Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 748789, 15 pages
doi:10.1155/2010/748789

Research Article

Exponential Stability and Global Attractors for a Thermoelastic Bresse System

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Received 13 September 2010; Accepted 29 October 2010

Academic Editor: E. Thandapani

We consider the stability properties for thermoelastic Bresse system which describes the motion of a linear planar shearable thermoelastic beam. The system consists of three wave equations and two heat equations coupled in certain pattern. The two wave equations about the longitudinal displacement and shear angle displacement are effectively damped by the dissipation from the two heat equations. We use multiplier techniques to prove the exponential stability result when the wave speed of the vertical displacement coincides with the wave speed of the longitudinal or of the shear angle displacement. Moreover, the existence of the global attractor is firstly achieved.

1. Introduction

In this paper, we will consider the following system:

$$\rho h w_{1tt} = (Eh(w_{1x} - kw_3) - \alpha\theta_{1t})_x - kGh(\phi_2 + w_{3x} + kw_1), \tag{1.1}$$

$$\rho h w_{3tt} = Gh(\phi_2 + w_{3x} + kw_1)_x + kEh(w_{1x} - kw_3) - k\alpha\theta_{1t}, \tag{1.2}$$

$$\rho I \phi_{2tt} = EI\phi_{2xx} - Gh(\phi_2 + w_{3x} + kw_1) - \alpha\theta_{3x}, \tag{1.3}$$

$$\rho c\theta_{1tt} = \theta_{1xxt} + \theta_{1xx} - \alpha T_0(w_{1tx} - kw_{3t}), \tag{1.4}$$

$$\rho c\theta_{3t} = \theta_{3xx} - \alpha T_0\phi_{2tx}, \tag{1.5}$$

together with initial conditions

$$\begin{gathered} w_1(x,0) = u_0(x), \qquad w_{1t}(x,0) = v_0(x), \qquad \phi_2(x,0) = \phi_0(x), \\ \phi_{2t}(x,0) = \psi_0(x), \qquad w_3(x,0) = w_0(x), \qquad w_{3t}(x,0) = \varphi_0(x), \\ \theta_1(x,0) = \theta_0(x), \qquad \theta_{1t}(x,0) = \eta_0(x), \qquad \theta_3(x,0) = \xi_0(x) \end{gathered} \tag{1.6}$$

and boundary conditions

$$w_1(x,t) = w_{3x}(x,t) = \phi_2(x,t) = \theta_1(x,t) = \theta_3(x,t) = 0, \quad \text{for } x = 0, 1, \tag{1.7}$$

where w_1, w_3, and ϕ_2 are the longitudinal, vertical, and shear angle displacement, θ_1 and θ_3 are the temperature deviations from the T_0 along the longitudinal and vertical directions, E, G, ρ, I, m, k, h, and c are positive constants for the elastic and thermal material properties.

From this seemingly complicated system, very interesting special cases can be obtained. In particular, the isothermal system is exactly the system obtained by Bresse [1] in 1856. The Bresse system, (1.1)–(1.3) with θ_1, θ_3 removed, is more general than the well-known Timoshenko system where the longitudinal displacement w_1 is not considered. If both θ_1 and w_1 are neglected, the Bresse thermoelastic system simplifies to the following Timoshenko thermoelastic system:

$$\begin{aligned}
\rho h w_{3tt} &= Gh(\phi_2 + w_{3x})_x, \\
\rho I \phi_{2tt} &= EI\phi_{2xx} - Gh(\phi_2 + w_{3x}) - \alpha\theta_{3tx}, \\
\rho c \theta_{3tt} &= \theta_{3xxt} + \theta_{3xx} - \alpha T_0 \phi_{2tx},
\end{aligned} \tag{1.8}$$

which was studied by Messaoudi and Said-Houari [2]. For the boundary conditions

$$w_3(x,t) = \phi_2(x,t) = \theta_{3x}(x,t) = 0, \quad \text{at } x = 0, l, \tag{1.9}$$

they obtained exponential stability for the thermoelastic Timoshenko system (1.8) when $E = G$; later, they proved energy decay for a Timoshenko-type system with history in thermoelasticity of type III [3], and this paper is similar to [2] with an extra damping that comes from the presence of a history term; it improves the result of [2] in the sense that the case of nonequal wave speed has been considered and the relaxation function g is allowed to decay exponentially or polynomially. We refer the reader to [4–10] for the Timoshenko system with other kinds of damping mechanisms such as viscous damping, viscoelastic damping of Boltzmann type acting on the motion equation of w_3 or ϕ_2. In all these cases, the rotational displacement ϕ_2 of the Timoshenko system is effectively damped due to the thermal energy dissipation. In fact, the energy associated with this component of motion decays exponentially. The transverse displacement w_3 is only indirectly damped through the coupling, which can be observed from (1.2). The effectiveness of this damping depends on the type of coupling and the wave speeds. When the wave speeds are the same ($E = G$), the indirect damping is actually strong enough to induce exponential stability for the Timoshenko system, but when the wave speeds are different, the Timoshenko system loses the exponential stability. This phenomenon has been observed for partially damped second-order evolution equations. We would like to mention other works in [11–15] for other related models.

