.11)$$

6 Positive solutions of difference equations

Finally, we show that

$$P(\alpha, a) \neq \emptyset, \quad \alpha(Ty) > a \quad \forall y \in \partial P(\alpha, a). \quad (2.12)$$

It is obvious that $P(\alpha, a) \neq \emptyset$. On the other hand, $\alpha(y) = y(h) = a$ implies that

$$\begin{aligned} a &\leq \|y\| \leq 2a \quad \text{for } t \in [h, N], \\ a &\leq \|y_n\|_C \leq 2a \quad \text{for } n \in [h + \tau, N]. \end{aligned} \quad (2.13)$$

Thus,

$$\begin{aligned} \alpha(Ty) &= B_0 \left(\phi_q \left(\sum_{n=0}^N r(n) f(y_n + \bar{x}_n) \right) \right) + \sum_{m=0}^{h-1} \phi_q \left(\sum_{n=m}^N r(n) f(y_n + \bar{x}_n) \right) \\ &\geq \alpha \phi_q \left(\sum_{n=0}^N r(n) f(y_n + \bar{x}_n) \right) + \sum_{m=0}^{h-1} \phi_q \left(\sum_{n=m}^N r(n) f(y_n + \bar{x}_n) \right) \\ &\geq \alpha \phi_q \left(\sum_{n=h+\tau}^N r(n) f(y_n) \right) + \phi_q \left(\sum_{n=h+\tau}^N r(n) f(y_n) \right) \\ &= (\alpha + 1) \phi_q \left(\sum_{n=h+\tau}^N r(n) f(y_n) \right) > (\alpha + 1) \phi_q \left(\sum_{n=h+\tau}^N r(n) \phi_p \left(\frac{a}{\sigma} \right) \right) \\ &= \frac{a}{\sigma} (\alpha + 1) \phi_q \left(\sum_{n=h+\tau}^N r(n) \right) = a. \end{aligned} \quad (2.14)$$

Hence by Lemma 1.1, T has at least two different fixed points y_1 and y_2 . Let $x_i = y_i + \bar{x}$ ($i = 1, 2$), which are twin positive solutions of BVP (1.1) such that (2.7) holds. The proof is complete. \square

THEOREM 2.3. *Suppose that there are positive numbers $0 < a < b < c$ such that*

$$0 < 2a + d < b < \frac{\sigma}{\rho} c. \quad (2.15)$$

Assume that $f(\varphi)$ satisfies the following conditions:

$$(A') \quad f(\varphi) < \phi_p(c/\rho) \text{ for } 0 \leq \|\varphi\|_C \leq 2c + d,$$

$$(B') \quad f(\varphi) > \phi_p(b/\sigma) \text{ for } b \leq \|\varphi\|_C \leq 2b,$$

$$(C') \quad f(\varphi) < \phi_p(a/\rho) \text{ for } 0 \leq \|\varphi\|_C \leq 2a + d.$$

Then BVP (1.1) has at least two positive solutions x_1 and x_2 such that

$$a < \max_{t \in [0, h]} x_1(t) < b < \max_{t \in [0, h]} x_2(t) < c. \quad (2.16)$$

The proof is omitted since it is similar to that of Theorem 2.2.

Now, we give theorems which may be considered as the corollaries of Theorems 2.2 and 2.3.

Let

$$f_0 = \lim_{\|\varphi\|_C \rightarrow 0} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}}; \quad f_\infty = \lim_{\|\varphi\|_C \rightarrow \infty} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}}, \quad (2.17)$$

and choose k_1, k_2, k_3 such that

$$k_i \sigma > 1, \quad i = 1, 2, \quad 0 < k_3 \rho < 1. \quad (2.18)$$

THEOREM 2.4. *Let the following conditions be satisfied:*

(D) $f_0 > k_1^{p-1}, f_\infty > k_2^{p-1}$;

(E) *there exists a $p_1 > 0$ such that for $0 \leq \|\varphi\|_C \leq 2p_1 + d$, one has $f(\varphi) < (p_1/\rho)^{p-1}$.*

Then BVP (1.1) has at least two positive solutions.

Proof. Firstly, choose $b = p_1$, then

$$f(\varphi) < \left(\frac{2p_1}{\rho}\right)^{p-1} = \phi_p\left(\frac{b}{\rho}\right) \quad \text{for } 0 \leq \|\varphi\|_C \leq 2b + d. \quad (2.19)$$

Secondly, since $f_0 > k_1^{p-1}$, there is $R_1 > 0$ sufficiently small such that

$$f(\varphi) > (k_1 \|\varphi\|_C)^{p-1} \quad \text{for } 0 \leq \|\varphi\|_C \leq R_1. \quad (2.20)$$

Without loss of generality, suppose that

$$R_1 \leq \frac{2\sigma}{\rho} b. \quad (2.21)$$

Choose $a > 0$ so that $a < (1/2)R_1$. For $a \leq \|\varphi\|_C \leq 2a$, we have $\|\varphi\|_C \leq R_1$ and $a < (\sigma/\rho)b$. Thus,

$$f(\varphi) > (k_1 \|\varphi\|_C)^{p-1} \geq (k_1 a)^{p-1} > \phi_p\left(\frac{a}{\sigma}\right) \quad \text{for } a \leq \|\varphi\|_C \leq 2a. \quad (2.22)$$

Thirdly, since $f_\infty > k_2^{p-1}$, there is $R_2 > 0$ sufficiently large such that

$$f(\varphi) > (k_2 \|\varphi\|_C)^{p-1} \quad \text{for } \|\varphi\|_C \geq R_2. \quad (2.23)$$

