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Research Article

Approximating Fixed Points of Some Maps in Uniformly Convex Metric Spaces

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We study strong convergence of the Ishikawa iterates of qasi-nonexpansive (generalized nonexpansive) maps and some related results in uniformly convex metric spaces. Our work improves and generalizes the corresponding results existing in the literature for uniformly convex Banach spaces.

1. Introduction and Preliminaries

Let C be a nonempty subset of a metric space (X, d) and let $T : C \to C$ be a map. Denote the set of fixed points of $T, \{x \in C : T(x) = x\}$ by F. The map T is said to be (i) quasi-nonexpansive if $F \neq \phi$ and $d(Tx, p) \leq d(x, p)$ for all $x \in C$ and $p \in F$, (ii) k-Lipschitz if for some $k > 0$, we have $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in C$; for $k = 1$, it becomes nonexpansive, and (iii) generalized nonexpansive (cf. [1] and the references therein) if

$$d(Tx, Ty) \leq ad(x, y) + b\{d(x, Tx) + d(y, Ty)\} + c\{d(x, Ty) + d(y, Tx)\} \tag{$*$}$$

for all $x, y \in C$ where $a, b, c \geq 0$ with $a + 2b + 2c \leq 1$.

The concept of quasi-nonexpansiveness is more general than that of nonexpansiveness. A nonexpansive map with at least one fixed point is quasi-nonexpansive but there are quasi-nonexpansive maps which are not nonexpansive [2].

Mann and Ishikawa type iterates for nonexpansive and quasi-nonexpansive maps have been extensively studied in uniformly convex Banach spaces [1, 3–6]. Senter and Dotson [7] established convergence of Mann type iterates of quais-nonexpansive maps under a condition in uniformly convex Banach spaces. In 1973, Goebel et al. [8] proved that generalized nonexpansive self maps have fixed points in uniformly convex Banach spaces. Based on their work, Bose and Mukerjee [1] proved theorems for the convergence of Mann type iterates of generalized nonexpansive maps and obtained a result of Kannan [9] under relaxed conditions. Maiti and Ghosh [6] generalized the results of Bose and Mukerjee [1] for Ishikawa iterates by using modified conditions of Senter and Dotson [7] (see, also [10]). For the sake of completeness, we state the result of Kannan [9] and its generalization by Bose and Mukerjee [1].

Theorem 1.1 (see [9]). *Let C be a nonempty, bounded, closed, and convex subset of a uniformly convex Banach space. Let T be a map of C into itself such that*

(i) $\|Tx - Ty\| \leq (1/2)\|x - Tx\| + (1/2)\|y - Ty\|$ *for all* $x, y \in C$,

(ii) $\sup_{z\in K}\|z - Tz\| \leq \delta(K)/2$, *where K is any nonempty convex subset of C which is mapped into itself by T and $\delta(K)$ is the diameter of K.*

Then the sequence $\{x_n\}$ defined by $x_{n+1} = (1/2)x_n + (1/2)Tx_n$ converges to the fixed point of T, where x_1 is any arbitrary point of C.

Theorem 1.2 (see [1]). *Let C be a nonempty, bounded, closed, and convex subset of a uniformly convex Banach space. Let T be a map of C into itself such that*

$$\|Tx - Ty\| \leq a\|x - y\| + b\{\|x - Tx\| + \|y - Ty\|\} + c\{\|x - Ty\| + \|y - Tx\|\} \tag{1.1}$$

for all $x, y \in C$ where $a, b, c \geq 0$ and $3a+2b+4c \leq 1$. Define a sequence $\{x_n\}$ in C for $x_1 \in C$, $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$, for all $n \geq 1$, where $0 < \beta \leq \alpha_n \leq \gamma < 1$. Then $\{x_n\}$ converges to a fixed point of T.

In Theorem 1.2, taking $a = c = 0$, $b = 1/2$, and $\alpha_n = 1/2$ for all $n \geq 1$, it becomes Theorem 1.1 without requiring condition (ii).