Recently, Liu and Rao [16] considered a similar system; they used semigroup method and showed that the exponentially decay rate is preserved when the wave speed of the vertical displacement coincides with the wave speed of longitudinal displacement or of the shear angle displacement. Otherwise, only a polynomial-type decay rate can be obtained; their main tools are the frequency-domain characterization of exponential decay obtained by Prüss [17] and Huang [18] and of polynomial decay obtained recently by Muñoz Rivera and Fernández Sare [5]. For the attractors, we refer to [19–24].

In this paper, we consider system (1.1)–(1.5); that is, we use multiplier techniques to prove the exponential stability result only for $E = G$. However, from the theory of elasticity, E and G denote Young's modulus and the shear modulus, respectively. These two elastic moduli are not equal since

$$G = \frac{E}{2(1+\nu)}, \tag{1.10}$$

where $\nu \in (0, 1/2)$ is the Poisson's ratio. Thus, the exponential stability for the case of $E = G$ is only mathematically sound. However, it does provide useful insight into the study of similar models arising from other applications.

2. Equal Wave Speeds Case: $E = G$

Here we state and prove a decay result in the case of equal wave speeds propagation.

Define the state spaces

$$\mathscr{H} = H_0^1 \times H_*^1 \times H_0^1 \times H_0^1 \times \left(L^2\right)^5, \tag{2.1}$$

where

$$H_*^1 = \left\{ f \in H^1(0,1) \mid \int_0^1 f(x) = 0 \right\}. \tag{2.2}$$

The associated energy term is given by

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^1 \Bigg(&\left[Eh(w_{1x} - kw_3)^2 + Gh(\phi_2 + w_{3x} + kw_1)^2 + EI\phi_{2x}^2 \right] \\ &+ \left[\rho h\left(w_{1t}^2 + w_{3t}^2\right) + \rho I\phi_{2t}^2 \right] + \frac{\rho c}{T_0}\left(\theta_{1t}^2 + \theta_{1x}^2 + \theta_3^2\right) \Bigg) dx. \end{aligned} \tag{2.3}$$

By a straightforward calculation, we have

$$\frac{dE(t)}{dt} = -\frac{1}{T_0}\left(\|\theta_{1xt}\|^2 + \|\theta_{3x}\|^2 \right). \tag{2.4}$$

From semigroup theory [25, 26], we have the following existence and regularity result; for the explicit proofs, we refer the reader to [16].

Lemma 2.1. *Let $u_0(x), w_0(x), \varphi_0(x), \theta_0(x), v_0(x), \phi_0(x), \psi_0(x), \eta_0(x), \xi_0(x) \in \mathscr{H}$ be given. Then problem* (1.1)–(1.5) *has a unique global weak solution (φ, ψ, θ) verifying*

$$\begin{gathered} w_3(x,t) \in C\left(R^+, H^1_*(0,1)\right) \cap C^1\left(R^+, L^2(0,1)\right), \\ \left(w_1(x,t), \phi_2(x,t), \theta_1(x,t), \theta_3(x,t)\right) \in C\left(R^+, H^1_0(0,1)\right) \cap C^1\left(R^+, L^2(0,1)\right). \end{gathered} \tag{2.5}$$

We are now ready to state our main stability result.

Theorem 2.2. *Suppose that $E = G$ and $u_0(x), w_0(x), \varphi_0(x), \theta_0(x), v_0(x), \phi_0(x), \psi_0(x), \eta_0(x), \xi_0(x) \in \mathscr{H}$. Then the energy $E(t)$ decays exponentially as time tends to infinity; that is, there exist two positive constants C and μ independent of the initial data and t, such that*

$$E(t) \le CE(0)e^{-\mu t}, \quad \forall t > 0. \tag{2.6}$$

The proof of our result will be established through several lemmas.

Let

$$I_1 = \int_0^1 \rho I \phi_{2t}\phi_2 + \rho h w_{3t} f, \tag{2.7}$$

where f is the solution of

$$-f_{xx} = \phi_{2x}, \qquad f(0) = f(1) = 0. \tag{2.8}$$

Lemma 2.3. *Letting w_1, w_3, ϕ_2, θ_1, θ_3 be a solution of* (1.1)–(1.5), *then one has, for all $\varepsilon_1 > 0$,*

$$\begin{aligned} \frac{I_1(t)}{dt} \le & -\frac{EI}{2}\|\phi_{2x}\|^2 + \rho I\|\phi_{2t}\|^2 + \varepsilon_1\left(\|w_{3t}\|^2 + \|(w_{1x} - kw_3)\|^2\right) \\ & + C(\varepsilon_1)\left(\|\theta_{3x}\|^2 + \|\theta_{1xt}\|^2 + \|\phi_{2t}\|^2\right). \end{aligned} \tag{2.9}$$

Proof.

