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Research Article

Generalized Lazarevic's Inequality and Its Applications—Part II

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A generalized Lazarevic's inequality is established. The applications of this generalized Lazarevic's inequality give some new lower bounds for logarithmic mean.

1. Introduction

Lazarević [1] (or see Mitrinović [2]) gives us the following result.

Theorem 1.1. *Let $x \neq 0$. Then*

$$\left(\frac{\sinh x}{x}\right)^q > \cosh x \tag{1.1}$$

holds if and only if $q \geq 3$.

Recently, the author of this paper gives a new proof of the inequality (1.1) in [3] and extends the inequality (1.1) to the following result in [4].

Theorem 1.2. *Let $p > 0$, and $x \in (0, +\infty)$. Then*

$$\left(\frac{\sinh x}{x}\right)^q > \frac{\sinh x}{x} + \frac{p}{2}\left(\cosh x - \frac{\sinh x}{x}\right) = \frac{2-p}{2}\frac{\sinh x}{x} + \frac{p}{2}\cosh x \tag{1.2}$$

holds if and only if $q \geq p + 1$.

Moreover, the inequality (1.1) can be extended as follows.

Theorem 1.3. *Let $p > 1$ or $p \leq 8/15$, and $x \in (0, +\infty)$. Then*

$$\left(\frac{\sinh x}{x}\right)^q > p + (1-p)\cosh x \tag{1.3}$$

holds if and only if $q \geq 3(1-p)$.

2. Three Lemmas

Lemma 2.1 (see [5–8]). *Let $f, g : [a,b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a,b). Further, let $g' \neq 0$ on (a,b). If f'/g' is increasing (or decreasing) on (a,b), then the functions $(f(x) - f(b))/(g(x) - g(b))$ and $(f(x) - f(a))/(g(x) - g(a))$ are also increasing (or decreasing) on (a,b).*

Lemma 2.2 (see [9–11]). *Let a_n and b_n $(n = 0, 1, 2, \ldots)$ be real numbers, and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$. If $b_n > 0$ for $n = 0, 1, 2, \ldots$, and if a_n/b_n is strictly increasing (or decreasing) for $n = 0, 1, 2, \ldots$, then the function $A(x)/B(x)$ is strictly increasing (or decreasing) on $(0, R)$.*

Lemma 2.3. *Let $p < 1$ and $x > 0$. Then the function $[p + (1-p)\cosh x]^{1/(1-p)}$ strictly increases as p increases.*

3. A Concise Proof of Theorem 1.3

Let $F(x) = \log[p + (1-p)\cosh x] / \log(\sinh x/x) = f_1(x)/g_1(x)$, where $f_1(x) = \log[p + (1-p)\cosh x]$, and $g_1(x) = \log(\sinh x/x)$. Then

$$\frac{f_1'(x)}{g_1'(x)} = (1-p)\frac{f_2(x)}{g_2(x)}, \tag{3.1}$$

where $f_2(x) = x^2 \sinh x$, and $g_2(x) = (x\cosh x - \sinh x)[p + (1-p)\cosh x]$.

We compute

$$\frac{f_2'(x)}{g_2'(x)} = \frac{\sinh x + 2x\cosh x}{x\left[p + (1-p)\cosh x\right] + (1-p)(x\cosh x - \sinh x)} = \frac{A(x)}{B(x)}, \tag{3.2}$$

where

$$\begin{aligned} A(x) &= \sinh x + 2x\cosh x = 3x + \sum_{n=1}^{\infty} a_n x^{2n+1}, \\ B(x) &= x\left[p + (1-p)\cosh x\right] + (1-p)(x\cosh x - \sinh x) = x + \sum_{n=1}^{\infty} b_n x^{2n+1}, \end{aligned} \tag{3.3}$$

and $a_n = (4n+3)/(2n+1)!, b_n = (1-p)((4n+1)/(2n+1)!)$.

We obtain results in two cases.

