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Research Article
Some Caccioppoli Estimates for Differential Forms

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We prove the global Caccioppoli estimate for the solution to the nonhomogeneous A-harmonic equation $d^*A(x,u,du) = B(x,u,du)$, which is the generalization of the quasilinear equation $\operatorname{div} A(x,u,\nabla u) = B(x,u,\nabla u)$. We will also give some examples to see that not all properties of functions may be deduced to differential forms.

1. Introduction

The main work of this paper is study the properties of the solutions to the nonhomogeneous A-harmonic equation for differential forms

$$d^* \; A(x,u,du) = B(x,u,du). \tag{1.1}$$

When u is a 0-form, that is, u is a function, (1.1) is equivalent to

$$\operatorname{div} A(x,u,\nabla u) = B(x,u,\nabla u). \tag{1.2}$$

In [1], Serrin gave some properties of (1.2) when the operator satisfies some conditions. In [2, chapter 3], Heinonen et al. discussed the properties of the quasielliptic equations $-\operatorname{div} A(x,\nabla u) = 0$ in the weighted Sobolev spaces, which is a particular form of (1.2). Recently, a large amount of work on the A-harmonic equation for differential forms has been done. In 1992, Iwaniec introduced the p-harmonic tensors and the relations between quasiregular mappings and the exterior algebra (or differential forms) in [3]. In 1993, Iwaniec and Lutoborski discussed the Poincaré inequality for differential forms when $1 < p < n$ in [4], and the Poincaré inequality for differential forms was generalized to $p > 1$ in [5]. In 1999, Nolder gave the reverse Hölder inequality for the solution to the A-harmonic equation in [6], and different versions of the Caccioppoli estimates have been established in [7–9]. In 2004, Ding proved the Caccioppli estimates for the solution to the nonhomogeneous A-harmonic equation $d^*A(x,du) = B(x,du)$ in [10], where the operator B satisfies $|B(x,\xi)| \le |\xi|^{p-1}$. In 2004, D'Onofrio and Iwaniec introduced the p-harmonic type system in [11], which is an important extension of the conjugate A-harmonic equation. Lots of work on the solution to the p-harmonic type system have been done in [5, 12].

As prior estimates, the Caccioppoli estimate, the weak reverse Hölder inequality, and the Harnack inequality play important roles in PDEs. In this paper, we will prove some Caccioppoli estimates for the solution to (1.1), where the operators $A : \Omega \times \Lambda^l \times \Lambda^{l+1} \to \Lambda^{l+1}$ and $B : \Omega \times \Lambda^l \times \Lambda^{l+1} \to \Lambda^l$ satisfy the following conditions on a bounded convex domain Ω:

$$\begin{aligned} |A(x,u,\xi)| &\le a|\xi|^{p-1} + b(x)|u-u_\Omega|^{p-1} + e(x), \\ |B(x,u,\xi)| &\le c(x)|\xi|^{p-1} + d(x)|u|^{p-1} + f(x), \\ \langle \xi, A(x,u,du)\rangle &\ge |\xi|^p - d(x)|u-u_\Omega| - g(x) \end{aligned} \tag{1.3}$$

for almost every $x \in \Omega$, all l-differential forms u and $(l+1)$-differential forms ξ. Where a is a positive constant and $b(x)$ through $g(x)$ are measurable functions on Ω satisfying:

$$b, e \in L^m(\Omega), \qquad c \in L^{n/(1-\varepsilon)}, \qquad d, f, g \in L^t(\Omega) \tag{1.4}$$

with some $0 < \varepsilon \le 1$, $1/m = 1 - 1/p - (p-1)/\chi p$, $1/t = 1 - \varepsilon/p - (p-\varepsilon)/\chi p$, and χ is the Poincaré constant.