$$\begin{aligned} \frac{dI_1}{dt} = & -EI\|\phi_{2x}\|^2 + \rho I\|\phi_{2t}\|^2 - \int_0^1 \alpha\theta_{3x}\phi_2 dx \\ & - kEh\int_0^1 (w_{1x} - kw_3) f dx - k\alpha\int_0^1 \theta_{1t} f dx + \rho h\int_0^1 w_{3t} f_t dx, \end{aligned} \tag{2.10}$$

By using the inequalities

$$\begin{gathered} \int_0^1 f_x^2 dx \le \int_0^1 \phi_2^2 dx \le \int_0^1 \phi_{2x}^2 dx, \\ \int_0^1 f_t^2 dx \le \int_0^1 f_{tx}^2 dx \le \int_0^1 \phi_{2t}^2 dx, \end{gathered} \tag{2.11}$$

and Young's inequality, the assertion of the lemma follows.

Let

$$I_2 = \rho c \rho h \int_0^1 \left(\int_0^x \theta_{1t} dy \right) w_{1t} dx. \tag{2.12}$$

□

Lemma 2.4. *Letting w_1, w_3, ϕ_2, θ_1, θ_3 be a solution of* (1.1)–(1.5), *then one has, for all $\varepsilon_2 > 0$,*

$$\begin{aligned} \frac{dI_2(t)}{dt} \leq \frac{-\alpha\rho h T_0}{2} \int_0^1 w_{1t}^2 dx + C(\varepsilon_2)\left(\|\theta_{1xt}\|^2 + \|w_{3t}\|^2 \right) \\ + \varepsilon_2 \left(\|(w_{1x} - kw_3)\|^2 + \|\phi_2 + w_{3x} + kw_1\|^2 \right). \end{aligned} \tag{2.13}$$

Proof. Using (1.4) and (1.1), we get

$$\begin{aligned} \frac{I_2(t)}{dt} &= \rho c \rho h \int_0^1 \left(\int_0^x \theta_{1tt} dy \right) w_{1t} dx + \rho c \rho h \int_0^1 \left(\int_0^x \theta_{1t} dy \right) w_{1tt} dx \\ &= \rho h \int_0^1 \left(\int_0^x \theta_{1xxt} + \theta_{1xx} - \alpha T_0 (w_{1tx} - kw_{3t}) dy \right) w_{1t} dx \\ &\quad + \int_0^1 \left(\int_0^x \theta_{1t} dy \right) \left((Eh(w_{1x} - kw_3) - \alpha\theta_{1t})_x - KGh(\phi_2 + w_{3x} + kw_1) \right) dx \\ &= \rho h \int_0^1 (\theta_{1xt} + \theta_{1x}) w_{1t} dx - \rho h \alpha T_0 \int_0^1 w_{1t}^2 dx + \rho h k \int_0^1 \left(\int_0^x w_{3t} dy \right) w_{1t} dx \\ &\quad + \rho h E h \int_0^1 \left(\theta_{1xt} w_1 + k\theta_{1t} w_3 + \alpha\theta_{1t}^2 \right) dx \\ &\quad - \rho c K G h \int_0^1 \left(\int_0^x \theta_{1t} dy \right) (\phi_2 + w_{3x} + kw_1) dx. \end{aligned} \tag{2.14}$$

The assertion of the lemma then follows, using Young's and Poincaré's inequalities.

Let

$$I_3 = \rho c \rho I \int_0^1 \left(\int_0^x \theta_3 dy \right) \phi_{2t} dx. \tag{2.15}$$

□

Lemma 2.5. *Letting w_1, w_3, ϕ_2, θ_1, θ_3 be a solution of* (1.1)–(1.5), *then one has, for all $\varepsilon_3 > 0$,*

$$\frac{dI_3}{dt} \leq -\frac{\alpha\rho I T_0}{2} \|\phi_{2t}\|^2 + C(\varepsilon_3)\|\theta_{3x}\|^2 + \varepsilon_3 \|\phi_{2x}\|^2 + \varepsilon_3 \|\phi_2 + w_{3x} + kw_1\|^2. \tag{2.16}$$

Proof. Using (1.3) and (1.5), we have

$$\begin{aligned}
\frac{dI_3}{dt} &= \rho c\rho I\int_0^1\left(\int_0^x \theta_{3t}dy\right)\phi_{2t}dx + \rho c\rho I\int_0^1\left(\int_0^x \theta_3 dy\right)\phi_{2tt}dx \\
&= \rho I\int_0^1\int_0^x (\theta_{3xx} - \alpha T_0\phi_{2xt})dy\phi_{2t}dx \\
&\quad + \rho c\int_0^1\left(\int_0^x \theta_3 dy\right)(EI\phi_{2xx} - Gh(\phi_2 + w_{3x} + kw_1) - \alpha\theta_{3x})dx \\
&= \rho I\int_0^1 \theta_{3x}\phi_{2t}dx - \alpha IT_0\int_0^1 \phi_{2t}^2 dx + \rho cEI\int_0^1 \theta_3\phi_{2x}dx \\
&\quad - \rho cGh\int_0^1\left(\int_0^x \theta_3 dy\right)(\phi_2 + w_{3x} + kw_1)dx - \alpha\rho c\int_0^1 \theta_3^2 dx.
\end{aligned} \tag{2.17}$$

Then, using Young's and Poincaré's inequalities, we can obtain the assertion.

