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Research Article

Common Fixed Points for Multimaps in Metric Spaces

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We discuss the existence of common fixed points in uniformly convex metric spaces for single-valued pointwise asymptotically nonexpansive or nonexpansive mappings and multivalued nonexpansive, $*$-nonexpansive, or ε-semicontinuous maps under different conditions of commutativity.

1. Introduction

Fixed point theory for nonexpansive and related mappings has played a fundamental role in many aspects of nonlinear functional analysis for many years. The notion of asymptotic pointwise nonexpansive mapping was introduced and studied in [1, 2]. Very recently, in [3], techniques developed in [1, 2] were applied in metric spaces and CAT(0) spaces where the authors attend to the Bruhat-Tits inequality for CAT(0) spaces in order to obtain such results. In [4] it has been shown that these results hold even for a more general class of uniformly convex metric spaces than CAT(0) spaces. Here, we take advantage of this recent progress on asymptotic pointwise nonexpansive mappings and existence of fixed points for multivalued nonexpansive mappings in metric spaces to discuss the existence of common fixed points in either uniformly convex metric spaces or $\mathbb{R}$-trees for this kind of mappings, as well as for $*$-nonexpansive or ε-semicontinuous multivalued mappings under different kinds of commutativity conditions.

2. Basic Definitions and Results

First let us start by making some basic definitions.

Definition 2.1. Let (M, d) be a metric space. A mapping $T : M \to M$ is called nonexpansive if $d(T(x), T(y)) \leq d(x, y)$ for any $x, y \in M$. A fixed point of T will be a point $x \in M$ such that $T(x) = x$.

At least something else is stated, the set of fixed points of a mapping T will be denoted by $\mathrm{Fix}(T)$.

Definition 2.2. A point $z \in M$ is called a center for the mapping $T : M \to M$ if for each $x \in M$, $d(z, T(x)) \leq d(z, x)$. The set $Z(t)$ denotes the set of all centers of the mapping T.

Definition 2.3. Let (M, d) be a metric space. $T : M \to M$ will be said to be an asymptotic pointwise nonexpansive mapping if there exists a sequence of mappings $\alpha_n : M \to [0, \infty)$ such that

$$d(T^n(x), T^n(y)) \leq \alpha_n(x) d(x, y), \tag{2.1}$$

and

$$\limsup_{n \to \infty} \alpha_n(x) \leq 1 \tag{2.2}$$

for any $x, y \in M$.

This notion comes from the notion of asymptotic contraction introduced in [1]. Asymptotic pointwise nonexpansive mappings have been recently studied in [2–4].

In this paper we will mainly work with uniformly convex geodesic metric space. Since the definition of convexity requires the existence of midpoint, the word geodesic is redundant and so, for simplicity, we will omit it.

Definition 2.4. A geodesic metric space (M, d) is said to be *uniformly convex* if for any $r > 0$ and any $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $a, x, y \in M$ with $d(x, a) \leq r$, $d(y, a) \leq r$ and $d(x, y) \geq \varepsilon r$ it is the case that

$$d(m, a) \leq (1 - \delta) r, \tag{2.3}$$

where m stands for any midpoint of any geodesic segment $[x, y]$. A mapping $\delta : (0, +\infty) \times (0, 2] \to (0, 1]$ providing such a $\delta = \delta(r, \varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0, 2]$ is called a *modulus of uniform convexity*.

A particular case of this kind of spaces was studied by Takahashi and others in the 90s [5]. To define them we first need to introduce the notion of convex metric.

Definition 2.5. Let (M, d) be a metric space, then the metric is said to be convex if for any x, y, and z in M, and m a middle point in between x and y (that is, m is such that $d(m, x) = d(m, y) = 1/2 d(x, y)$), it is the case that

$$d(z, m) \leq \frac{1}{2}\left(d(z, x) + d(z, y)\right). \tag{2.4}$$

Definition 2.6. A uniformly convex metric space will be said to be of type (T) if it has a modulus of convexity which does not depend on r and its metric is convex.

Notice that some of the most relevant examples of uniformly convex metric spaces, as it is the case of uniformly convex Banach spaces or CAT(0) spaces, are of type (T).

Another situation where the geometry of uniformly convex metric spaces has been shown to be specially rich is when certain conditions are found in at least one of their modulus of convexity even though it may depend on r. These cases have been recently studied in [4, 6, 7]. After these works we will say that given a uniformly convex metric space, this space will be of type (M) (or [L]) if it has an adequate monotone (lower semicontinuous from the right) with respect to r modulus of convexity (see [4, 6, 7] for proper definitions). It is immediate to see that any space of type (T) is also of type (M) and (L). CAT(1) spaces with small diameters are of type (M) and (L) while their metric needs not to be convex.