In 1970, Takahashi [11] introduced a notion of convexity in a metric space (X, d) as follows: a map $W : X \times X \times I \to X$ is a convex structure in X if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y) \tag{1.2}$$

for all $x, y \in X$ and $\lambda \in I = [0, 1]$. A metric space together with a convex structure is said to be convex metric space. A nonempty subset C of a convex metric space is convex if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in I$. In fact, every normed space and its convex subsets are convex metric spaces but the converse is not true, in general (see [11]). Later on, Shimizu and Takahashi [12] obtained some fixed point theorems for nonexpansive maps in convex metric spaces. This notion of convexity has been used in [13–15] to study Mann and Ishikawa iterations in convex metric spaces. For other fixed point results in the closely related classes of spaces, namely, hyperbolic and hyperconvex metric spaces, we refer to [16–19].

In the sequel, we assume that C is a nonempty convex subset of a convex metric space X and T is a selfmap on C. For an initial value $x_1 \in C$, we define the Ishikawa iteration scheme in C as follows:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = W(Ty_n, x_n, \alpha_n), \\ &y_n = W(Tx_n, x_n, \beta_n) \quad \forall n \geq 1, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are control sequences in $[0,1]$.

If we choose $\beta_n = 0$, then (1.3) reduces to the following Mann iteration scheme:

$$x_1 \in C, \quad x_{n+1} = W(Tx_n, x_n, \alpha_n), \quad \forall n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$ is a control sequence in $[0,1]$.

If X is a normed space with C as its convex subset, then $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ is a convex structure in X; consequently (1.3) and (1.4), respectively, become

$$\begin{aligned} &x_1 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \\ &y_n = (1-\beta_n)x_n + \beta_n Tx_n \quad \forall n \geq 1. \\ &x_1 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 1, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are control sequences in $[0,1]$.

A convex metric space X is said to be uniformly convex [11] if for arbitrary positive numbers ϵ and r, there exists $\alpha(\epsilon) > 0$ such that

$$d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq r(1-\alpha) \tag{1.6}$$

whenever $x, y, z \in X, d(z,x) \leq r, d(z,y) \leq r$ and $d(x,y) \geq r\epsilon$.

In 1989, Maiti and Ghosh [6] generalized the two conditions due to Senter and Dotson [7]. We state all these conditions in convex metric spaces:

Let T be a map with nonempty fixed point set F and $d(x,F) = \inf_{p \in F} d(x,p)$. *Then T is said to satisfy the following Condotions*.

Condition 1. If there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x,Tx) \geq f(d(x,F))$ for $x \in C$.

Condition 2. If there exists a real number $k > 0$ such that $d(x,Tx) \geq kd(x,F)$ for $x \in C$.

Condition 3. If there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x,Ty) \geq f(d(x,F))$ for $x \in C$ and all corresponding $y = W(Tx,x,t)$ where $0 \leq t \leq \beta < 1$.

Condition 4. If there exists a real number $k > 0$ such that $d(x,Ty) \geq kd(x,F)$ for $x \in C$ and all corresponding $y = W(Tx,x,t)$ where $0 \leq t \leq \beta < 1$.

Note that if T satisfies Condition 1 (resp., 3), then it satisfies Condition 2 (resp., 4). We also note that Conditions 1 and 2 become Conditions A and B, respectively, of Senter and Dotson [7] while Conditions 3 and 4 become Conditions I and II, respectively, of Maiti and Ghosh [6] in a normed space. Further, Conditions 3 and 4 reduce to Conditions 1 and 2, respectively, when $t = 0$.

In this note, we present results under relaxed control conditions which generalize the corresponding results of Kannan [9], Bose and Mukerjee [1], and Maiti and Ghosh [6] from uniformly convex Banach spaces to uniformly convex metric spaces. We present sufficient conditions for the convergence of Ishikawa iterates of k–Lipschitz maps to their fixed points in convex metric spaces and improve [3, Lemma 2]. A necessary and sufficient condition is obtained for the convergence of a sequence to fixed point of a generalized nonexpansive map in metric spaces.

We need the following fundamental result for the developmant of our results.

Theorem 1.3 (see [20]). *Let X be a uniformly convex metric space with a continuous convex structure $W : X \times X \times 0, 1] \to X$. Then for arbitrary positive numbers ϵ and r, there exists $\alpha(\epsilon) > 0$ such that*

$$d(z, W(x, y, \lambda)) \le r(1 - 2\min\{\lambda, 1 - \lambda\}\alpha) \tag{1.7}$$

for all $x, y, z \in X, d(z, x) \le r, d(z, y) \le r, d(x, y) \ge r\epsilon$ and $\lambda[\in 0, 1]$.