Next, we set

$$I_4 = h\rho I\int_0^1 \phi_{2t}(\phi_2 + w_{3x} + kw_1)dx + h\rho I\int_0^1 \phi_{2x}\mathrm{w}_{3t}dx. \tag{2.18}$$

□

Lemma 2.6. *Letting w_1, w_3, ϕ_2, θ_1, θ_3 be a solution of* (1.1)–(1.5)*, then one has, for all $\varepsilon_4 > 0$,*

$$\begin{aligned}
\frac{dI_4}{dt} \le &-\frac{Gh^2}{2}\int_0^1 (\phi_2 + w_{3x} + kw_1)^2 dx + C(\varepsilon_4)\left(\|\theta_{3x}\|^2 + \|\theta_{1xt}\|^2\right) \\
&+ \frac{kh\rho I}{2}\left(\|\phi_{2t}\|^2 + \|w_{1t}\|^2\right) + C(\varepsilon_4)\|\phi_{2x}\|^2 + \varepsilon_4\|w_{1x} - kw_3\|^2.
\end{aligned} \tag{2.19}$$

Proof. Letting (1) $= I\int_0^1 \phi_{2t}(\phi_2 + w_{3x} + kw_1)dx$, (2) $= h\rho I\int_0^1 \phi_{2x}w_{3t}dx$, then using (1.2), (1.3), we have

$$\begin{aligned}
(1)' &= \rho I\int_0^1 \phi_{2tt}(\phi_2 + w_{3x} + kw_1)dx + h\rho I\int_0^1 \phi_{2t}(\phi_2 + w_{3x} + kw_1)_t dx \\
&= hEI\int_0^1 \phi_{2xx}(\phi_2 + w_{3x} + kw_1)dx - Gh^2\int_0^1 (\phi_2 + w_{3x} + kw_1)^2 dx \\
&\quad - \alpha h\int_0^1 \theta_{3x}(\phi_2 + w_{3x} + kw_1)dx + h\rho I\int_0^1 \phi_{2t}^2 dx + h\rho I\int_0^1 \phi_{2t}(w_{3x} + kw_1)_t dx,
\end{aligned}$$

$$
\begin{aligned}
(2)' &= I\rho h \int_0^1 \phi_{2xt} w_{3t}\,dx + I\rho h \int_0^1 \phi_{2x} w_{3tt}\,dx \\
&= -I\rho h \int_0^1 \phi_{2t} w_{3xt}\,dx + IGh \int_0^1 \phi_{2x} \left(\phi_2 + w_{3x} + kw_1\right)_x dx \\
&\quad + IkEh \int_0^1 \phi_{2x}(w_{1x} - kw_3)\,dx - \alpha Ik \int_0^1 \phi_{2x}\theta_{1t}\,dx.
\end{aligned}
\tag{2.20}
$$

Noticing that $E = G$, then

$$
\begin{aligned}
I_4' &= (1)' + (2)' \\
&= -Gh^2 \int_0^1 \left(\phi_2 + w_{3x} + kw_1\right)^2 dx - \alpha h \int_0^1 \theta_{3x}\left(\phi_2 + w_{3x} + kw_1\right) dx + h\rho I \int_0^1 \phi_{2t}^2\,dx \\
&\quad + khI\rho \int_0^1 \phi_{2t} w_{1t}\,dx + IkEh \int_0^1 \phi_{2x}(w_{1x} - kw_3)\,dx - \alpha Ik \int_0^1 \phi_{2x}\theta_{1t}\,dx.
\end{aligned}
\tag{2.21}
$$

Then, using Young's inequality, we can obtain the assertion.

We set

$$
I_5 = -h\rho \int_0^1 w_{3t}(w_{1x} - kw_3)\,dx - h\rho \int_0^1 w_{1t}\left(\phi_2 + w_{3x} + kw_1\right) dx. \tag{2.22}
$$

□

Lemma 2.7. *Letting w_1, w_3, ϕ_2, θ_1, θ_3 be a solution of* (1.1)–(1.5), *then one has, for all $\varepsilon_5 > 0$,*

$$
\begin{aligned}
\frac{dI_5}{dt} &\le -\frac{kEh}{2}\|(w_{1x} - kw_3)\|^2 - \frac{\rho h}{2}\|w_{1t}\|^2 + k\rho h\|w_{3t}\|^2 \\
&\quad + \frac{\rho h}{2}\|\phi_{2t}\|^2 + C(\varepsilon_5)\|\theta_{1xt}\|^2 + (kGh + \varepsilon_5)\|\left(\phi_2 + w_{3x} + kw_1\right)\|^2.
\end{aligned}
\tag{2.23}
$$