$\mathbb{R}$-trees are largely studied and their class is a very important within the class of CAT(0)-spaces (and so of uniformly convex metric spaces of type (T)). $\mathbb{R}$-trees will be our main object in Section 4.

Definition 2.7. An $\mathbb{R}$-tree is a metric space M such that:

(i) there is a unique geodesic segment $[x, y]$ joining each pair of points $x, y \in M$;
(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.

It is easy to see that uniform convex metric spaces are unique geodesic; that is, for each two points there is just one geodesic joining them. Therefore midpoints and geodesic segments are unique. In this case there is a natural way to define convexity. Given two points x and y in a geodesic space, the (metric) segment joining x and y is the geodesic joining both points and it is usually denoted by $[x, y]$. A subset C of a (unique) geodesic space is said to be convex if $[x, y] \subseteq C$ for any $x, y \in C$. For more about geodesic spaces the reader may check [8].

The following theorem is relevant to our results. Recall first that given a metric space M and $C \subseteq M$, the metric projection P_C from M onto C is defined by $P_C(x) = \{y \in C : d(x, y) = \operatorname{dist}(x, C)\}$, where $\operatorname{dist}(x, C) = \inf\{d(x, y) : y \in C\}$.

Theorem 2.8 (see [4, 6]). *Let M be a uniformly convex metric space of type (M) or (L), let $C \subseteq M$ nonempty complete and convex. Then the metric projection $P_C(x)$ of $x \in M$ onto C is a singleton for any $x \in M$.*

These spaces have also been proved to enjoy very good properties regarding the existence of fixed points [4, 5] for both single and multivalued mappings. In [2] we can find the central fixed point result for asymptotic pointwise nonexpansive mappings in uniformly convex Banach spaces. This result was later extended to CAT(0) spaces in [3] and more recently to uniformly convex metric spaces of type either (M) or (L) in [4].

Theorem 2.9. *Let C be a closed bounded convex subset of a complete uniformly convex metric space of type either (M) or (L) and suppose that $I : C \to C$ is a pointwise asymptotically nonexpansive mapping. Then the fixed point set* $\operatorname{Fix}(I)$ *is nonempty closed and convex.*

Before introducing more fixed point results, we need to present some notations and definitions. Given a geodesic metric space M we will denote by $K(M)$ the family of nonempty compact subsets of M and by $KC(M)$ the family of nonempty compact and convex subsets of M. If $U$and V are bounded subsets of M, let H denote the Hausdorff metric defined as usual by

$$H(U,V) = \inf\{\varepsilon > 0 : U \subset N_\varepsilon(V),\ \text{and}\ V \subset N_\varepsilon(U)\}, \tag{2.5}$$

where $N_\varepsilon(V) = \{y \in M : d(y,V) \le \varepsilon\}$. Let C be a subset of a metric space M. A mapping $T : C \to 2^C$ with nonempty bounded values is nonexpansive provided that $H(T(x),T(y)) \le d(x,y)$ for all $x,y \in C$.

Theorem 2.10 (see [5]). *Let M be a complete uniformly convex metric space of type (T) and $C \subseteq M$ nonempty bounded closed and convex. Let $T : C \to KC(C)$ be a nonexpansive multivalued mapping, then the set of fixed points of T is nonempty.*

We next give the definition of those uniformly convex metric spaces for which most of the results in the present work will apply.

Definition 2.11. A uniformly convex metric space with the fixed point property for nonexpansive multivalued mappings (FPPMM) will be any such space of type either (M) or (L) or both verifying the above theorem.

The problem of studying whether more general uniformly convex metric spaces than those of type (T) enjoy that the FPPMM has been recently taken up in [4], where it has been shown that under additional geometrical conditions certain spaces of type (M) and (L) also enjoy the FPPMM.

The following notion of semicontinuity for multivalued mappings has been considered in [9] to obtain different results on coincidence fixed points in $\mathbb{R}$-trees and will play a main role in our last section.

Definition 2.12. For a subset C of M, a set-valued mapping $T : C \to 2^M$ is said to be ε-semicontinuous at $x_0 \in C$ if for each $\varepsilon > 0$ there exists an open neighborhood U of x_0 in C such that

$$T(x) \cap N_\varepsilon(T(x_0)) \ne \emptyset \tag{2.6}$$

for all $x \in U$.

It is shown in [9] that ε-semicontinuity of multivalued mappings is a strictly weaker notion than upper semicontinuity and almost lower semicontinuity. Similar results to those presented in [9] had been previously obtained under these other semicontinuity conditions in [10, 11].