2. Convergence Analysis

We prove a lemma which plays key role to establish strong convergence of the iterative schemes (1.3) and (1.4).

Lemma 2.1. *Let X be a uniformly convex metric space. Let C be a nonempty closed convex subset of $X, T : C \to C$ a quasi-nonexpansive map and $\{x_n\}$ as in* (1.3). *If $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $0 \le \beta_n \le \beta < 1$, then $\liminf_{n\to\infty} d(x_n, Ty_n) = 0$.*

Proof. For $p \in F$, we consider

$$\begin{aligned} d(x_{n+1}, p) &= d(p, W(Ty_n, x_n, \alpha_n)) \\ &\le \alpha_n d(p, Ty_n) + (1 - \alpha_n)d(p, x_n) \\ &\le \alpha_n d(p, y_n) + (1 - \alpha_n)d(p, x_n) \\ &= \alpha_n d(p, W(Tx_n, x_n, \beta_n)) + (1 - \alpha_n)d(p, x_n) \\ &\le \alpha_n\beta_n d(p, Tx_n) + \alpha_n(1 - \beta_n)d(p, x_n) + (1 - \alpha_n)d(p, x_n) \\ &\le \alpha_n\beta_n d(p, x_n) + \alpha_n(1 - \beta_n)d(p, x_n) + (1 - \alpha_n)d(p, x_n) \\ &= d(x_n, p). \end{aligned} \tag{2.1}$$

This implies that the sequence $\{d(x_n, p)\}$ is nonincreasing and bounded below. Thus $\lim_{n\to\infty} d(x_n, p)$ exists. We may assume that $c = \lim_{n\to\infty} d(x_n, p) > 0$.

For any $p \in F$, we have that

$$
\begin{aligned}
d(x_n, Ty_n) &\leq d(x_n, p) + d(Ty_n, p) \\
&\leq d(x_n, p) + d(y_n, p) \\
&= d(x_n, p) + d(p, W(Tx_n, x_n, \beta_n)) \\
&\leq d(x_n, p) + \beta_n d(Tx_n, p) + (1-\beta_n) d(x_n, p) \\
&\leq d(x_n, p) + \beta_n d(x_n, p) + (1-\beta_n) d(x_n, p) \\
&= 2d(x_n, p).
\end{aligned} \tag{2.2}
$$

Since $\lim_{n\to\infty} d(x_n, p)$ exists, so $d(x_n, Ty_n)$ is bounded and hence $\inf_{n\geq 1} d(x_n, Ty_n)$ exists. We show that $\inf_{n\geq 1} d(x_n, Ty_n) = 0$. Assume that $\inf_{n\geq 1} d(x_n, Ty_n) = \sigma > 0$.

Then

$$
\begin{aligned}
d(x_n, Ty_n) &\geq d(x_n, p) \cdot \frac{\sigma}{d(x_n, p)} \\
&\geq d(x_n, p) \cdot \frac{\sigma}{d(x_1, p)}.
\end{aligned} \tag{2.3}
$$

Hence by Theorem 1.3, there exists $\alpha(\sigma/d(x_1, p)) > 0$ such that

$$
\begin{aligned}
d(x_{n-1,} p) &= d(W(Ty_n, x_n, \alpha_n), p) \\
&\leq d(x_n, p)(1 - 2\min\{\alpha_n, 1-\alpha_n\}\alpha) \\
&\leq d(x_n, p)(1 - 2\alpha_n(1-\alpha_n)\alpha).
\end{aligned} \tag{2.4}
$$

That is,

$$
2c\alpha_n(1-\alpha_n)\alpha \leq d(x_n, p) - d(x_{n+1}, p). \tag{2.5}
$$

Taking $m \geqslant 1$ and summing up the $(m+1)$ terms on the both sides in the above inequality, we have

$$
2c\alpha \sum_{n=1}^{m} \alpha_n(1-\alpha_n) \leq d(p, x_1) - d(p, x_m) \quad \forall m \geq 1. \tag{2.6}
$$

Let $m \to \infty$. Then, we have

$$
\infty \leq d(p, x_1) < \infty. \tag{2.7}
$$

This is contradiction and hence $\inf_{n\geq 1} d(x_n, Ty_n) = 0$. □

In the light of above result, we can construct subsequences $\{x_{n_i}\}$ and $\{y_{n_i}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that $\lim_{i\to\infty} d(x_{n_i}, Ty_{n_i}) = 0$ and hence $\lim\inf_{n\to\infty} d(x_n, Ty_n) = 0$.