Proof. Let $(1) = -h\rho \int_0^1 w_{3t}(w_{1x} - kw_3)\,dx$, $(2) = -h\rho \int_0^1 w_{1t}(\phi_2 + w_{3x} + kw_1)\,dx$, then using (1.1), (1.2), we have

$$
\begin{aligned}
(1)' &= -Gh \int_0^1 \left(\phi_2 + w_{3x} + kw_1\right)_x (w_{1x} - kw_3)\,dx - kEh \int_0^1 (w_{1x} - kw_3)^2 dx \\
&\quad + \alpha k \int_0^1 \theta_{1t}(w_{1x} - kw_3)\,dx + k\rho h \int_0^1 w_{3t}^2 - \rho h \int_0^1 w_{3t} w_{1xt}\,dx, \\
(2)' &= -Eh \int_0^1 (w_{1x} - kw_3)_x \left(\phi_2 + w_{3x} + kw_1\right) dx + \alpha \int_0^1 \theta_{1xt}\left(\phi_2 + w_{3x} + kw_1\right) dx \\
&\quad + kGh \int_0^1 \left(\phi_2 + w_{3x} + kw_1\right)^2 dx - \rho h \int_0^1 w_{1t}^2\,dx - \rho h \int_0^1 w_{1t}\phi_{2t}\,dx + \rho h \int_0^1 w_{1tx} w_{3t}\,dx.
\end{aligned}
\tag{2.24}
$$

Then, noticing $E = G$, again, from the above two equalities and Young's inequality, we can obtain the assertion.

Next, we set

$$I_6 = -\rho h \int_0^1 w_{3t} w_3 dx - \rho h \int_0^1 w_{1t} w_1 dx. \tag{2.25}$$

□

Lemma 2.8. *Letting w_1, w_3, ϕ_2, θ_1, θ_3 be a solution of* (1.1)–(1.5), *then one has*

$$\frac{dI_6}{dt} \leq -\rho h \left(\|w_{3t}\|^2 + \|w_{1t}\|^2 \right) + C\|\theta_{1xt}\|^2 + C\|\phi_{2x}\|^2. \tag{2.26}$$

Proof. Using (1.1), (1.2), we have

$$\begin{aligned} I_6' = {} & -\rho h \int_0^1 w_{3t}^2 dx - \rho h \int_0^1 w_{1t}^2 dx + Eh \int_0^1 (w_{1x} - k w_3)^2 dx \\ & + Gh \int_0^1 (\phi_2 + w_{3x} + k w_1)(w_{3x} + k w_1) dx - \alpha \int_0^1 \theta_{1t} (w_{1x} - k w_3) dx. \end{aligned} \tag{2.27}$$

Noticing (2.3) and (2.4), we have that $\exists C_1 > 0$ satisfy the following:

$$-\alpha \int_0^1 \theta_{1t} (w_{1x} - k w_3) dx \leq C_1 \|\theta_{1xt}\|^2 - Eh \|w_{1x} - k w_3\|^2. \tag{2.28}$$

Similarly,

$$\begin{aligned} & Gh \int_0^1 (\phi_2 + w_{3x} + k w_1)(w_{3x} + k w_1) dx \\ & \quad = Gh \|\phi_2 + w_{3x} + k w_1\|^2 - Gh \int_0^1 (\phi_2 + w_{3x} + k w_1) \phi_2 dx \\ & \quad \leq C_1 \|\phi_{2x}\|^2. \end{aligned} \tag{2.29}$$

Then, insert (2.28) and (2.29) into (2.27), and the assertion of the lemma follows.

Now, we set

$$I_7 = \rho c \int_0^1 \theta_{1t} \theta_1 dx + \frac{1}{2} \|\theta_{1x}\|^2. \tag{2.30}$$

□

Lemma 2.9. *Letting w_1, w_3, ϕ_2, θ_1, θ_3 be a solution of* (1.1)–(1.5), *then one has, for all $\varepsilon_7 > 0$,*

$$\frac{dI_7}{dt} \leq -\frac{1}{2} \|\theta_{1x}\|^2 + \rho c \|\theta_{3xt}\|^2 + C(\varepsilon_7) \left(\|w_{1t}\|^2 + \|w_{3t}\|^2 \right). \tag{2.31}$$

Proof. Using (1.5), we have

$$\frac{dI_7}{dt} = -\|\theta_{1x}\|^2 + \alpha T_0 \int_0^1 w_{1t}\theta_{1x}dx + \alpha T_0 k \int_0^1 w_{3t}\theta_{1x}dx + \rho c\|\theta_{1t}\|^2. \tag{2.32}$$

Then, using Young's and Poincaré's inequalities, we can obtain the assertion.