Let C be a nonempty subset of a metric space M. Let $I : C \to C$ and $T : C \to 2^C$ with $T(x) \ne \emptyset$ for $x \in C$. Then I and T are said to be *commuting mappings* if $I(T(x)) \subset T(I(x))$ for all $x \in C$. I and T are said to *commute weakly* [12] if $I(\partial_C(T(x))) \subset T(I(x))$ for all $x \in C$, where $\partial_C Y$ denotes the relative boundary of $Y \subset C$ with respect to C. We define a subclass of weakly commuting pair which is different than that of commuting pair as follows.

Definition 2.13. If I and T are as what is previously mentioned, then they are said to commute subweakly if $I(\partial_C(T(x))) \subset \partial_C(TI(x))$ for all $x \in C$.

Notice that saying that $I : C \to C$ and $T : C \to 2^C$ commute subweakly is equivalent to saying that I and $\partial_C T : C \to 2^{\partial C}$ commute.

Recently, Chen and Li [13] introduced the class of *Banach operator pairs* as a new class of noncommuting maps which has been further studied by Hussain [14] and Pathak and Hussain [15]. Here we extend this concept to multivalued mappings.

Definition 2.14. Let $I : C \to C$ and $T : C \to 2^C$ with $T(x) \neq \emptyset$ for $x \in C$. The ordered pair (T, I) is a Banach operator pair if $Tx \subseteq F(I)$ for each $x \in F(I)$.

Next examples show that Banach operator pairs need not be neither commuting nor weakly commuting.

Example 2.15. Let $X = \mathbb{R}$ with the usual norm and $C = [1, \infty)$. Let $T(x) = \{x^2\}$ and $I(x) = 2x - 1$, for all $x \in C$. Then $F(I) = \{1\}$. Note that (T, I) is a Banach operator pair but T and I are not commuting.

Example 2.16. Let $X = [0, 1]$ with the usual metric. Let $T : X \to Cl(X)$ be defined by

$$T(x) = \begin{cases} \left\{\dfrac{1}{2}x^2\right\}, & \text{for } x \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right], \\ \left\{\dfrac{17}{96}, \dfrac{1}{4}\right\}, & \text{for } x = \dfrac{15}{32}. \end{cases} \tag{2.7}$$

Define $I : X \to X$ by

$$I(x) = \begin{cases} 0, & \text{for } x \in \left[0, \dfrac{15}{32}\right) \cup \left(\dfrac{15}{32}, 1\right], \\ 1, & \text{for } x = \dfrac{15}{32}. \end{cases} \tag{2.8}$$

Then $F(I) = \{0\}$ and $T(0) = \{0\} \subseteq F(I)$ imply that (T, I) is a Banach operator pair. Further, $TI(15/32) = T(1) = \{1/2\}$ and $IT(15/32) = I(\{17/96, 1/4\}) = \{0\}$. Thus T and I are neither commuting nor weakly commuting.

In 2005, Dhompongsa et al. [16] proved the following fixed point result for commuting mappings.

Theorem DKP. *Let X be a nonempty closed bounded convex subset of a complete* CAT*(0) space M, f a nonexpansive self-mapping of X, and $T : X \to 2^X$ nonexpansive, where for any $x \in X$, $T(x)$ is* *nonempty compact convex. Assume that for some* $p \in \mathrm{Fix}(f)$

$$\alpha \cdot p \oplus (1-\alpha)Tx \tag{2.9}$$

is convex for all $x \in X$ *and* $\alpha \in [0,1]$*. If* f *and* T *commute, then there exists an element* $z \in X$ *such that* $z = f(z) \in T(z)$*.*

This result has been recently improved by Shahzad in [17, Theorem 3.3]. More specifically, the same coincidence result was achieved in [17] for quasi-nonexpansive mappings (i.e., mappings for which its fixed points are centers) with nonempty fixed point sets in CAT (0) spaces and dropping the condition given by (2.9) at the time that the commutativity condition was weakened to weakly commutativity. Our main results provide further extensions of this result for asymptotic pointwise nonexpansive mappings and for nonexpansive multivalued mappings T with convex and nonconvex values. Earlier versions of such results for asymptotically nonexpansive mappings can already be found in [3, 4].

Summarizing, in this paper we prove some common fixed point results either in uniformly convex metric space with the FPPMM (Section 3) or $\mathbb{R}$-trees (Section 4) for single-valued asymptotic pointwise nonexpansive or nonexpansive mappings and multivalued nonexpansive, $*$-nonexpansive, or ε-semicontinuous maps which improve and/or complement Theorem DKP, [17, Theorem 3.3], and many others.

3. Main Results

Our first result gives the counterpart of [17, Theorem 3.3] to asymptotic pointwise nonexpansive mappings.