Now we introduce some notations and operations about exterior forms. Let $e_1, e_2, \ldots, e_n$ denote the standard orthogonal basis of $\mathbb{R}^n$. For $l = 0, 1, \ldots, n$, we denote the linear space of all l-vectors by $\Lambda^l = \Lambda^l(\mathbb{R}^n)$, spanned by the exterior product $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$. The Grassmann algebra $\Lambda = \oplus \Lambda^l$ is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha_I e_I \in \Lambda$ and $\beta = \sum \beta_I e_I \in \Lambda$, then its inner product is obtained by

$$\langle \alpha, \beta \rangle = \sum \alpha_I \beta_I \tag{1.5}$$

with the summation over all $I = (i_1, i_2, \ldots, i_l)$ and all integers $l = 0, 1, \ldots, n$. The Hodge star operator $*$:$\Lambda \to \Lambda$ is defined by the rule

$$\begin{gathered} *1 = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}, \\ \alpha \wedge *\beta = \beta \wedge *\alpha = \langle \alpha, \beta \rangle (*1) \end{gathered} \tag{1.6}$$

for all $\alpha, \beta \in \Lambda$. Hence the norm of $\alpha \in \Lambda$ can be given by

$$|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge *\alpha) \in \Lambda_0 = \mathbb{R}. \tag{1.7}$$

Throughout this paper, $\Omega \subset \mathbb{R}^n$ is an open subset. For any constant $\sigma > 1$, Q denotes a cube such that $Q \subset \sigma Q \subset \Omega$, where σQ denotes the cube which center is as same as Q, and $\operatorname{diam}(\sigma Q) = \sigma \operatorname{diam} Q$. We say $\alpha = \sum \alpha_I e_I \in \Lambda$ is a differential l-form on Ω, if every coefficient

α_I of α is Schwartz distribution on Ω. We denote the space spanned by differential l-form on Ω by $D'(\Omega, \Lambda^l)$. We write $L^p(\Omega, \Lambda^l)$ for the l-form $\alpha = \sum \alpha_I dx_I$ on Ω with $\alpha_I \in L^p(\Omega)$ for all ordered l-tuple I. Thus $L^p(\Omega, \Lambda^l)$ is a Banach space with the norm

$$\|\alpha\|_{p,\Omega} = \left(\int_\Omega |\alpha|^p\right)^{1/p} = \left(\int_\Omega \left(\sum_I |\alpha_I|^2\right)^{p/2}\right)^{1/p}. \tag{1.8}$$

Similarly, $W^{k,p}(\Omega, \Lambda^l)$ denotes those l-forms on Ω which all coefficients belong to $W^{k,p}(\Omega)$. The following definition can be found in [3, page 596].

Definition 1.1 ([3]). We denote the exterior derivative by

$$d : D'\left(\Omega, \Lambda^l\right) \longrightarrow D'\left(\Omega, \Lambda^{l+1}\right), \tag{1.9}$$

and its formal adjoint (the Hodge co-differential) is the operator

$$d^* : D'\left(\Omega, \Lambda^l\right) \longrightarrow D'\left(\Omega, \Lambda^{l-1}\right). \tag{1.10}$$

The operators d and d^* are given by the formulas

$$d\alpha = \sum_I d\alpha_I \wedge dx_I, \qquad d^* = (-1)^{nl+1} * d * . \tag{1.11}$$

By [3, Lemma 2.3], we know that a solution to (1.1) is an element of the Sobolev space $W^{1,p}_{\text{loc}}(\Omega, \Lambda^{l-1})$ such that

$$\int_\Omega \langle A(x, u, du), d\varphi\rangle + \langle B(x, u, du), \varphi\rangle \equiv 0 \tag{1.12}$$

for all $\varphi \in W^{1,p}_0(\Omega, \Lambda^{l-1})$ with compact support.

Remark 1.2. In fact, the usual A-harmonic equation is the particular form of the equation (1.1) when $B = 0$ and A satisfies

$$|A(x, \xi)| \le K|\xi|^{p-1}, \qquad \langle A(x, \xi), \xi\rangle \ge |\xi|^p. \tag{1.13}$$

We notice that the nonhomogeneous A-harmonic equation $d^*A(x, du) = B(x, du)$ and the p-harmonic type equation are special forms of (1.1).