Now, letting $N, N_1, N_2, N_3, N_4, N_5, N_6, N_7 > 0$, we define the Lyapunov functional $\mathcal{F}$ as follows:

$$\mathcal{F} = NE + N_1I_1 + N_2I_2 + N_3I_3 + N_4I_4 + N_5I_5 + N_6I_6 + N_7I_7. \tag{2.33}$$

By using (2.4), (2.9), (2.13), (2.16), (2.19), (2.23), (2.26), and (2.31), we have

$$\begin{aligned} \frac{d\mathcal{F}}{dt} \le\ & \Upsilon_1\|\theta_{1xt}\|^2 + \Upsilon_2\|\theta_{3x}\|^2 + \Upsilon_3\|\phi_{2x}\|^2 + \Upsilon_4\|w_{1t}\|^2 + \Upsilon_5\|\phi_{2t}\|^2 \\ & + \Upsilon_6\|\phi_2 + w_{3x} + kw_1\|^2 + \Upsilon_7\|w_{1x} - kw_3\|^2 + \Upsilon_8\|w_{3t}\|^2 + \Upsilon_9\|\theta_{1x}\|^2, \end{aligned} \tag{2.34}$$

where

$$\begin{aligned} \Upsilon_1 &= -\frac{N}{T_0} + C(\varepsilon_1)N_1 + N_2C(\varepsilon_2) + N_4C(\varepsilon_4) + N_5C(\varepsilon_5) + N_7C_1 + \rho cN_7, \\ \Upsilon_2 &= -\frac{N}{T_0} + C(\varepsilon_1)N_1 + N_3C(\varepsilon_3) + N_4C(\varepsilon_4), \\ \Upsilon_3 &= -\frac{N_1EI}{2} + \varepsilon_3N_3 + C(\varepsilon_4)N_4 + C_1N_6, \\ \Upsilon_4 &= -\frac{\alpha\rho hT_0N_2}{2} + \frac{kh\rho IN_4}{2} - \frac{\rho hN_5}{2} - \rho hN_6 + N_7C(\varepsilon_7), \\ \Upsilon_5 &= -\frac{\alpha\rho hT_0N_3}{2} + \frac{kh\rho IN_4}{2} + N_1\rho I + N_1C(\varepsilon_1) + \frac{\rho hN_5}{2}, \\ \Upsilon_6 &= -\frac{Gh^2N_4}{2} + kGhN_5 + N_5\varepsilon_5 + N_3\varepsilon_3 + N_2\varepsilon_2, \\ \Upsilon_7 &= -\frac{kEhN_5}{2} + N_4\varepsilon_4 + N_1\varepsilon_1 + N_2\varepsilon_2, \\ \Upsilon_8 &= -N_6\rho h + N_5k\rho h + C(\varepsilon_2)N_2 + N_1\varepsilon_1 + N_7C(\varepsilon_7), \\ \Upsilon_9 &= -\frac{N_7}{2} + N_2C(\varepsilon_2). \end{aligned} \tag{2.35}$$

We can choose N big enough, $\varepsilon_1, \ldots, \varepsilon_8$ small enough, and

$$
\begin{gathered}
N_1 \gg N_4, N_6, \\
N_2 \gg N_4, \\
N_3 \gg N_1, N_6, \\
N_4 \gg N_5, \\
N_6 \gg N_2, N_5, N_7, \\
N_7 \gg N_2.
\end{gathered}
\tag{2.36}
$$

Then $\Upsilon_1, \ldots, \Upsilon_9$ are all negative constants; at this point, there exists a constant $\omega > 0$, and (2.34) takes the form

$$
\begin{aligned}
\frac{d\mathcal{F}}{dt} \leq -\omega\Big(& \|\theta_{1xt}\|^2 + \|\theta_{1x}\|^2 + \|\theta_{3x}\|^2 + \|\phi_{2x}\|^2 + \|w_{1t}\|^2 \\
&+ \|\phi_{2t}\|^2 + \|\phi_2 + w_{3x} + kw_1\|^2 + \|w_{1x} - kw_3\|^2 + \|w_{3t}\|^2\Big).
\end{aligned}
\tag{2.37}
$$

We are now ready to prove Theorem 2.2. □

Proof of Theorem 2.2. Firstly, from the definition of $\mathcal{F}$, we have

$$
\mathcal{F} \sim E(t),
\tag{2.38}
$$

which, from (2.37) and (2.38), leads to

$$
\frac{d}{dt}\mathcal{F} \leq -\mu\mathcal{F}.
\tag{2.39}
$$

Integrating (2.39) over $(0, t)$ and using (2.38) lead to (2.6). This completes the proof of Theorem 2.2. □

3. Global Attractors

In this section, we establish the existence of the global attractor for system (1.1)–(1.5).

Setting $v = w_{1t}$, $\varphi = w_{3t}$, $\psi = \phi_{2t}$, $\eta = \theta_{1t}$, then, (1.1)–(1.5) can be transformed into the system

$$\begin{aligned}
w_{1t} &= v,\\
w_{3t} &= \varphi,\\
\phi_{2t} &= \psi,\\
\theta_{1t} &= \eta,\\
\rho h v_t &= (Eh(w_{1x} - kw_3) - \alpha\theta_1)_x - kGh(\phi_2 + w_{3x} + kw_1),\\
\rho h \varphi_t &= Gh(\phi_2 + w_{3x} + kw_1)_x + kEh(w_{1x} - kw_3) - k\alpha\theta_1,\\
\rho I \psi_t &= EI\phi_{2xx} - Gh(\phi_2 + w_{3x} + kw_1) - \alpha\theta_{3x},\\
\rho c \eta_t &= \theta_{1xxt} + \theta_{1xx} - \alpha T_0(w_{1tx} - kw_{3t}),\\
\rho c \theta_{3t} &= \theta_{3xx} - \alpha T_0 \phi_{2tx}.
\end{aligned} \tag{3.1}$$