Theorem 3.1. *Let* M *be a complete uniformly convex metric space with FPPMM, and,* C *be a bounded closed convex subset of* M*. Assume that* $I : C \to C$ *is an asymptotic pointwise nonexpansive mapping and* $T : C \to 2^C$ *a nonexpansive mapping with* $T(x)$ *a nonempty compact convex subset of* C *for each* $x \in C$*. If the mappings* T *and* I *commute then there is* $z \in C$ *such that* $z = I(z) \in T(z)$*.*

Proof. By Theorem 2.9, the fixed point set A of I of a bounded closed convex subset is a nonempty closed and convex subset of M. By the commutativity of T and I, $T(x)$ is I-invariant for any $x \in A$ and so $T(x) \cap A \neq \emptyset$ and convex for any $x \in A$. Therefore, the mapping $S(x) := T(\cdot) \cap A : A \to KC(A)$ is well defined.

We will show next that S is also nonexpansive as a multivalued mapping. Before that, we claim that $\mathrm{dist}(x, T(y)) = \mathrm{dist}(x, S(y))$ for any $x, y \in A$. In fact, by convexity of $T(y)$ and Theorem 2.8, we can take $a_x \in T(y)$ to be the unique point in $T(y)$ such that $d(x, a_x) = \mathrm{dist}(x, T(y))$. Now consider the sequence $\{I^n(a_x)\}$. Since T and I commute we know that $I^n(a_x) \in T(y)$ for any n. Therefore, by the compactness of $T(y)$, it has a convergent subsequence $\{I^{n_k}(a_x)\}$. Let $p \in T(x)$ be the limit of $\{I^{n_k}(a_x)\}$, then we have that

$$\begin{aligned} d(p,x) &= \lim_{k\to\infty} d(I^{n_k}(a_x), x) = \lim_{k\to\infty} d(I^{n_k}(a_x), I^{n_k}(x)) \\ &\leq \lim_{k\to\infty} \alpha_{n_k}(x) d(a_x, x) \leq \mathrm{dist}(x, T(y)), \end{aligned} \tag{3.1}$$

from where, by the uniqueness of a_x, $p = a_x$. Consequently, $\lim I^n(a_x) = a_x$ and so $a_x \in A$. This, in particular, shows that $\operatorname{dist}(x, T(y)) = \operatorname{dist}(x, S(y))$ and explains equality (3.1) below. Now, we can argue as follows:

$$\begin{aligned} H(S(x), S(y)) &= \max\left\{ \sup_{u\in S(x)} \operatorname{dist}(u, S(y)), \sup_{v\in S(y)} \operatorname{dist}(v, S(x)) \right\} \\ &= \max\left\{ \sup_{u\in S(x)} \operatorname{dist}(u, T(y)), \sup_{v\in S(y)} \operatorname{dist}(v, T(x)) \right\} \\ &\leq \max\left\{ \sup_{u\in T(x)} \operatorname{dist}(u, T(y)), \sup_{v\in T(y)} \operatorname{dist}(v, T(x)) \right\} \\ &= H(T(x), T(y)) \\ &\leq d(x, y). \end{aligned} \tag{$\star$}$$

Finally, since M has the FPPMM, there exists $z \in A$ such that $z \in T(z) \cap A$. Therefore, $z = I(z) \in T(z)$. □

Remark 3.2. The proof of our result is inspired on that one [17, Theorem 3.3]. Notice, however, that equality (3.1) is given as trivial in [17] while this is not the case. Notice also that there is no direct relation between the families of quasi-nonexpansive mappings and asymptotically pointwise nonexpansive mappings which make both results independent and complementary to each other.

The condition that $T : C \to 2^C$ is a mapping with convex values is crucial to get the desired conclusion in the previous theorem, Theorem DKP and all the results in [17]. Next we give conditions under which this hypothesis can be dropped. A self-map I of a topological space M is said to satisfy condition (C) [15, 18] provided $B \cap \operatorname{Fix}(I) \neq \emptyset$ for any nonempty I-invariant closed set $B \subseteq M$.

Theorem 3.3. *Let M be a complete uniformly convex metric space with FPPMM and C a bounded closed convex subset of M. Assume that $I : C \to C$ is asymptotically pointwise nonexpansive and $T : C \to 2^C$ is nonexpansive with $T(x)$ a nonempty compact subset of C for each $x \in C$. If the mappings T and I commute and I satisfies condition (C), then there is $z \in C$ such that $z = I(z) \in T(z)$.*

Proof. We know that the fixed point set A of I is a nonempty closed and convex subset of M. Since I and T commute then $T(x)$ is I-invariant for $x \in A$, and also, since I satisfies condition (C), the mapping $S(x) := T(\cdot) \cap A : A \to K(A)$ is well defined. We prove next that the mapping S is nonexpansive.