Now we state and prove Ishikawa type convergence result in uniformly convex metric spaces.

Theorem 2.2. *Let X be a uniformly convex complete metric space with continuous convex structure and let C be its nonempty closed convex subset. Let T be a continuous quasi-nonexpansive map of C into itself satisfying Condition 3. If $\{x_n\}$ is as in* (1.3)*, where $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $0 \le \beta_n \le \beta < 1$, then $\{x_n\}$ converges to a fixed point of T.*

Proof. In Lemma 2.1, we have shown that $d(x_{n+1}, p) \le d(x_n, p)$. Therefore $d(x_{n+1}, F) \le d(x_n, F)$. This implies that the sequence $\{d(x_n, F)\}$ is nonincreasing and bounded below. Thus $\lim_{n\to\infty} d(x_n, F)$ exists. Now by Condition 3, we have

$$\liminf_{n\to\infty} f(d(x_n, F)) \le \liminf_{n\to\infty} d(Ty_n, x_n) = 0. \tag{2.8}$$

Using the properties of f, we have $\lim\inf_{n\to\infty} d(x_n, F) = 0$. As $\lim_{n\to\infty} d(x_n, F)$ exists, therefore $\lim_{n\to\infty} d(x_n, F) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. For $\epsilon > 0$, there exists a constant n_0 such that for all $n \ge n_0$, we have $d(x_n, F) < \epsilon/4$. In particular, $d(x_{n_0}, F) < \epsilon/4$. That is, $\inf\{d(x_{n_0}, p) : p \in F\} < \epsilon/4$. There must exist $p^* \in F$ such that $d(x_{n_0}, p^*) < \epsilon/2$. Now, for $m, n \ge n_0$, we have that

$$\begin{aligned} d(x_{n+m}, x_n) &\le d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\le 2d(x_{n_0}, p^*) \\ &< \epsilon. \end{aligned} \tag{2.9}$$

This proves that $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset of a complete metric space X, therefore it must converge to a point q in C.

Finally, we prove that q is a fixed point of T.

Since

$$d(q, F) \le d(q, x_n) + d(x_n, F), \tag{2.10}$$

therefore $d(q, F) = 0$. As F is closed, so $q \in F$. □

Choose $\beta_n = 0$ for all $n \ge 1$, in the above theorem; it reduces to the following Mann type convergence result.

Theorem 2.3. *Let X be a uniformly convex complete metric space with continuous convex structure and let C be its nonempty closed convex subset. Let T be a continuous quasi-nonexpansive map of C into itself satisfying Condition 1. If $\{x_n\}$ is as in* (1.4)*, where $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then $\{x_n\}$ converges to a fixed point of T.*

Next we establish strong convergence of Ishikawa iterates of a generalized nonexpansive map.

Theorem 2.4. *Let X and C be as in Theorem 2.3. Let T be a continuous generalied nonexpansive map of C into itself with at least one fixed point. If $\{x_n\}$ is as in (1.3), where $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n)=\infty$ and $0\leq\beta_n\leq\beta<1$,then $\{x_n\}$ converges to a fixed point of T.*

Proof. Let p be any fixed point of T. Then setting $y=p$ in $(*)$, we have

$$\begin{aligned} d(Tx,p) &\leq (a+c)d(x,p)+bd(x,Tx)+cd(Tx,p) \\ &\leq (a+b+c)d(x,p)+(b+c)d(Tx,p), \end{aligned} \tag{2.11}$$

which implies

$$d(Tx,p)\leq\frac{a+b+c}{1-b-c}d(x,p)\leq d(x,p). \tag{2.12}$$

Thus T is quasi-nonexpansive.