We consider the problem in the following Hilbert space:

$$\mathscr{H} = H_0^1 \times H_*^1 \times H_0^1 \times H_0^1 \times \left(L^2\right)^5. \tag{3.2}$$

Recall that the global attractor of $S(t)$ acting on $\mathscr{H}$ is a compact set $\mathscr{A} \subset \mathscr{H}$ enjoying the following properties:

(1) $\mathscr{A}$ is fully invariant for $S(t)$, that is, $S(t)\mathscr{A} = \mathscr{A}$ for every $t \geq 0$;

(2) $\mathscr{A}$ is an attracting set, namely, for any bounded set $\mathcal{R} \subset \mathscr{H}$,

$$\lim_{t\to\infty} \delta_{\mathscr{H}}(S(t)\mathcal{R}, \mathscr{A}) = 0, \tag{3.3}$$

where $\delta_{\mathscr{H}}$ denotes the Hausdorff semidistance on $\mathscr{H}$.

More details on the subject can be found in the books [23, 26, 27].

Remark 3.1. The uniform energy estimate (2.6) implies the existence of a bounded absorbing set $\mathcal{R}^* \subset \mathscr{H}$ for the C_0 semigroup $S(t)$. Indeed, if $\mathcal{R}^*$ is any ball of $\mathscr{H}$, then for any bounded set $\mathcal{R} \subset \mathscr{H}$, it is immediate to see that there exists $t(\mathcal{R}) \geq 0$ such that

$$S(t)\mathcal{R} \subset \mathcal{R}^* \tag{3.4}$$

for every $t \geq t(\mathcal{R})$.

Moreover, if we define

$$\mathcal{R}_0 = \bigcup_{t \geq 0} S(t)\mathcal{R}^*, \tag{3.5}$$

it is clear that $\mathcal{R}_0$ is still a bounded absorbing set which is also invariant for $S(t)$, that is, $S(t)\mathcal{R}_0 \subset \mathcal{R}_0$ for every $t \geq 0$.

In the sequel, we define the operator A as $Af = -f_{xx}$ with Dirichlet boundary conditions. It is well known that A is a positive operator on L^2 with domain $\mathfrak{D}(A) = H^2 \cap H_0^1$. Moreover, we can define the powers A^s of A for $s \in R$. The space $V_{2s} = \mathfrak{D}(A^s)$ turns out to be a Hilbert space with the inner product

$$\langle u, v \rangle_{V_{2s}} = \langle A^s u, A^s v \rangle, \tag{3.6}$$

where $\langle \cdot \rangle$ stands for L^2-inner product on L^2.

In particular, $V_{-1} = H^{-1}$, $V_0 = L^2$, and $V_1 = H_0^1$. The injection $V_{s_1} \hookrightarrow V_{s_2}$ is compact whenever $s_1 > s_2$. For further convenience, for $s \in R$, introduce the Hilbert space

$$\mathscr{H}_s = V_{1+s} \times V_{1+s} \times V_{1+s} \times V_{1+s} \times (V_s)^5. \tag{3.7}$$

Clearly, $\mathscr{H}_0 = \mathscr{H}$.

Now, let $z_0 = (u_0, w_0, \phi_0, \theta_0, v_0, \varphi_0, \psi_0, \eta_0, \xi_0)$, where $\mathcal{R}_0$ is the invariant, bounded absorbing set of $S(t)$ given by Remark 3.1, and take the inner product in $\mathscr{H}_0$ of (3.1) and $(A^\sigma w_1, A^\sigma w_3, A^\sigma \phi_2, A^\sigma \theta_1, A^\sigma v, A^\sigma \varphi, A^\sigma \psi, A^\sigma \eta, A^\sigma \theta_3)$ to get

$$\begin{aligned}
&\frac{d}{dt}\Big(Eh\|(w_{1x} - kw_3)\|_\sigma^2 + Gh\|(\phi_2 + w_{3x} + kw_1)\|_\sigma^2 + EI\|\phi_2\|_{1+\sigma}^2 \\
&\quad + \rho h\Big(\|w_{1t}\|_\sigma^2 + \|w_{3t}\|_\sigma^2\Big) + \rho I\|\phi_{2t}\|_\sigma^2 + \frac{\rho c}{T_0}\Big(\|\theta_{1t}\|_\sigma^2 + \|\theta_1\|_{1+\sigma}^2 + \|\theta_3\|_\sigma^2\Big)\Big) \\
&= -\frac{2}{T_0}\Big(\|\theta_{1t}\|_{1+\sigma}^2 + \|\theta_3\|_{(1+\sigma)}^2\Big).
\end{aligned} \tag{3.8}$$