As in the above proof, we need to show that for any $x, y \in A$ it is the case that $\operatorname{dist}(x, T(y)) = \operatorname{dist}(x, S(y))$. Since T and I commute, we know that $T(y)$ is I-invariant. Take $a_x \in T(y)$ such that $d(x, a_x) = \operatorname{dist}(x, T(y))$ and consider the sequence $\{I^n(a_x)\}$. Let B be the set of limit points of $\{I^n(a_x)\}$, then B is a nonempty and closed subset of $T(y)$. Consider now

$b \in B$, then

$$d(b,x) = \lim_{k\to\infty} d(I^{n_k}(a_x), x) = \lim_{k\to\infty} d(I^{n_k}(a_x), I^{n_k}(x)) \\ \le \lim_{k\to\infty} \alpha_{n_k}(x)\ d(a_x, x) \le d(a_x, x), \tag{3.2}$$

and, therefore, $d(b,x) = d(a_x,x) = \mathrm{dist}(x, T(y))$. But B is also I-invariant, so, by condition (C), I has a fixed point in B and so $\mathrm{dist}(x,T(y)) = \mathrm{dist}(x,S(y))$. The rest of the proof follows as in Theorem 3.1. □

For the next corollary we need to recall some definitions about orbits. *The orbit* $\{I^n(x)\}$ *of* I *at* x *is proper* if $\{I^n(x)\} = \{x\}$ or there exists $n_x \in \mathbb{N}$ such that $\mathrm{cl}(\{I^n(I^{n_x}(x))\})$ is a proper subset of $\mathrm{cl}(\{I^n(x)\})$. If $\{I^n(x)\}$ is proper for each $x \in C \subset M$, we will say that I has *proper orbits* on C [19].

Condition (C) in Theorem 3.3 may seem restrictive, however it looks weaker if we recall that the values of T are compact. This is shown in the next corollary.

Corollary 3.4. *Under the same conditions of the previous theorem, if condition (C) is replaced with I having proper orbits then the same conclusion follows.*

Proof. The idea now is that the orbits through I of points in A are relatively compact, then, by [19, Theorem 3.1], I satisfies condition (C). □

For any nonempty subset C of a metric space M, *the diameter of* C is denoted and defined by $\delta(B) = \sup\{d(x,y) : x, y \in B\}$. A mapping $I : M \to M$ has *diminishing orbital diameters* (*d.o.d.*) [19, 20] if for each $x \in M$, $\delta(\{I^n(x)\}) < \infty$ and whenever $\delta(\{I^n(x)\}) > 0$, there exists $n_x \in \mathbb{N}$ such that $\delta(\{I^n(x)\}) > \delta(\{I^n(I^{n_x}(x))\})$. Observe that in a metric space M if I has d.o.d. on X, then I has proper orbits [15, 19]; consequently, we obtain the following generalization of the corresponding result of Kirk [20].

Corollary 3.5. *Under the same conditions of the previous theorem, if condition (C) is replaced with I having d.o.d. then the same conclusion follows.*

In our next result we also drop the condition on the convexity of the values of T but, this time, we ask the geodesic space M not to have *bifurcating geodesics*. That is, for any two segments starting at the same point and having another common point, this second point is a common endpoint of both or one segment that includes the other. This condition has been studied by Zamfirescu in [21] in order to obtain stronger versions of the next lemma which is the one we need and which proof is immediate.

Lemma 3.6. *Let M be a geodesic space with no bifurcating geodesics and let C be a nonempty subset of M. Let $x \in M \setminus C$, $a_x \in C$ such that $d(x, a_x) = \mathrm{dist}(x, C)$, and $I_x = \{a \in X : a = tx + (1-t)a_x$ with $t \in [0,1)\}$. Then the metric projection of $a \in I_x$ onto C is the singleton $\{a_x\}$ for any $a \in I_x$.*

Now we give another version of Theorem 3.1 without assuming that the values of T are convex.

Theorem 3.7. *Let M be a complete uniformly convex metric space with FPPMM and with no bifurcating geodesics and C a bounded closed convex subset of M. Assume that $I : C \to C$ is* *asymptotically pointwise nonexpansive and $T : C \to 2^C$ nonexpansive with $T(x)$ a nonempty compact subset of C for each $x \in C$. Assume further that the fixed point set A of I is such that its topological interior (in M) is dense in A. If the mappings T and I commute, then there exists $z \in C$ such that $z = I(z) \in T(z)$.*

Proof. Just as before, we know that the fixed point set A of I is a nonempty closed and convex subset of M. We are going to see that $S(x) := T(\cdot) \cap A : A \to K(A)$ is well defined. Take $x \in \text{int}(A)$ and let us see that $T(x) \cap A \neq \emptyset$. Consider $a_x \in T(x)$ such that $d(x, a_x) = \text{dist}(x, T(x))$ and let p be a limit point of $\{I^n(a_x)\}$. Fix $y \in I_x \cap A$, then

$$\begin{aligned} d(p, y) &= \lim_{k \to \infty} d\left(I^{m_k}(a_x), y\right) = \lim_{k \to \infty} d\left(I^{m_k}(a_x), I^{m_k}(y)\right) \\ &\leq \lim_{k \to \infty} \alpha_{m_k}(y) d(a_x, y) \leq d(a_x, y). \end{aligned} \tag{3.3}$$

