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Research Article

A New Projection Algorithm for Generalized Variational Inequality

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We propose a new projection algorithm for generalized variational inequality with multivalued mapping. Our method is proven to be globally convergent to a solution of the variational inequality problem, provided that the multivalued mapping is continuous and pseudomonotone with nonempty compact convex values. Preliminary computational experience is also reported.

1. Introduction

We consider the following generalized variational inequality. To find $x^* \in C$ and $\xi \in F(x^*)$ such that

$$\langle \xi, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where C is a nonempty closed convex set in $\mathbb{R}^n$, F is a multivalued mapping from C into $\mathbb{R}^n$ with nonempty values, and $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the inner product and norm in $\mathbb{R}^n$, respectively.

Theory and algorithm of generalized variational inequality have been much studied in the literature [1–9]. Various algorithms for computing the solution of (1.1) are proposed. The well-known proximal point algorithm [10] requires the multivalued mapping F to be monotone. Relaxing the monotonicity assumption, [1] proved if the set C is a box and F is order monotone, then the proximal point algorithm still applies for problem (1.1). Assume that F is pseudomonotone, and [11] described a combined relaxation method for solving (1.1); see also [12, 13]. Projection-type algorithms have been extensively studied in the literature; see [14–17] and the references therein. Recently, [15] proposes a projection algorithm for generalized variational inequality with pseudomonotone mapping. In [15], choosing $\xi_i \in F(x_i)$ needs solving a single-valued variational inequality and hence is computationally expensive; see expression (2.1) in [15]. In this paper, we introduce a different projection algorithm for generalized variational inequality. In our method, $\xi_i \in F(x_i)$ can be taken arbitrarily. Moreover, the main difference of our method from that of [15] is the procedure of Armijo-type linesearch; see expression (2.2) in [15] and expression (2.2) in the next section.

Let S be the solution set of (1.1), that is, those points $x^* \in C$ satisfying (1.1). Throughout this paper, we assume that the solution set S of problem (1.1) is nonempty and F is continuous on C with nonempty compact convex values satisfying the following property:

$$\langle \zeta, y - x \rangle \geq 0, \quad \forall y \in C,\ \zeta \in F(y),\ \forall x \in S. \tag{1.2}$$

Property (1.2) holds if F is pseudomonotone on C in the sense of Karamardian [18]. In particular, if F is monotone, then (1.2) holds.

The organization of this paper is as follows. In the next section, we recall the definition of continuous multivalued mapping, present the algorithm details, and prove the preliminary result for convergence analysis in Section 3. Numerical results are reported in the last section.

2. Algorithms

Let us recall the definition of continuous multivalued mapping. F is said to be upper semicontinuous at $x \in C$ if for every open set V containing $F(x)$, there is an open set U containing x such that $F(y) \subset V$ for all $y \in C \cap U$. F is said to be lower semicontinuous at $x \in C$, if we give any sequence x_k converging to x and any $y \in F(x)$, there exists a sequence $y_k \in F(x_k)$ that converges to y. F is said to be continuous at $x \in C$ if it is both upper semicontinuous and lower semicontinuous at x. If F is single valued, then both upper semicontinuity and lower semicontinuity reduce to the continuity of F.

Let Π_C denote the projector onto C and let $\mu > 0$ be a parameter.

Proposition 2.1. *$x \in C$ and $\xi \in F(x)$ solve problem* (1.1) *if and only if*

$$r_\mu(x, \xi) := x - \Pi_C(x - \mu\xi) = 0. \tag{2.1}$$

Algorithm 2.2. Choose $x_0 \in C$ and three parameters $\sigma > 0$, $0 < \mu < min\{1, 1/\sigma\}$, and $\gamma \in (0,1)$. Set $i = 0$.

Step 1. If $r_\mu(x_i, \xi) = 0$ for some $\xi \in F(x_i)$, stop; else take arbitrarily $\xi_i \in F(x_i)$.

Step 2. Let k_i be the smallest nonnegative integer satisfying

$$\langle \xi_i - y_i, r_\mu(x_i, \xi_i) \rangle \leq \sigma \| r_\mu(x_i, \xi_i) \|^2, \tag{2.2}$$

where $y_i = \Pi_{F(x_i - \gamma^{k_i} r_\mu(x_i, \xi_i))}(\xi_i)$. Set $\eta_i = \gamma^{k_i}$.