2. The Caccioppoli Estimate

In this section we will prove the global and the local Caccioppoli estimates for the solution to (1.1) which satisfies (1.3). In the proof of the global Caccioppoli estimate, we need the following three lemmas.

Lemma 2.1 ([1]). *Let α be a positive exponent, and let α_i, β_i, $i = 1, 2, \ldots, N$, be two sets of N real numbers such that $0 < \alpha_i < \infty$ and $0 \le \beta_i < \alpha$. Suppose that z is a positive number satisfying the inequality*

$$z^{\alpha} \le \sum \alpha_i z^{\beta_i} \tag{2.1}$$

then

$$z \le C \sum (\alpha_i)^{\gamma_i}, \tag{2.2}$$

where C depends only on N, α, β_i, and where $\gamma_i = (\alpha - \beta_i)^{-1}$.

By the inequalities (2.13) and (3.28) in [5], One has the following lemma.

Lemma 2.2 ([5]). *Let Ω be a bounded convex domain in $\mathbb{R}^n$, then for any differential form u, one has*

$$|d|u - u_{\Omega}|| \le C(n, p)|du|. \tag{2.3}$$

Lemma 2.3 ([5]). *If $f, g \ge 0$ and for any nonnegative $\eta \in C_0^{\infty}(\Omega)$, one has*

$$\int_{\Omega} \eta f \, dx \le \int_{\Omega} g \, dx, \tag{2.4}$$

then for any $h \ge 0$, one has

$$\int_{\Omega} \eta f h \, dx \le \int_{\Omega} g h \, dx. \tag{2.5}$$

Theorem 2.4. *Suppose that Ω is a bounded convex domain in $\mathbb{R}^n$, and u is a solution to* (1.1) *which satisfies* (1.3)*, and $p > 1$, then for any $\eta \in C_0^{\infty}(\Omega)$, there exist constants C and k, such that*

$$\begin{aligned} \|\eta du\|_{p,\Omega} \le C\Big\{ &\Big((\operatorname{diam}\Omega)^{s(p-1)} + 1\Big)\|(u - u_{\Omega})d\eta\|_{p,\Omega} + (\operatorname{diam}\Omega)^{s(p/\varepsilon-1)}\|\eta(u - u_{\Omega})\|_{p,\Omega} \\ &+ k\Big((\operatorname{diam}\Omega)^{s(p-1)} + 1\Big)\|d\eta\|_{p,\Omega} + k(\operatorname{diam}\Omega)^{sp/\varepsilon}\Big\}, \end{aligned} \tag{2.6}$$

where $s = n/\chi p + 1 - n/p$, $C = C(n, p, l, a, \|b\|, \|d\|, \varepsilon)$, $k = \|e\|^{1/(p-1)} + \|g\|^{1/p}$, and χ is the Poincaré constant. (i.e., $\chi = 2$ when $p \ge n$, and $\chi = np/(n-p)$ when $1 < p < n$).

Proof. We assume that $B(x,u,du) = \sum_I \omega_I dx_I$. For any nonnegative $\eta \in C_0^{\infty}(\Omega)$, we let $\varphi_1 = -\sum_I \eta \,\mathrm{sign}(\omega_I) dx_I$, then we have $d\varphi_1 = -\sum_I \mathrm{sign}(\omega_I) d\eta \wedge dx_I$. By using $\varphi = \varphi_1$ in the equation (1.12), we can obtain

$$\int_\Omega \left\langle B(x,u,du), \sum_I \eta \,\mathrm{sign}(\omega_I) dx_I \right\rangle dx = \int_\Omega \left\langle A(x,u,du), -\sum_I \mathrm{sign}(\omega_I) d\eta \wedge dx_I \right\rangle dx, \tag{2.7}$$

that is,

$$\int_\Omega \sum_I \eta |\omega_I| dx = \int_\Omega \left\langle A(x,u,du), -\sum_I \mathrm{sign}(\omega_I) d\eta \wedge dx_I \right\rangle dx. \tag{2.8}$$