Without loss of generality, suppose that $R_2 > 2b$. Choose $c \geq R_2 + d$. Then,

$$f(\varphi) > (k_2 \|\varphi\|_C)^{p-1} \geq (k_2 c)^{p-1} > \phi_p\left(\frac{c}{\sigma}\right) \quad \text{for } c \leq \|\varphi\|_C \leq 2c. \quad (2.24)$$

We then have $0 < a < (\sigma/\rho)b < (\sigma/2\rho)(c - d)$, and now the conditions in Theorem 2.2 are all satisfied. By Theorem 2.2, BVP (1.1) has at least two positive solutions. The proof is complete. \square

8 Positive solutions of difference equations

THEOREM 2.5. *Let the following conditions be satisfied:*

(F) $f_0 < k_3^{p-1}$;

(G) *there exists a $p_2 > 0$ such that for $0 \leq \|\varphi\|_C \leq 2p_2$, one has $f(\varphi) > (p_2/\sigma)^{p-1}$.*

Then BVP (1.1) has at least two positive solutions.

The following corollaries are obvious.

COROLLARY 2.6. *Let the following conditions be satisfied:*

(D') $f_0 = \infty, f_\infty = \infty$;

(E) *there exists a $p_1 > 0$ such that for $0 \leq \|\varphi\|_C \leq 2p_1 + d$, one has $f(\varphi) < (p_1/\rho)^{p-1}$.*

Then BVP (1.1) has at least two positive solutions.

COROLLARY 2.7. *Let the following conditions be satisfied:*

(F') $f_0 = 0$;

(G) *there exists a $p_2 > 0$ such that for $0 \leq \|\varphi\|_C \leq 2p_2$, one has $f(\varphi) > (p_2/\sigma)^{p-1}$.*

Then BVP (1.1) has at least two positive solutions.

3. Example

Example 3.1. Consider BVP

$$\begin{aligned} \Delta \phi_p(\Delta x(t)) + r[x^{1/9}(t-1) + x^{1/3}(t-1)] &= 0, \quad t \in [0, 4], \\ x(t) = \psi(t), \quad t = -1, \quad x(0) = 0, \quad x(5) = x(6) = 1, \end{aligned} \quad (3.1)$$

where $\tau = 1, k = -1, N = 4, h = 3, \alpha = \beta = 0, r > 0$ is a constant satisfying $\sum_{n=h+\tau}^N r > 0, \psi(t) \geq 0, d = \|\psi\|_C = \max_{k=-1}^N |\psi(k)| > 0, p = 7/6, q = 7$, and $f(\varphi) = \varphi^{1/9}(-1) + \varphi^{1/3}(-1)$.

Suppose that $\varphi \in C^+$, then $\|\varphi\|_C = \varphi(-1)$.

As $\|\varphi\|_C \rightarrow 0$ or $\|\varphi\|_C \rightarrow +\infty$, we get

$$\begin{aligned} \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} &= \frac{\varphi^{1/9}(-1) + \varphi^{1/3}(-1)}{\|\varphi\|_C^{p-1}} \\ &= \|\varphi\|_C^{(10-9p)/9} + \|\varphi\|_C^{(4-3p)/3} \rightarrow +\infty. \end{aligned} \quad (3.2)$$

We deduce that

$$\rho = (\beta + h)\phi_q\left(\sum_{n=0}^N r(n)\right) = 3\left[\sum_{n=0}^4 r\right]^6 = 46875r, \quad (3.3)$$

thus, for all $m > 0$ and $0 \leq \|\varphi\|_C \leq m + d$, one has

$$0 \leq f(\varphi) \leq (m+d)^{1/9} + (m+d)^{1/3} = (m+d)^{1/9} \left(m^{1-p} + \frac{(m+d)^{2/9}}{m^{p-1}}\right) m^{p-1}. \quad (3.4)$$

Define $H(m) = (m+d)^{1/9}(m^{1-p} + (m+d)^{2/9}/m^{p-1})$.

Suppose that r and d satisfy

$$(2d)^{1/9}(d^{-1/6} + 2^{2/9}d^{1/18}) < \left(\frac{1}{2\rho}\right)^{p-1}; \quad (3.5)$$

then $H(d) = (2d)^{1/9}(d^{-1/6} + 2^{2/9}d^{1/18}) < (1/2\rho)^{p-1}$ holds. So, we can find a $p_1 = d/2$ such that $f(\varphi) \leq H(2p_1)(2p_1)^{p-1} < (p_1/\rho)^{p-1}$ for $0 \leq \|\varphi\|_C \leq 2p_1 + d$. By Corollary 2.6, we know that BVP (3.1) has at least two positive solutions.

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References

- [1] R. P. Agarwal and J. Henderson, *Positive solutions and nonlinear eigenvalue problems for third-order difference equations*, Computers & Mathematics with Applications **36** (1998), no. 10–12, 347–355.
- [2] R. I. Avery, C. J. Chyan, and J. Henderson, *Twin solutions of boundary value problems for ordinary differential equations and finite difference equations*, Computers & Mathematics with Applications **42** (2001), no. 3–5, 695–704.
- [3] A. Cabada, *Extremal solutions for the difference ϕ -Laplacian problem with nonlinear functional boundary conditions*, Computers & Mathematics with Applications **42** (2001), no. 3–5, 593–601.
- [4] J. Henderson, *Positive solutions for nonlinear difference equations*, Nonlinear Studies **4** (1997), no. 1, 29–36.
- [5] Y. Liu and W. Ge, *Twin positive solutions of boundary value problems for finite difference equations with p -Laplacian operator*, Journal of Mathematical Analysis and Applications **278** (2003), no. 2, 551–561.

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