Therefore, by Lemma 3.6, $p = a_x$ and so $\{I^n(a_x)\}$ is a convergent sequence to a_x and $I(a_x) = a_x$. Take now $x \in A$, then, by hypothesis, there exists a sequence $\{x_n\} \subseteq \text{int}(A)$ converging to x. Consider the sequence of points $\{a_{x_n}\}$ given by the above reasoning such that $a_{x_n} \in A \cap T(x_n)$. Define, for each $n \in \mathbb{N}$, $b_n \in T(x)$ such that $d(b_n, a_{x_n}) = \text{dist}(T(x), a_{x_n})$. Since T is nonexpansive, $d(b_n, a_{x_n}) \leq d(x, x_n)$. Now, since $T(x)$ is compact, take b a limit point of $\{b_n\}$. Then $b \in A$ because it is also a limit point of $\{a_{x_n}\}$ and $b \in T(x)$. Therefore our claim that S is well defined is correct. Let us see now that S is also nonexpansive.

As in the previous theorems, we show that for $x, u \in A$ we have that $\text{dist}(x, S(u)) = \text{dist}(x, T(u))$. Take $x \in \text{int}(A)$ and consider $a_x \in T(u)$ such that $d(x, a_x) = \text{dist}(x, T(u))$ and p a limit point of $\{I^n(a_x)\}$. Take $y \in I_x \cap A$. Then, repeating the same reasoning as above, $p = a_x$ and so a_x is a fixed point of T which proves that $\text{dist}(x, T(u)) = \text{dist}(x, S(u))$ for $x \in \text{int}(A)$ and $u \in A$. For $x \in A$ we apply a similar argument as above using that $\text{int}(A)$ is dense in A. Now the result follows as in Theorem 3.1. □

Remark 3.8. The condition about the commutativity of I and T has been used to guarantee that the orbits $\{I^n(a_x)\}$ for x in the fixed point set of I remain in a certain compact set and so they are relatively compact. The same conclusion can be reached if we require I and T to commute subweakly. Therefore, Theorems 3.1, 3.3 and 3.7, and stated corollaries remain true under this other condition.

In the next result the convexity condition on the multivalued mappings is also removed.

Theorem 3.9. *Let M be a complete uniformly convex metric space with FPPMM, and, let C be a bounded closed convex subset of M. Assume that $I : C \to C$ is asymptotically pointwise nonexpansive and $T : C \to 2^C$ nonexpansive with $T(x)$ a nonempty compact subset of C for each $x \in C$. If the pair (T, I) is a Banach operator pair, then there is $z \in C$ such that $z = I(z) \in T(z)$.*

Proof. By Theorem 2.9 the fixed point set A of I is a nonempty closed and convex subset of M. Since the pair (T, I) is a Banach operator pair, $T(x) \subset A$ for each $x \in A$, and therefore, $T(x) \cap A \neq \emptyset$ for $x \in A$. The mapping $T(\cdot) \cap A : A \to K(A)$ being the restriction of T on A is nonexpansive. Now the proof follows as in Theorem 3.1. □

Remark 3.10. Since asymptotically nonexpansive and nonexpansive maps are asymptotically pointwise nonexpansive maps, all the so far obtained results also apply for any of these mappings.

A set-valued map $T : C \to 2^C$ is called $*$-nonexpansive [22] if for all $x, y \in C$ and $a_x \in T(x)$ with $d(x, a_x) = \operatorname{dist}(x, T(x))$, there exists $a_y \in T(y)$ with $d(y, a_y) = \operatorname{dist}(y, T(y))$ such that $d(a_x, a_y) \leq d(x, y)$. Define $P_T : C \to 2^C$ by

$$P_T(x) = \{a_x \in T(x) : d(x, a_x) = \operatorname{dist}(x, T(x))\} \quad \text{for each } x \in C. \tag{3.4}$$

Husain and Latif [22] introduced the class of $*$-nonexpansive multivalued maps and it has been further studied by Hussain and Khan [23] and many others. The concept of a $*$-nonexpansive multivalued mapping is different from that one of continuity and nonexpansivity, as it is clear from the following example [23].