By the elementary inequality

$$\left(\sum_{i=1}^n {a_i}^2 \right)^{1/2} \le \sum_{i=1}^n |a_i|, \tag{2.9}$$

(2.8) becomes

$$\begin{aligned}
\int_\Omega \eta |B(x,u,du)| dx &= \int_\Omega \eta \left(\sum_I \omega_I^2 \right)^{1/2} dx \\
&\le \int_\Omega \eta \sum_I |\omega_I| dx \\
&= \int_\Omega \left\langle A(x,u,du), -\sum_I \mathrm{sign}(\omega_I) d\eta \wedge dx_I \right\rangle dx \\
&\le \int_\Omega \left| \left\langle A(x,u,du), -\sum_I \mathrm{sign}(\omega_I) d\eta \wedge dx_I \right\rangle \right| dx.
\end{aligned} \tag{2.10}$$

Using the inequality

$$|\langle a,b\rangle| \le |a|\,|b|, \tag{2.11}$$

then (2.10) becomes

$$\begin{aligned}
\int_\Omega \eta |B(x,u,du)| dx &\le \int_\Omega |A(x,u,du)| \left| \sum_I \mathrm{sign}(\omega_I) d\eta \wedge dx_I \right| dx \\
&\le \int_\Omega |A(x,u,du)| \cdot \sum_I \left| \mathrm{sign}(\omega_I) d\eta \wedge dx_I \right| dx \\
&= \int_\Omega |A(x,u,du)| \sum_I \left| d\eta \right| dx.
\end{aligned} \tag{2.12}$$

Since $B(x,u,du) \in \Lambda^{l-1}$,so we can deduce

$$\int_\Omega \eta\, |B(x,u,du)| dx \le C_n^{l-1} \int_\Omega |A(x,u,du)| \big|d\eta\big| dx. \tag{2.13}$$

Now we let $\varphi_2 = -(u-u_\Omega)\eta^p$, then $d\varphi_2 = -p\eta^{p-1} d\eta \wedge (u-u_\Omega) - \eta^p du$. We use $\varphi = \varphi_2$ in (1.12), then we can obtain

$$-\int_\Omega \Big\langle A(x,u,du), p\eta^{p-1} d\eta \wedge u + \eta^p du \Big\rangle dx - \int_\Omega \langle B(x,u,du), \eta^p u\rangle dx \equiv 0. \tag{2.14}$$

So we have

$$\begin{aligned} \int_\Omega \langle A(x,u,du), \eta^p du\rangle dx = &-\int_\Omega \Big\langle A(x,u,du), p\eta^{p-1} d\eta \wedge (u-u_\Omega) \Big\rangle dx \\ &- \int_\Omega \langle B(x,u,du), \eta^p u\rangle dx. \end{aligned} \tag{2.15}$$

By (1.3), (2.13), (2.15) and Lemma 2.2, we have

$$\begin{aligned} 0 &\le \int_\Omega \eta^p |du|^p dx \le \left|\int_\Omega \langle A(x,u,du), \eta^p u\rangle dx\right| + \int_\Omega \left(|d(x)|\,|u-u_\Omega|^p + \big|g\big|\right) dx \\ &\le \int_\Omega \left|\Big\langle A(x,u,du), p\eta^{p-1} d\eta \wedge (u-u_\Omega)\Big\rangle\right| dx + \int_\Omega \left|\langle B(x,u,du), \eta^p (u-u_\Omega)\rangle\right| dx \\ &\quad + \int_\Omega \eta^p \left(|d(x)|\,|u-u_\Omega|^p + \big|g\big|\right) dx \\ &\le \int_\Omega |A(x,u,du)| p\eta^{p-1} \big|d\eta\big|\, |u-u_\Omega| dx + \int_\Omega |B(x,u,du)| \eta^p |u-u_\Omega| dx \\ &\quad + \int_\Omega \eta^p \left(|d(x)|\,|u|^p + \big|g\big|\right) dx \\ &\le \left(C_n^{l-1} + p\right) \int_\Omega |A(x,u,du)| \eta^{p-1} \big|d\eta\big| |u-u_\Omega| dx + \int_\Omega \eta^p \left(|d(x)|\,|u-u_\Omega|^p + \big|g\big|\right) dx \\ &\le C_1 \bigg(\int_\Omega \eta^{p-1} \big|d\eta\big| |u-u_\Omega| |du|^{p-1} dx + \int_\Omega \eta^{p-1} \big|d\eta\big| |u-u_\Omega| \Big(|b(x)|\,|u-u_\Omega|^{p-1} + |e|\Big) dx \\ &\qquad + \int_\Omega \eta^p \left(|d(x)|\,|u-u_\Omega|^p + \big|g\big|\right) dx \bigg), \end{aligned} \tag{2.16}$$