Here, the boundary term of integration by parts is neglected since we are working with more regular functions. We denote

$$
\begin{aligned}
E_2(t) &= Eh\|(w_{1x} - kw_3)\|_\sigma^2 + Gh\|(\phi_2 + w_{3x} + kw_1)\|_\sigma^2 + EI\|\phi_2\|_{1+\sigma}^2 \\
&\quad + \rho h\left(\|w_{1t}\|_\sigma^2 + \|w_{3t}\|_\sigma^2\right) + \rho I\|\phi_{2t}\|_\sigma^2 + \frac{\rho c}{T_0}\left(\|\theta_1\|_\sigma^2 + \|\theta_3\|_\sigma^2\right), \\
F_1(t) &= \int_0^1 \rho I A^\sigma \phi_{2t}\phi_2 dx + \int_0^1 \rho h A^\sigma w_{3t} f dx, \\
F_2(t) &= \rho c \rho h \int_0^1 \left(\int_0^x \theta_1 dy\right) A^\sigma w_{1t} dx, \\
F_3 &= \rho c \rho I \int_0^1 \left(\int_0^x \theta_3 dy\right) A^\sigma \phi_{2t} dx, \\
F_4(t) &= h\rho I \int_0^1 A^\sigma \phi_{2t}(\phi_2 + w_{3x} + kw_1) dx + h\rho I \int_0^1 A^\sigma \phi_{2x} w_{3t} dx, \\
F_5(t) &= -h\rho \int_0^1 A^\sigma w_{3t}(w_{1x} - kw_3) dx - h\rho \int_0^1 A^\sigma w_{1t}(\phi_2 + w_{3x} + kw_1) dx, \\
F_6(t) &= -\rho h \int_0^1 A^\sigma w_{3t} w_3 dx - \rho h \int_0^1 A^\sigma w_{1t} w_1 dx, \\
F_7 &= \rho c \int_0^1 A^\sigma \theta_{1t}\theta_1 dx + \frac{1}{2}\int_0^1 A^\sigma \theta_{1x}^2 dx.
\end{aligned}
\tag{3.9}
$$

Then, introduce the functional

$$
\mathcal{Q}(t) = NE_2(t) + N_1F_1 + N_2F_2 + N_3F_3 + N_4F_4 + N_5F_5 + N_6F_6 + N_7F_7. \tag{3.10}
$$

By repeating similar argument as in the proofs of Lemmas 2.3–2.9 and (3.8), choosing our constants very carefully and properly, we get

$$
\frac{d}{dt}\mathcal{Q}(t) + cE_2(t) \le 0. \tag{3.11}
$$

On the other hand,

$$
\mathcal{Q}(t) \sim E_2(t), \tag{3.12}
$$

so that

$$
\frac{d}{dt}\mathcal{Q}(t) + c_1\mathcal{Q}(t) \le 0, \tag{3.13}
$$

which gives

$$E_2(t) \sim \mathcal{J}(t) \le c_2 e^{-c_1 t}, \qquad \|z(t)\|_{\mathcal{H}_\sigma} \le c_2 e^{-c_1 t}. \tag{3.14}$$

Let $\mathcal{R}(t)$ be the ball of $V_{3/2} \times V_{3/2} \times V_{3/2} \times V_{3/2} \times (V_{1/2})^5$; from the compact embedding $V_{3/2} \times V_{3/2} \times V_{3/2} \times V_{3/2} \times (V_{1/2})^5 \hookrightarrow H_0^1 \times H_*^1 \times H_0^1 \times H_0^1 \times (L^2)^5$, $\mathcal{R}(t)$ is compact in $\mathcal{H}$. Then, due to the compactness of $\mathcal{R}(t)$, for every fixed $t \ge 0$ and every $d > c_2 e^{-c_1 t}$, there exist finitely many balls of $\mathcal{H}$ of radius d such that $z(t)$ belongs to the union of such balls, for every $z_0 \in \mathcal{R}_0$. This implies that

$$\alpha_{\mathcal{H}}(S(t)\mathcal{R}_0) \le c_2 e^{-c_1 t}, \quad \forall t \ge 0, \tag{3.15}$$

where $\alpha_{\mathcal{H}}$ is the Kuratowski measure of noncompactness, defined by

$$\alpha_{\mathcal{H}}(\mathcal{R}) = \inf\{d : \mathcal{R} \text{ has a finite cover of balls of } \mathcal{H} \text{ of diameter less than } d\}. \tag{3.16}$$

Since the invariant, connected, bounded absorbing set $\mathcal{R}_0$ fulfills (3.15), exploiting a classical result of the theory of attractors of semigroups (see, e.g., [28]), we conclude that the ω-limit set of $\mathcal{R}_0$, that is,

$$\mathcal{A} \equiv \omega(\mathcal{R}_0) = \bigcap_{t \ge 0} \overline{\bigcup_{s \ge t} S(s)\mathcal{R}_0}, \tag{3.17}$$

is a connected and compact global attractor of $S(t)$. Therefore, we have proved the following result.

Theorem 3.2. *Under the assumption of* $(H_1) - (H_2)$, *problem* (3.1) *possesses a unique global attractor* $\mathcal{A}$.

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