Example 3.11. Let $T : [0,1] \to 2^{[0,1]}$ be the multivalued map defined by

$$T(x) = \begin{cases} \left\{\dfrac{1}{2}\right\}, & \text{for } x \in \left[0, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, 1\right], \\ \left[\dfrac{1}{4}, \dfrac{3}{4}\right], & \text{for } x = \dfrac{1}{2}. \end{cases} \tag{3.5}$$

Then $P_T(x) = \{1/2\}$ for every $x \in [0,1]$. This implies that T is a $*$-nonexpansive map. However,

$$H\left(T\left(\frac{1}{3}\right), T\left(\frac{1}{2}\right)\right) = \frac{1}{4} > \frac{1}{6} = \left\|\frac{1}{3} - \frac{1}{2}\right\|, \tag{3.6}$$

which implies that T is not nonexpansive. Let $V_{1/4}$ be any small open neighborhood of $1/4$, then

$$T^{-1}(V_{1/4}) = \left\{\frac{1}{2}\right\} \tag{3.7}$$

which is not open. Thus T is not continuous. Note also that $1/2$ is a fixed point of T.

Theorem 3.12. *Let M be a complete uniformly convex metric space with FPPMM and C be a bounded closed convex subset of M. Assume that $I : C \to C$ is asymptotically pointwise nonexpansive and $T : C \to 2^C$ $*$-nonexpansive with $T(x)$ a compact subset of C for each $x \in C$. If the pair (T, I) is a Banach operator pair, then there is $z \in C$ such that $z = I(z) \in T(z)$.*

Proof. As above, the set A of fixed points of I is nonempty closed convex subset of M. Since $T(x)$ is compact for each x, $P_T(x)$ is well defined and a multivalued nonexpansive selector of T [23]. We also have that $T(x) \subset A$ and $P_T(x) \subset T(x)$ for each $x \in A$, so $P_T(x) \subset A$ for each

$x \in A$. Thus the pair (P_T, I) is a Banach operator pair. By Theorem 3.9, the desired conclusion follows. □

The following corollary is a particular case of Theorem 3.12.

Corollary 3.13. *Let M be a complete uniformly convex metric space with FPPMM, and, let C be a bounded closed convex subset of M. Assumethat $I : C \to C$ is a nonexpansive map and $T : C \to 2^C$ is a $*$-nonexpansive mapping with $T(x)$ a compact subset of C for each $x \in C$. If the pair (T, I) is a Banach operator pair, then there is $z \in C$ such that $z = I(z) \in T(z)$.*

4. Coincidence Results in $\mathbb{R}$-Trees

In this section we present different results on common fixed points for a family of commuting asymptotic pointwise nonexpansive mappings. As it can be seen in [9–11], existence of fixed points for multivalued mappings happens under very weak conditions if we are working in $\mathbb{R}$-tree spaces. This allows us to find much weaker results for $\mathbb{R}$-trees than those in the previos section. Close results to those presented in this section can be found in [17]. We begin with the adaptation of Theorem 3.1 to $\mathbb{R}$-trees.

Theorem 4.1. *Let M be a complete $\mathbb{R}$-tree, and suppose that C is a bounded closed convex subset of M. Assume that $I : C \to C$ is asymptotically pointwise nonexpansive and $T : C \to 2^C$ is ε-semicontinuous mapping with $T(x)$ a nonempty closed and convex subset of C for each $x \in C$. If the mappings T and I commute then there is $z \in C$ such that $z = I(z) \in T(z)$.*

Proof. We know that the fixed point set A of I is a nonempty closed and convex subset of C. From the commutativity condition we also have that $T(x)$ is I-invariant for any $x \in A$ and so the mapping defined by $S(x) = T(x) \cap A$ is well defined on A and takes closed and convex values. By [9, Lemma 2], S is a ε-semicontinuous mapping and so, by [9, Theorem 4] applied to S, the conclusion follows. □

Remark 4.2. Actually the only condition we need in the above theorem from I is that its set of fixed points is nonempty bounded closed and convex. In the case I is nonexpansive then C may be supposed to be geodesically bounded instead of bounded, as shown in [24].

The next theorem is the counterpart of Espínola and Kirk [24, Theorem 4.3] to asymptotic pointwise nonexpansive mappings.

Theorem 4.3. *Let M be a complete $\mathbb{R}$-tree, and suppose C is a bounded closed convex subset of M. Then every commuting family $\mathfrak{F}$ of asymptotic pointwise nonexpansive self-mappings of C has a nonempty closed and convex common fixed point set.*

Proof. Let $f \in \mathfrak{F}$. Then by Theorem 2.9, the set $\mathrm{Fix}(f) = F$ of fixed points of f is nonempty closed and convex and hence again an $\mathbb{R}$-tree. Now suppose $g \in \mathfrak{F}$. Since g and f commute it follows $g : F \to F$, and, by applying the preceding argument to g and F, we conclude that g has a nonempty fixed point set in F. In particular the fixed point set of f and the fixed point set of g intersect. The rest of the proof is similar to that of Espínola and Kirk [24, Theorem 4.3] and so is omitted. □

In the next result, we combine a family of commuting asymptotic pointwise nonexpansive mappings with a multivalued mapping.