Step 3. Compute $x_{i+1} := \Pi_{C_i}(x_i)$, where $C_i := \{x \in C : h_i(x) \le 0\}$, and

$$h_i(x) := \langle y_i + \eta_i r_\mu(x_i, \xi_i), x - x_i \rangle + \eta_i \langle y_i - \mu\xi_i + r_\mu(x_i, \xi_i), r_\mu(x_i, \xi_i) \rangle. \tag{2.3}$$

Let $i := i + 1$ and go to Step 1.

Remark 2.3. Since F has compact convex values, F has closed convex values. Therefore, y_i in Step 2 is uniquely determined by k_i.

Remark 2.4. If F is a single-valued mapping, the Armijo-type linesearch procedure (2.2) becomes that of Algorithm 2.2 in [14].

We show that Algorithm 2.2 is well defined and implementable.

Proposition 2.5. *If x_i is not a solution of problem* (1.1)*, then there exists a nonnegative integer k_i satisfying* (2.2).

Proof. Suppose that for all k, we have

$$\langle \xi_i - y_k, r_\mu(x_i, \xi_i) \rangle > \sigma \| r_\mu(x_i, \xi_i) \|^2, \tag{2.4}$$

where $y_k = \Pi_{F(x_i - \gamma^k r_\mu(x_i, \xi_i))}(\xi_i)$. Since F is lower semicontinuous, $\xi_i \in F(x_i)$, and $x_i - \gamma^k r_\mu(x_i, \xi_i) \to x_i$ as $k \to \infty$, for each k, there is $u_k \in F(x_i - \gamma^k r_\mu(x_i, \xi_i))$ such that $\lim_{k\to\infty} u_k = \xi_i$. Since $y_k = \Pi_{F(x_i - \gamma^k r_\mu(x_i, \xi_i))}(\xi_i)$,

$$\| y_k - \xi_i \| \le \| u_k - \xi_i \| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{2.5}$$

So $\lim_{k\to\infty} y_k = \xi_i$. Let $k \to \infty$ in (2.4), we have $0 = \|\xi_i - \xi_i\| \ge \sigma \| r_\mu(x_i, \xi_i)\| > 0$. This contradiction completes the proof. □

Lemma 2.6. *For every $x \in C$ and $\xi \in F(x)$,*

$$\langle \xi, r_\mu(x, \xi) \rangle \ge \mu^{-1} \| r_\mu(x, \xi) \|^2. \tag{2.6}$$

Proof. See [15, Lemma 2.3]. □

Lemma 2.7. *Let C be a closed convex set in $\mathbb{R}^n$, h a real-valued function on $\mathbb{R}^n$, and K the set $\{x \in C : h(x) \le 0\}$. If K is nonempty and h is Lipschitz continuous on C with modulus $\theta > 0$, then*

$$\operatorname{dist}(x, K) \ge \theta^{-1} \max\{h(x), 0\}, \quad \forall\, x \in C, \tag{2.7}$$

where $\operatorname{dist}(x, K)$ *denotes the distance from x to K.*

Proof. See [14, Lemma 2.3]. □

Lemma 2.8. *Let x^* solve the variational inequality* (1.1) *and let the function h_i be defined by* (2.3)*. Then $h_i(x_i) \ge \eta_i(\mu^{-1} - \sigma)\| r_\mu(x_i, \xi_i)\|^2$ and $h_i(x^*) \le 0$. In particular, if $r_\mu(x_i, \xi_i) \ne 0$, then $h_i(x_i) > 0$.*