where $C_1 = (C_n^{l-1} + p)\max(1,a)$.

We suppose that $u - u_\Omega = \sum_I u_I dx_I$, $k = \|e\|^{1/(p-1)} + \|g\|^{1/p}$ and let $u_1 = \sum_I (u_I + k\,\mathrm{sign}(u_I))dx_I$, then we have $du_1 = du$ and

$$|u - u_\Omega| + k \le |u_1| = \left(\sum_I (|u_I| + k)^2\right)^{1/2} \le |u - u_\Omega| + C_n^{l-1}k. \tag{2.17}$$

Combining (2.16) and (2.17), we have

$$\begin{aligned}
&\int_\Omega \eta^p |du_1|^p dx \\
&\le C_1\Big(\int_\Omega \eta^{p-1}\big|d\eta\big|\,|u - u_\Omega|\,|du|^{p-1}dx + \int_\Omega \eta^{p-1}\big|d\eta\big||u - u_\Omega|\Big(|b(x)|\,|u - u_\Omega|^{p-1} + |e|\Big)dx \\
&\qquad + \int_\Omega \eta^p\left(|d(x)|\,|u - u_\Omega|^p + \big|g\big|\right)dx\Big) \\
&\le C_1\Big(\int_\Omega \eta^{p-1}\big|d\eta\big|\,|u_1|\,|du_1|^{p-1}dx + \int_\Omega \eta^{p-1}\big|d\eta\big|\,|u_1|\Big(|b(x)| + k^{1-p}|e|\Big)\Big(|u - u_\Omega|^{p-1} + k^{p-1}\Big)dx \\
&\qquad + \int_\Omega \eta^p\left(|d(x)| + k^{-p}\big|g\big|\right)\left(|u - u_\Omega|^p + k^p\right)dx\Big) \\
&\le C_2\Big(\int_\Omega \eta^{p-1}\big|d\eta\big|\,|u_1|\,|du_1|^{p-1}dx + \int_\Omega \eta^{p-1}\big|d\eta\big|\,|u_1|\Big(|b(x)| + k^{1-p}|e|\Big)(|u - u_\Omega| + k)^{p-1}dx \\
&\qquad + \int_\Omega \eta^p\left(|d(x)| + k^{-p}\big|g\big|\right)(|u - u_\Omega| + k)^p dx\Big) \\
&\le C_2\Big(\int_\Omega \eta^{p-1}\big|d\eta\big|\,|u_1||du_1|^{p-1}dx + \int_\Omega \eta^{p-1}b_1(x)\big|d\eta\big|\,|u_1|^p dx + \int_\Omega \eta^p d_1(x)|u_1|^p dx\Big),
\end{aligned} \tag{2.18}$$

where $C_2 = C_1 2^{p-1}$, $b_1(x) = |b(x)| + k^{1-p}|e|$ and $d_1(x) = |d(x)| + k^{-p}|g|$. By simple computations, we get $\|b_1(x)\| \le \|b(x)\| + 1$ and $\|d_1(x)\| \le \|d(x)\| + 1$.