For any $y\in C$, we also observe that

$$d(Ty,p)\leq(a+c)d(y,p)+bd(y,Ty)+cd(Ty,p). \tag{2.13}$$

If $y=W(Tx,x,t)$, where $0\leq t\leq\beta<1$, then

$$\begin{aligned} d(y,p) &= d(W(Tx,x,t),p) \\ &\leq td(Tx,p)+(1-t)d(x,p) \\ &\leq d(x,p), \end{aligned} \tag{2.14}$$

$$\begin{aligned} d(y,x) &= d(W(Tx,x,t),x) \\ &\leq td(x,Tx)+(1-t)d(x,x) \\ &= td(x,Tx) \\ &\leq t\left[d(x,p)+d(Tx,p)\right] \\ &\leq 2td(x,p). \end{aligned} \tag{2.15}$$

Using (2.14) in (2.13), we have

$$\begin{aligned} d(Ty,p) &\leq (a+c)d(y,p)+bd(y,Ty)+cd(Ty,p) \\ &\leq (a+c)d(y,p)+c\{d(x,p)+d(x,Ty)\}+b\{d(x,y)+d(x,Ty)\} \\ &\leq (a+2c)d(x,p)+bd(x,y)+(b+c)d(x,Ty). \end{aligned} \tag{2.16}$$

Also it is obvious that

$$d(Ty,p)\geq d(x,p)-d(x,Ty). \tag{2.17}$$

Combining (2.16) and (2.17), we get that

$$\begin{aligned} bd(x,y)+(1+b+c)d(x,Ty) &\geq (1-a-2c)d(x,p) \\ &\geq 2bd(x,p). \end{aligned} \tag{2.18}$$

Now inserting (2.15) in (2.18), we derive

$$\begin{aligned} (1+b+c)d(x,Ty) &\geq 2bd(x,p)-bd(x,y) \\ &\geq 2b(1-t)d(x,p). \end{aligned} \tag{2.19}$$

That is,

$$d(x,Ty) \geq \frac{2b(1-t)}{1+b+c}d(x,p) \geq \frac{2b(1-\beta)}{1+b+c}d(x,p), \tag{2.20}$$

where $2b(1-t)/(1+b+c) > 0$. Thus T satisfies Condition 4 (and hence Condition 3). The result now follows from Theorem 2.2. □

Remark 2.5. In the above theorem, we have assumed that the generalied nonexpansive map T has a fixed point. It remains an open questions: what conditions on a, b, and c in $(*)$ are sufficient to guarantee the existence of a fixed point of T even in the setting of a metric space.

Choose $\beta_n = 0$ for all $n \geq 1$ in Theorem 2.4 to get the following Mann type convergence result.

Theorem 2.6. *Let X, C,and T be as in Theorem 2.4. If* $\{x_n\}$ *is as in* (1.4)*, where* $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$*,then* $\{x_n\}$ *converges to a fixed point of* T*.*

Proof. For $\beta_n = 0$ for all $n \geq 1$, $y = W(Tx, x, 0) = x$, the inequality (2.20) in the proof of Theorem 2.4 becomes

$$d(x,Tx) \geq \frac{2b}{1+b+c}d(x,p). \tag{2.21}$$

Thus T satisfies Condition 2 (and hence Condition 1) and so the result follows from Theorem 2.3. □

The analogue of Kannan result in uniformly convex metric space can be deduced from Theorem 2.6 (by taking $a = c = 0, b = 1/2$, and $\alpha_n = 1/2$ for all $n \geq 1$) as follows.

Theorem 2.7. *Let X be a uniformly convex complete metric space with continuous convex structure and let C be its nonempty closed convex subset. Let T be a continuous map of C into itself with at least one fixed point such that* $d(Tx, Ty) \leq (1/2)d(x, Tx) + (1/2)d(y, Ty)$ *for all* $x, y \in C$*. Then the sequence* $\{x_n\}$ *where* $x_1 \in C$ *and* $x_{n+1} = W(Tx_n, x_n, 1/2)$ *converges to a fixed point of* T*.*

Next we give sufficient conditions for the existence of fixed point of a k-Lipschitz map in terms of the Ishikawa iterates.

Theorem 2.8. *Let (X,d) be a convex metric space and let C be its nonempty convex subset. Let T be a k-Lipschitz selfmap of C. Let $\{x_n\}$ be the sequence as in* (1.3)*, where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy* (i)$0 \le \alpha_n, \beta_n \le 1$ *for all $n \ge 1$*(ii) $\liminf_{n\to\infty}\alpha_n > 0$ *and* (iii) $\liminf_{n\to\infty}\beta_n < k^{-1}$. *If $d(x_{n+1}, x_n) = \alpha_n d(x_n, Ty_n)$ and $x_n \to p$, then p is a fixed point of T.*