Proof. It follows from (2.3) that

$$
\begin{aligned}
h_i(x_i) &= \eta_i \langle y_i - \mu\xi_i + r_\mu(x_i,\xi_i), r_\mu(x_i,\xi_i)\rangle \\
&= \eta_i \langle y_i, r_\mu(x_i,\xi_i)\rangle - \mu\eta_i\langle \xi_i, r_\mu(x_i,\xi_i)\rangle + \eta_i \left\| r_\mu(x_i,\xi_i)\right\|^2 \\
&\geq \eta_i(1-\mu)\langle \xi_i, r_\mu(x_i,\xi_i)\rangle + \eta_i(1-\sigma)\left\| r_\mu(x_i,\xi_i)\right\|^2 \\
&\geq \left(\mu^{-1}-\sigma\right)\eta_i \left\| r_\mu(x_i,\xi_i)\right\|^2,
\end{aligned}
\tag{2.8}
$$

where the first inequality follows from (2.2) and the last one follows from Lemma 2.6 and $\mu < 1$. If $r_\mu(x_i,\xi_i) \neq 0$, then $h_i(x_i) > 0$ because $\mu < 1/\sigma$. It remains to be proved that $h_i(x^*) \leq 0$. Since $r_\mu(x_i,\xi_i) = x_i - \Pi_C(x_i - \mu\xi_i)$, we have

$$
\langle r_\mu(x_i,\xi_i) - \mu\xi_i, x^* - x_i + r_\mu(x_i,\xi_i)\rangle \leq 0,
\tag{2.9}
$$

on the other hand, assumption (1.2) implies that

$$
\langle \mu\xi_i, x^* - x_i\rangle = \mu\langle \xi_i, x^* - x_i\rangle \leq 0.
\tag{2.10}
$$

Adding the last two expressions, we obtain that

$$
\langle r_\mu(x_i,\xi_i), x^* - x_i\rangle \leq \langle r_\mu(x_i,\xi_i), \mu\xi_i - r_\mu(x_i,\xi_i)\rangle.
\tag{2.11}
$$

It follows that

$$
\begin{aligned}
\langle y_i + \eta_i r_\mu(x_i,\xi_i), x^* - x_i\rangle &= \langle y_i, x^* - x_i\rangle + \eta_i\langle r_\mu(x_i,\xi_i), x^* - x_i\rangle \\
&\leq \langle y_i, x^* - x_i\rangle + \eta_i\langle r_\mu(x_i,\xi_i), \mu\xi_i - r_\mu(x_i,\xi_i)\rangle \\
&= \langle y_i, x^* - x_i + \eta_i r_\mu(x_i,\xi_i)\rangle - \eta_i\langle y_i - \mu\xi_i + r_\mu(x_i,\xi_i), r_\mu(x_i,\xi_i)\rangle \\
&\leq -\eta_i\langle y_i - \mu\xi_i + r_\mu(x_i,\xi_i), r_\mu(x_i,\xi_i)\rangle,
\end{aligned}
\tag{2.12}
$$

where the second inequality follows from assumption (1.2) and $y_i \in F(x_i - \eta_i r_\mu(x_i,\xi_i))$. Thus $h_i(x^*) \leq 0$ is verified. □

3. Main Results

Theorem 3.1. *If $F : C \to 2^{\mathbb{R}^n}$ is continuous with nonempty compact convex values on C and condition* (1.2) *holds, then either Algorithm 2.2 terminates in a finite number of iterations or generates an infinite sequence $\{x_i\}$ converging to a solution of* (1.1).

Proof. Let x^* be a solution of the variational inequality problem. By Lemma 2.8, $x^* \in C_i$. We assume that Algorithm 2.2 generates an infinite sequence $\{x_i\}$. In particular, $r_\mu(x_i, \xi_i) \neq 0$ for every i. By Step 3, it follows from Lemma 2.4 in [14] that

$$\|x_{i+1} - x^*\|^2 \leq \|x_i - x^*\|^2 - \|x_{i+1} - x_i\|^2 \leq \|x_i - x^*\|^2 - \text{dist}^2(x_i, C_i), \tag{3.1}$$

where the last inequality is due to $x_{i+1} \in C_i$. It follows that the squence $\{\|x_{i+1} - x^*\|^2\}$ is nonincreasing, and hence is a convergent sequence. Therefore, $\{x_i\}$ is bounded and

$$\lim_{i \to \infty} \text{dist}^2(x_i, C_i) = 0. \tag{3.2}$$

By the boundedness of $\{x_i\}$, there exists a convergent subsequence $\{x_{i_j}\}$ converging to $\overline{x}$.