Theorem 4.4. *Let C be a nonempty bounded closed convex subset of a complete $\mathbb{R}$-tree M, $\mathcal{F}$ a commuting family of asymptotic pointwise nonexpansive self-mappings on C. Assume that $T : C \to 2^C$ is ε-semicontinuous mapping on C with nonempty closed and convex values and such that f and T commute weakly for any $f \in \mathcal{F}$. If for each $f \in \mathcal{F}$*

$$d\big(f(u), x\big) \leq d(u, x) \quad \textit{for each } x \in \operatorname{Fix}(\mathcal{F}),\ u \in \partial_C(T(x)), \tag{4.1}$$

then there exists an element $z \in C$ such that $z = f(z) \in T(z)$ for all $f \in \mathcal{F}$.

Proof. By Theorem 4.3, $\operatorname{Fix}(\mathcal{F})$ is nonempty closed and convex. Let $x \in \operatorname{Fix}(\mathcal{F})$. Then for any $f \in \mathcal{F}$,

$$f(\partial_C(T(x))) \subset T\big(f(x)\big) = T(x). \tag{4.2}$$

Let $u \in \partial_C(T(x))$ the unique closest point to x from $T(x)$. Now (4.2) implies that $f(u)$ is in $T(x)$. Further, (4.1) implies that $d(f(u), x) \leq d(u, x)$ and so, by the uniqueness of u, $f(u) = u$. Thus u is a common fixed point of $\mathcal{F}$, which implies

$$T(x) \cap \operatorname{Fix}(\mathcal{F}) \neq \emptyset \tag{4.3}$$

for each $x \in \operatorname{Fix}(\mathcal{F})$. Let $S(x) = T(x) \cap \operatorname{Fix}(\mathcal{F})$. Now the proof follows as the proof of Theorem 4.1. □

Remark 4.5. Note that condition (4.1) is satisfied if each $f \in \mathcal{F}$ is nonexpansive with respect to $\operatorname{Fix}(\mathcal{F})$.

Corollary 4.6. *Let C be a nonempty bounded closed convex subset of a complete $\mathbb{R}$-tree M and $\mathcal{F}$ a commuting family of asymptotic pointwise nonexpansive self-mappings of C. Assume that $T : C \to 2^C$ is ε-semicontinuous, where for any $x \in C$, $T(x)$ is nonempty closed and convex and $\mathcal{F}$ and T commute weakly. If*

$$\operatorname{Fix}(\mathcal{F}) \subset Z(f), \tag{4.4}$$

then there exists an element $z \in C$ such that $z = f(z) \in T(z)$ for all $f \in \mathbb{F}$.

Proof. Condition (4.4) implies (4.1). The desired conclusion now follows from the previous theorem. □

In our next result we make use of the fact that convex subsets of $\mathbb{R}$-trees are gated; that is, if C is a closed and convex subset of the $\mathbb{R}$-tree M, $x \in M \setminus C$ and a_x is the metric projection of x onto C then a_x is in the metric segment joining x and y for any $y \in C$. Notice that condition (4.1) is dropped in the next theorem.

Theorem 4.7. *Let C be a nonempty bounded closed convex subset of a complete $\mathbb{R}$-tree M, $\mathcal{F}$ a commuting family of asymptotic pointwise nonexpansive self-mappings on C. Assume that $T : C \to 2^C$ is ε-semicontinuous mapping on C with nonempty closed and convex values and such that f and T commute for any $f \in \mathcal{F}$, then there exists an element $z \in C$ such that $z = f(z) \in T(z)$ for all $f \in \mathcal{F}$.*

Proof. As in the proof of Theorem 4.4, the only thing that really needs to be proved is that $T(x) \cap \mathrm{Fix}(\mathcal{F}) \neq \emptyset$ for each $x \in \mathrm{Fix}(\mathcal{F})$. From the commutativity condition we know that $T(x)$ is f-invariant for any $f \in \mathcal{F}$. Therefore each f has a fixed point $x_f \in T(x)$. But, since the fixed point set of f is convex and $x \in \mathrm{Fix}(\mathcal{F})$, then the metric segment joining x and x_f is contained in $\mathrm{Fix}(f)$. From the gated property, we know that the closest point u to x from $T(x)$ is in such segment for any f. In consequence, u is a fixed point for any f and, therefore, $u \in T(x) \cap \mathrm{Fix}(\mathcal{F})$. □

The next theorem follows as a consequence of Theorem 4.7.

Theorem 4.8. *If in the previous theorem T is supposed to be either upper semicontinuous or almost lower semicontinuous then the same conclusion follows.*

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