The terms on the right-hand side of the preceding inequality can be estimated by using the Hölder inequality, Minkowski inequality, Poincaré inequality and Lemma 2.2. Thus

$$\int_\Omega \eta^{p-1}\big|d\eta\big||u_1|\,|du_1|^{p-1}dx \le \|u_1 d\eta\|_{p,\Omega}\|\eta du_1\|_{p,\Omega}^{p-1}, \tag{2.19}$$

$$\begin{aligned}
\int_\Omega \eta^{p-1}b_1(x)\big|d\eta\big||u_1|^p dx &= \int_\Omega \eta^{p-1}b_1(x)\big|d\eta\big||u_1||u_1|^{p-1}dx \\
&\le \|b_1(x)\|_{m,\Omega}\left(\int_\Omega \left(\eta^{p-1}\big|d\eta\big||u_1||u_1|^{p-1}\right)^{m/(m-1)} dx\right)^{1-1/m}
\end{aligned}$$

$$\begin{aligned}
&\leq \|b_1(x)\|_{m,\Omega}\|u_1 d\eta\|_{p,\Omega}\|u_1\eta\|_{\chi p,\Omega}^{p-1} \\
&\leq C_3\|b_1(x)\|_{m,\Omega}(\operatorname{diam}\Omega)^{s(p-1)}\|u_1 d\eta\|_{p,\Omega}\|d|u_1\eta|\|_{p,\Omega}^{p-1} \\
&\leq C_4(\operatorname{diam}\Omega)^{s(p-1)}\|u_1 d\eta\|_{p,\Omega}\left(\|u_1 d\eta\|_{p,\Omega}+\|d|u_1|\eta\|_{p,\Omega}\right)^{p-1} \\
&\leq C_5(\operatorname{diam}\Omega)^{s(p-1)}\|u_1 d\eta\|_{p,\Omega}\left(\|u_1 d\eta\|_{p,\Omega}^{p-1}+\|\eta|du|\|_{p,\Omega}^{p-1}\right) \\
&= C_5(\operatorname{diam}\Omega)^{s(p-1)}\|u_1 d\eta\|_{p,\Omega}\left(\|u_1 d\eta\|_{p,\Omega}^{p-1}+\|\eta|du_1|\|_{p,\Omega}^{p-1}\right) \\
&= C_5(\operatorname{diam}\Omega)^{s(p-1)}\left(\|u_1 d\eta\|_{p,\Omega}^{p}+\|u_1 d\eta\|_{p,\Omega}\|\eta du_1\|_{p,\Omega}^{p-1}\right).
\end{aligned} \tag{2.20}$$

By the similar computation, we can obtain

$$\begin{aligned}
\int_\Omega \eta^p d_1(x)|u_1|^p dx &= \int_\Omega \eta^p d_1(x)|u_1|^{p-\varepsilon}|u_1|^\varepsilon dx \\
&\leq C_6(\operatorname{diam}\Omega)^{s(p-\varepsilon)}\|u_1\eta\|_{p,\Omega}^{\varepsilon}\left(\|u_1 d\eta\|_{p,\Omega}^{p-\varepsilon}+\|\eta du_1\|_{p,\Omega}^{p-\varepsilon}\right).
\end{aligned} \tag{2.21}$$

We insert the three previous estimates (2.19), (2.20) and (2.21) into the right-hand side of (2.15), and set

$$z=\frac{\|\eta du_1\|_{p,\Omega}}{\|u_1 d\eta\|_{p,\Omega}}, \qquad \zeta=\frac{\|\eta u_1\|_{p,\Omega}}{\|u_1 d\eta\|_{p,\Omega}}, \tag{2.22}$$

the result can be written

$$\begin{aligned}
z^p &\leq C_2 z^{p-1}+C_5(\operatorname{diam}\Omega)^{s(p-1)}\left(1+z^{p-1}\right)+C_6(\operatorname{diam}\Omega)^{s(p-\varepsilon)}\zeta^\varepsilon\left(1+z^{p-\varepsilon}\right) \\
&\leq C_7\left\{\left((\operatorname{diam}\Omega)^{s(p-1)}+1\right)\left(1+z^{p-1}\right)+(\operatorname{diam}\Omega)^{s(p-\varepsilon)}\zeta^\varepsilon\left(1+z^{p-\varepsilon}\right)\right\}
\end{aligned}. \tag{2.23}$$