If $\overline{x}$ is a solution of problem (1.1), we show next that the whole sequence $\{x_i\}$ converges to $\overline{x}$. Replacing x^* by $\overline{x}$ in the preceding argument, we obtain that the sequence $\{\|x_i - \overline{x}\|\}$ is nonincreasing and hence converges. Since $\overline{x}$ is an accumulation point of $\{x_i\}$, some subsequence of $\{\|x_i - \overline{x}\|\}$ converges to zero. This shows that the whole sequence $\{\|x_i - \overline{x}\|\}$ converges to zero, hence $\lim_{i \to \infty} x_i = \overline{x}$.

Suppose now that $\overline{x}$ is not a solution of problem (1.1). We show first that k_i in Algorithm 2.2 cannot tend to ∞. Since F is continuous with compact values, Proposition 3.11 in [19] implies that $\{F(x_i) : i \in N\}$ is a bounded set, and so the sequence $\{\xi_i\}$ is bounded. Therefore, there exists a subsequence $\{\xi_{i_j}\}$ converging to $\overline{\xi}$. Since F is upper semicontinuous with compact values, Proposition 3.7 in [19] implies that F is closed, and so $\overline{\xi} \in F(\overline{x})$. By the definition of k_i, we have

$$\left\langle \xi_i - u_i, r_\mu(x_i, \xi_i) \right\rangle > \sigma \left\| r_\mu(x_i, \xi_i) \right\|^2, \quad \forall u_i = \Pi_{F(x_i - \gamma^{k_i - 1} r_\mu(x_i, \xi_i))}(\xi_i). \tag{3.3}$$

If $k_{i_j} \to \infty$, then $x_{i_j} - \gamma^{k_{i_j} - 1} r_\mu(x_{i_j}, \xi_{i_j}) \to \overline{x}$. The lower continuity of F, in turn, implies the existence of $\overline{\xi}_{i_j} \in F(x_{i_j} - \gamma^{k_{i_j} - 1} r_\mu(x_{i_j}, \xi_{i_j}))$ such that $\overline{\xi}_{i_j}$ converges to $\overline{\xi}$. Since $u_{i_j} = \Pi_{F(x_{i_j} - \gamma^{k_{i_j} - 1} r_\mu(x_{i_j}, \xi_{i_j}))}(\xi_{i_j})$, $u_{i_j} \in F(x_{i_j} - \gamma^{k_{i_j} - 1} r_\mu(x_{i_j}, \xi_{i_j}))$, and $\|u_{i_j} - \xi_{i_j}\| \leq \|\overline{\xi}_{i_j} - \xi_{i_j}\|$. Therefore $\lim_{j \to \infty} u_{i_j} = \overline{\xi}$ and

$$\left\langle \xi_{i_j} - u_{i_j}, r_\mu\left(x_{i_j}, \xi_{i_j}\right) \right\rangle > \sigma \left\| r_\mu\left(x_{i_j}, \xi_{i_j}\right) \right\|^2. \tag{3.4}$$

Letting $j \to \infty$, we obtain the contradiction

$$0 \geq \sigma \left\| r_\mu\left(\overline{x}, \overline{\xi}\right) \right\|^2 > 0, \tag{3.5}$$

with $r_\mu(\cdot, \cdot)$ being continuous. Therefore, $\{k_i\}$ is bounded and so is $\{\eta_i\}$.

It follows from (2.3) that

$$\|h_i(x) - h_i(y)\| = \|\langle y_i + \eta_i r_\mu(x_i, \xi_i), x - y\rangle\| \le (\|y_i\| + \|\eta_i r_\mu(x_i, \xi_i)\|)\|x - y\|. \tag{3.6}$$

Since $\{x_i\}$ and $\{\xi_i\}$ are bounded, we have the sequence $\{r_\mu(x_i, \xi_i)\}$ and hence the sequence $\{F(x_i - \eta_i r_\mu(x_i, \xi_i))\}$ is bounded. Thus, for some $M > 0$,

$$\|y_i\| + \|\eta_i r_\mu(x_i, \xi_i)\| \le \sup_{\zeta \in F(x_i - \eta_i r_\mu(x_i, \xi_i))} \|\zeta\| + \|\eta_i r_\mu(x_i, \xi_i)\| \le M, \quad \forall i. \tag{3.7}$$