(a) Let $p \leq 8/15$, then $p < 1$ and $b_n > 0$. Let $c_n = a_n/b_n$ for $n = 0, 1, 2, \ldots$, we have that $c_0 = 3 \geq 7/(5(1-p)) = c_1$ and $c_n = (1/(1-p))((4n+3)/(4n+1)) = (1/(1-p))(2+(1/(4n+1)))$ is decreasing for $n = 1, 2, \ldots$; so c_n is decreasing for $n = 0, 1, \ldots$ and $A(x)/B(x)$ is decreasing on $(0, +\infty)$ by Lemma 2.2. Hence $f_2'(x)/g_2'(x) = A(x)/B(x)$ is decreasing on $(0, +\infty)$ and $f_1'(x)/g_1'(x) = (1-p)(f_2(x)/g_2(x)) = (1-p)((f_2(x)-f_2(0))/(g_2(x)-g_2(0)))$ is decreasing on $(0, +\infty)$ by Lemma 2.1. Thus $Q(x) = (f_1(x)-f_1(0^+))/(g_1(x)-g_1(0^+))$ is decreasing on $(0, +\infty)$ by Lemma 2.1.

(b) Let $p > 1$, then $p > 8/15$. Let $d_n = 1/c_n$ for $n = 0, 1, 2, \ldots$, we have that $d_0 = 1/3 > 7/(5(1-p)) = d_1$ and $d_n = (1-p)(1-2/(4n+1))$ is decreasing for $n = 1, 2, \ldots$; so d_n is decreasing for $n = 0, 1, \ldots$ and $B(x)/A(x)$ is decreasing on $(0, +\infty)$ by Lemma 2.2. Hence $f_2'(x)/g_2'(x) = A(x)/B(x)$ is increasing on $(0, +\infty)$ and $f_1'(x)/g_1'(x) = (1-p)(f_2(x)/g_2(x)) = (1-p)((f_2(x)-f_2(0))/(g_2(x)-g_2(0)))$ is decreasing on $(0, +\infty)$ by Lemma 2.1. Thus $Q(x) = (f_1(x)-f_1(0^+))/(g_1(x)-g_1(0^+))$ is decreasing on $(0, +\infty)$ by Lemma 2.1.

Since

$$\begin{aligned}\lim_{x\to 0^+} Q(x) &= \lim_{x\to 0^+} \frac{f_1(x)}{g_1(x)} = \lim_{x\to 0^+} \frac{f_1'(x)}{g_1'(x)} = \lim_{x\to 0^+} (1-p)\frac{f_2(x)}{g_2(x)} \\ &= \lim_{x\to 0^+} (1-p)\frac{f_2'(x)}{g_2'(x)} = \lim_{x\to 0^+} (1-p)\frac{A(x)}{B(x)} = (1-p)\frac{a_0}{b_0} = 3(1-p),\end{aligned} \tag{3.4}$$

the proof of Theorem 1.3 is complete.

4. Some New Lower Bounds for Logarithmic Mean

Assuming that x and y are two different positive numbers, let $A(x,y)$, $G(x,y)$, and $L(x,y)$ be the arithmetic, geometric, and logarithmic means, respectively. It is well known that (see [2, 12–16])

$$G < L < A. \tag{4.1}$$

Ostle and Terwilliger [17] (or see Leach and Sholander [18], Zhu [16]) gave bounds for $L(x,y)$ in terms of $G(x,y)$ and $A(x,y)$ as follows:

$$L > A^{1/3}G^{2/3}. \tag{4.2}$$

Without loss of generality, let $0 < x < y$ and $t = (1/2)\log(y/x)$, then $t > 0$. Replacing x with t in (1.3), we obtain the following new results for three classical means.

Theorem 4.1. *Let $p > 1$ or $p \leq 8/15$, and x and y be two positive numbers such that $x \neq y$. Then*

$$L > \left[p + (1-p)\frac{A}{G}\right]^{1/3(1-p)} G \tag{4.3}$$

holds if and only if $q \geq 3(1-p)$.

Now letting p in inequality (4.3) be $8/15, 1/2, 1/3$, and 0, respectively, by Theorem 4.1 and Lemma 2.3 we have the following inequalities:

$$L > \left(\frac{8G+7A}{15}\right)^{5/7} G^{2/7} > \left(\frac{G+A}{2}\right)^{2/3} G^{1/3} > \left(\frac{G+2A}{3}\right)^{1/2} G^{1/2} > A^{1/3}G^{2/3}. \tag{4.4}$$

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