Proof. Let $p \in C$. Then

$$\begin{aligned} d(p,Tp) &\le d(x_n,p) + d(x_n,Ty_n) + d(Ty_n,Tp) \\ &= d(x_n,p) + \frac{1}{a_n}d(x_{n+1},x_n) + d(Ty_n,Tp) \\ &\le d(x_n,p) + \frac{1}{a_n}d(x_{n+1},x_n) + kd(y_n,p) \\ &= d(x_n,p) + \frac{1}{a_n}d(x_{n+1},x_n) + kd(W(Tx_n,x_n,\beta_n),p) \\ &\le d(x_n,p) + \frac{1}{a_n}d(x_{n+1},x_n) + k\{\beta_n d(Tx_n,p) + (1-\beta_n)d(x_n,p)\} \\ &\le d(x_n,p) + \frac{1}{a_n}d(x_{n+1},x_n) + k\beta_n\{d(Tx_n,Tp) + d(p,Tp)\} \\ &\quad + k(1-\beta_n)d(x_n,p). \end{aligned} \tag{2.22}$$

That is,

$$(1-k\beta_n)d(p,Tp) \le \left(1 + k^2\beta_n + k(1-\beta_n)\right)d(x_n,p) + \frac{1}{a_n}d(x_{n+1},x_n), \tag{2.23}$$

Since $\liminf a_n > 0$, therefore there exists $a > 0$ such that $a_n > a$ for all $n \ge 1$. This implies that

$$(1-k\beta_n)d(p,Tp) \le \left(1 + k^2\beta_n + k(1-\beta_n)\right)d(x_n,p) + \frac{1}{a}d(x_{n+1},x_n), \tag{2.24}$$

Taking lim sup on both the sides in the above inequality and using the condition $\liminf_{n\to\infty}\beta_n < k^{-1}$, we have $d(p,Tp) = 0$. □

Finally, using a generalized nonexpansive map T on a metric space X, we provide a necessary and sufficient condition for the convergence of an arbitrary sequence $\{x_n\}$ in X to a fixed point of T in terms of the approximating sequence $\{d(x_n, Tx_n)\}$.

Theorem 2.9. *Suppose that C is a closed subset of a complete metric space (X,d) and $T : C \to C$ is a continuous map such that for some $a,b,c \ge 0$, $a + 2c < 1$, the following inequality holds:*

$$d(Tx,Ty) \le ad(x,y) + b\{d(x,Tx) + d(y,Ty)\} + c\{d(x,Ty) + d(y,Tx)\} \tag{2.25}$$

for all $x,y \in C$. Then a sequence $\{x_n\}$ in C converges to a fixed point of T if and only if $\lim_{n\to\infty} d(x_n,Tx_n) = 0$.

Proof. Suppose that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. First we show that $\{x_n\}$ is a Cauchy sequence in C. To acheive this goal, consider:

$$\begin{aligned}
d(Tx_n, Tx_m) &\le ad(x_n, x_m) + b\{d(x_n, Tx_n) + d(x_m, Tx_m)\} \\
&\quad + c\{d(x_n, Tx_m) + d(x_m, Tx_n)\} \\
&\le ad(x_n, x_m) + b\{d(x_n, Tx_n) + d(x_m, Tx_m)\} \\
&\quad + c\{d(x_n, x_m) + d(x_m, Tx_m) + d(x_m, x_n) + d(x_n, Tx_n)\} \\
&= (a + 2c)d(x_n, x_m) \\
&\quad + (b + c)\{d(x_n, Tx_n) + d(x_m, Tx_m)\} \\
&\le (a + b + 3c)\{d(x_n, Tx_n) + d(x_m, Tx_m)\} \\
&\quad + (a + 2c)d(Tx_n, Tx_m).
\end{aligned} \tag{2.26}$$

That is,

$$(1 - a - 2c)d(Tx_n, Tx_m) \le (a + b + 3c)\{d(x_n, Tx_n) + d(x_m, Tx_m)\}. \tag{2.27}$$

Since $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $a + 2c < 1$, therefore from the above inequality, it follows that $\{Tx_n\}$ is a Cauchy sequence in C. In view of closedness of C, this sequence converges to an element p of C. Also $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ gives that $\lim_{n\to\infty} x_n = p$. Now using the continuity of T, we have $T(p) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = p$. Hence p is a fixed point of T.