Applying Lemma 2.1 and simplifying the result, we obtain

$$z \leq C_7\left\{\left((\operatorname{diam}\Omega)^{s(p-1)}+1\right)+(\operatorname{diam}\Omega)^{s(p/\varepsilon-1)}\zeta\right\}, \tag{2.24}$$

or in terms of the original quantities

$$\|\eta du_1\|_{p,\Omega} \leq C_7\left\{\left((\operatorname{diam}\Omega)^{s(p-1)}+1\right)\|u_1 d\eta\|_{p,\Omega}+(\operatorname{diam}\Omega)^{s(p/\varepsilon-1)}\|\eta u_1\|_{p,\Omega}\right\}. \tag{2.25}$$

Combining (2.17) and (2.25), we can obtain

$$\|\eta du\|_{p,\Omega} \le C_7\Big\{ \Big((\operatorname{diam}\Omega)^{s(p-1)}+1\Big)\|(u-u_\Omega)d\eta\|_{p,\Omega} + k(\operatorname{diam}\Omega)^{sp/\varepsilon} + (\operatorname{diam}\Omega)^{s(p/\varepsilon-1)}\|\eta(u-u_\Omega)\|_{p,\Omega} + k\Big((\operatorname{diam}\Omega)^{s(p-1)}+1\Big)\|d\eta\|_{p,\Omega}\Big\}. \tag{2.26}$$

□

If $1 < p < n$ in Theorem 2.4, we can obtain the following.

Corollary 2.5. *Suppose that Ω is a bounded convex domain in $\mathbb{R}^n$, and u is a solution to* (1.1) *which satisfies* (1.3)*, and $1 < p < n$, then for any $\eta \in C_0^\infty(\Omega)$, there exist constants C and k, such that*

$$\|\eta du\|_{p,\Omega} \le C\Big\{ \big\|(u-u_\Omega)d\eta\big\|_{p,\Omega} + \big\|\eta(u-u_\Omega)\big\|_{p,\Omega} + k\big\|d\eta\big\|_{p,\Omega} + k|\Omega|\Big\}, \tag{2.27}$$

where $C = C(n,p,l,a,\|b\|,\|d\|,\varepsilon)$ and $k = \|e\|^{1/(p-1)} + \|g\|^{1/p}$.

When u is a 0-differential form, that is, u is a function, we have $|d|u|| \le |du|$. Now we use u in place of $u - u_\Omega$ in (1.3), then (1.1) satisfying (1.3) is equivalent to (5) which satisfies (6) in [1], we can obtain the following result which is the improving result of [1, Theorem 2].

Corollary 2.6. *Let u be a solution to the equation* div $A(x,u,\nabla u) = B(x,u,\nabla u)$ *in a domain Ω. For any $1 < p < n$, one denotes $\chi = n/(n-p)$. Suppose that the following conditions hold*

(i) $|A(x,u,\xi)| \le a|\xi|^{p-1} + b|u|^{p-1} + e$, *where $a > 0$ is a constant, $b, e \in L^q$ such that 2003$(p-1)/p\chi + 1/p + 1/q = 1$;*