Therefore, each function h_i is Lipschitz continuous on C with modulus M. Noting that $x_i \notin C_i$ and applying Lemma 2.7, we obtain that

$$\operatorname{dist}(x_i, C_i) \ge M^{-1} h_i(x_i), \quad \forall\, i. \tag{3.8}$$

It follows from (3.8) and Lemma 2.8 that

$$\operatorname{dist}(x_i, C_i) \ge M^{-1} h_i(x_i) \ge M^{-1}\left(\mu^{-1} - \sigma\right)\eta_i \|r_\mu(x_i, \xi_i)\|^2. \tag{3.9}$$

Then (3.2) implies that

$$\lim_{i \to \infty} \eta_i \|r_\mu(x_i, \xi_i)\|^2 = 0. \tag{3.10}$$

By the boundedness of $\{\eta_i\}$, we obtain that $\lim_{i\to\infty}\|r_\mu(x_i, \xi_i)\| = 0$. Since $r_\mu(\cdot, \cdot)$ is continuous and the sequences $\{x_i\}$ and $\{\xi_i\}$ are bounded, there exists an accumulation point $(\overline{x}, \overline{\xi})$ of $\{(x_i, \xi_i)\}$ such that $r_\mu(\overline{x}, \overline{\xi}) = 0$. This implies that $\overline{x}$ solves the variational inequality (1.1). Similar to the preceding proof, we obtain that $\lim_{i\to\infty} x_i = \overline{x}$. □

4. Numerical Experiments

In this section, we present some numerical experiments for the proposed algorithm. The MATLAB codes are run on a PC (with CPU Intel P-T2390) under MATLAB Version 7.0.1.24704(R14) Service Pack 1. We compare the performance of our Algorithm 2.2 and [15, Algorithm 1]. In the Tables 1 and 2, "It." denotes number of iteration, and "CPU" denotes the CPU time in seconds. The tolerance ε means when $\|r(x, \xi)\| \le \varepsilon$, the procedure stops.

Example 4.1. Let $n = 3$,

$$C := \left\{ x \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i = 1 \right\}, \tag{4.1}$$

and let $F : C \to 2^{\mathbb{R}^n}$ be defined by

$$F(x) := \{(t, t - x_1, t - x_2) : t \in [0, 1]\}. \tag{4.2}$$

Table 1: Example 4.1.

	Algorithm 2.2		[15, Algorithm 1]	
ε	It. (num.)	CPU (sec.)	It. (num.)	CPU (sec.)
10^{-7}	55	0.625	74	0.984375
10^{-5}	39	0.546875	51	0.75
10^{-3}	23	0.4375	27	0.5

Table 2: Example 4.2.

		Algorithm 2.2		[15, Algorithm 1]	
Initial point	ε	It. (num.)	CPU (sec.)	It. (num.)	CPU (sec.)
(0,0,0,1)	10^{-7}	53	0.75	61	0.90625
(0,0,1,0)	10^{-7}	47	0.625	79	1.28125
(0.5,0,0.5,0)	10^{-7}	42	0.53125	76	1.28125
(0,0,0,1)	10^{-5}	42	0.625	43	0.671875
(0,0,1,0)	10^{-5}	35	0.53125	56	0.921875
(0.5,0,0.5,0)	10^{-5}	31	0.5	53	0.890625

Then the set C and the mapping F satisfy the assumptions of Theorem 3.1 and (0,0,1) is a solution of the generalized variational inequality. Example 4.1 is tested in [15]. We choose $\sigma = 0.5$, $\gamma = 0.8$, and $\mu = 1$ for our algorithm; $\sigma = 0.8$, $\gamma = 0.6$, and $\mu = 1$ for Algorithm 1 in [15]. We use $x_0 = (0.3, 0.4, 0.3)$ as the initial point.

Example 4.2. Let $n = 4$,

$$C := \left\{ x \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i = 1 \right\}, \tag{4.3}$$

and $F : C \to 2^{\mathbb{R}^n}$ be defined by

$$F(x) = \{(t, t + 2x_2, t + 3x_3, t + 4x_4) : t \in [0,1]\}. \tag{4.4}$$

Then the set C and the mapping F satisfy the assumptions of Theorem 3.1 and (1,0,0,0) is a solution of the generalized variational inequality. We choose $\sigma = 0.5$, $\gamma = 0.8$, and $\mu = 1$ for the two algorithms.

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