Conversely, suppose that $\{x_n\}$ converges to a fixed point p of T. Using the continuity of T, we have that $\lim_{n\to\infty} Tx_n = p$. Thus $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

Remark 2.10. Theorem 2.8 improves Lemma 2 in [3] from real line to convex metric space setting. Theorem 2.9 is an extension of Theorem 4 in [21] to metric spaces. If we choose $c = 0$ in Theorem 2.9, it is still an improvement of [21, Theorem 4].

Remark 2.11. We have proved our results (2.1)–(2.8) in convex metric space setting. All these results, in particular, hold in Banach spaces if we set $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$.

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References

[1] R. K. Bose and R. N. Mukherjee, "Approximating fixed points of some mappings," *Proceedings of the American Mathematical Society*, vol. 82, no. 4, pp. 603–606, 1981.

[2] W. V. Petryshyn and T. E. Williamson Jr., "Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 43, pp. 459–497, 1973.

[3] L. Deng and X. P. Ding, "Ishikawa's iterations of real Lipschitz functions," *Bulletin of the Australian Mathematical Society*, vol. 46, no. 1, pp. 107–113, 1992.

[4] H. Fukhar-ud-din and A. R. Khan, "Approximating common fixed points of asymptotically nonexpansive maps in uniformly convex Banach spaces," *Computers & Mathematics with Applications*, vol. 53, no. 9, pp. 1349–1360, 2007.

[5] A. R. Khan, A.-A. Domlo, and H. Fukhar-ud-din, "Common fixed points Noor iteration for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 1–11, 2008.

[6] M. Maiti and M. K. Ghosh, "Approximating fixed points by Ishikawa iterates," *Bulletin of the Australian Mathematical Society*, vol. 40, no. 1, pp. 113–117, 1989.

[7] H. F. Senter and W. G. Dotson Jr., "Approximating fixed points of nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 44, pp. 375–380, 1974.

[8] K. Goebel, W. A. Kirk, and T. N. Shimi, "A fixed point theorem in uniformly convex spaces," *Bollettino dell'Unione Matematica Italiana*, vol. 7, pp. 67–75, 1973.

[9] R. Kannan, "Some results on fixed points. III," *Fundamenta Mathematicae*, vol. 70, no. 2, pp. 169–177, 1971.

[10] M. K. Ghosh and L. Debnath, "Convergence of Ishikawa iterates of quasi-nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 207, no. 1, pp. 96–103, 1997.

[11] W. Takahashi, "A convexity in metric space and nonexpansive mappings. I," *Kōdai Mathematical Seminar Reports*, vol. 22, pp. 142–149, 1970.

[12] T. Shimizu and W. Takahashi, "Fixed point theorems in certain convex metric spaces," *Mathematica Japonica*, vol. 37, no. 5, pp. 855–859, 1992.

[13] L. B. Cirić, "On some discontinuous fixed point mappings in convex metric spaces," *Czechoslovak Mathematical Journal*, vol. 43(118), no. 2, pp. 319–326, 1993.

[14] X. P. Ding, "Iteration processes for nonlinear mappings in convex metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 132, no. 1, pp. 114–122, 1988.

[15] L. A. Talman, "Fixed points for condensing multifunctions in metric spaces with convex structure," *Kōdai Mathematical Seminar Reports*, vol. 29, no. 1–2, pp. 62–70, 1977.

[16] R. Espinola and N. Hussain, "Common fixed points for multimaps in metric spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 204981, 14 pages, 2010.

[17] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, Pure and Applied Mathematics, Wiley-Interscience, New York, NY, USA, 2001.

[18] M. A. Khamsi, W. A. Kirk, and C. M. Yañez, "Fixed point and selection theorems in hyperconvex spaces," *Proceedings of the American Mathematical Society*, vol. 128, no. 11, pp. 3275–3283, 2000.

[19] W. A. Kirk, "Krasnoselskiĭ's iteration process in hyperbolic space," *Numerical Functional Analysis and Optimization*, vol. 4, no. 4, pp. 371–381, 1981/82.

[20] T. Shimizu, "A convergence theorem to common fixed points of families of nonexpansive mappings in convex metric spaces," in *Proceedings of the International Conference on Nonlinear and Convex Analysis*, pp. 575–585, 2005.

[21] Th. M. Rassias, "Some theorems of fixed points in nonlinear analysis," *Bulletin of the Institute of Mathematics. Academia Sinica*, vol. 13, no. 1, pp. 5–12, 1985.