(ii) $|B(x,u,\xi)| \le c|\xi|^{p-1} + d|u|^{p-1} + f$,

(iii) $\xi \cdot A(x,u,\xi) \ge |\xi|^p - d|u|^p - g$,

where $b \in L^{n/(p-1)}$; $c \in L^{n/(1-\varepsilon)}$; $d, f, g \in L^{n/(p-\varepsilon)}$ with for some $\varepsilon \in (0,1]$. Then for any $\sigma > 1$ and any cubes or balls Q such that $Q \subset \sigma Q \subset \Omega$, one has

$$\|\nabla u\|_{p,Q} \le C\Big(r^{-1}+1\Big)\Big(\|u\|_{p,\sigma Q} + kr^{n/p}\Big), \tag{2.28}$$

where C and k are constants depending only on the above conditions and r is the diameter of Q. One can write them

$$C = C(p,n,\sigma,\varepsilon;a,\|b\|,\|d\|), \qquad k = \|e\|^{1/(p-1)} + \|g\|^{1/p}. \tag{2.29}$$

If we let $\eta \in C_0^\infty(\sigma Q)$ and η is a bump function, then we have the following.

Corollary 2.7. *Suppose that Ω is a bounded convex domain in $\mathbb{R}^n$, and u is a solution to* (1.1) *which satisfies* (1.3)*, and $p > 1$, then for any $\sigma > 1$ and any cubes or balls Q such that $Q \subset \sigma Q \subset \Omega$, there exist constants C and k, such that*

$$\|du\|_{p,Q} \le C\left\{\|u - u_{\sigma Q}\|_{p,\sigma Q} + k\right\}, \tag{2.30}$$

where $C = C(n, p, l, a, \|b\|, \|d\|, \varepsilon, \operatorname{diam} Q)$, $k = \|e\|^{1/(p-1)} + \|g\|^{1/p}$, and χ is the Poincaré constant.

3. Some Examples

Example 3.1. The Sobolev inequality cannot be deduced to differential forms. For any $\eta \in C_0^\infty(B)$, we only let

$$u = \eta dx + \left(\int_B \frac{\partial \eta}{\partial y} dx\right) dy + \left(\int_B \frac{\partial \eta}{\partial z} dx\right) dz, \tag{3.1}$$

then $u \in C_0^\infty(B, \Lambda^1)$, and

$$\begin{aligned} du &= \left(\frac{\partial \eta}{\partial x} dx + \frac{\partial \eta}{\partial y} dy + \frac{\partial \eta}{\partial z} dz\right) \wedge dx \\ &\quad + \left(\frac{\partial \eta}{\partial y} dx + \left(\int_B \frac{\partial^2 \eta}{\partial y^2} dx\right) dy + \left(\int_B \frac{\partial^2 \eta}{\partial y \partial z} dx\right) dz\right) \wedge dy \\ &\quad + \left(\frac{\partial \eta}{\partial z} dx + \left(\int_B \frac{\partial^2 \eta}{\partial y \partial z} dx\right) dy + \left(\int_B \frac{\partial^2 \eta}{\partial z^2} dx\right) dz\right) \wedge dz \\ &= 0. \end{aligned} \tag{3.2}$$

So we cannot obtain

$$\left(\frac{1}{|B|}\int_B |u|^{p\chi} dx\right)^{1/p\chi} \le C \operatorname{diam}(B) \left(\frac{1}{|B|}\int_B |du|^p dx\right)^{1/p}. \tag{3.3}$$

Example 3.2. The Poincaré inequality can be deduced to differential forms. We can see the following lemma.

Lemma 3.3 ([5]). *Let $u \in D'(\mathbb{D}, \Lambda^l)$, and $du \in L^p(\mathbb{D}, \Lambda^{l+1})$, then $u - u_{\mathbb{D}}$ is in $L^{\chi p}(\mathbb{D}, \Lambda^l)$ and*

$$\left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} |u - u_{\mathbb{D}}|^{p\chi} dx\right)^{1/p\chi} \le C(n, p, l) \operatorname{diam}(\mathbb{D}) \left(\frac{1}{|\mathbb{D}|}\int_{\mathbb{D}} |du|^p dx\right)^{1/p}, \tag{3.4}$$

for any ball or cube $\mathbb{D} \in \mathbb{R}^n$, where $\chi = 2$ for $p \ge n$ and $\chi = np/(n-p)$ for $1 < p < n$.

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