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Research Article

Riemann-Stieltjes Operators between Vector-Valued Weighted Bloch Spaces

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Let X be a Banach space, we study Riemann-Stieltjes operators between X-valued weighted Bloch spaces. Some necessary and sufficient conditions for these operators induced by holomorphic functions to be weakly compact and weakly conditionally compact are given by certain growth properties of the inducing symbols and some structural properties of the abstract Banach space, which extend some previous results.

1. Introduction and statement of the main results

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$. For $g \in H(\mathbb{D})$, the Riemann-Stieltjes operator T_g is defined on $H(\mathbb{D})$ by

$$\left(T_g f\right)(z) = \int_0^z f(\zeta)\,dg(\zeta) = \int_0^1 f(tz)zg'(tz)\,dt, \quad z \in \mathbb{D}. \tag{1.1}$$

The Riemann-Stieltjes operator T_g can be viewed as a generalization of the well-known Cesàro operator defined by

$$\left(C[f]\right)(z) = \frac{1}{z}\int_0^z \frac{f(\zeta)}{1-\zeta}\,d\zeta = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{i=0}^{n} a_i\right)z^n, \quad z \in \mathbb{D} \tag{1.2}$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$.

Pommerenke [1] initiated the study of Riemann-Stieltjes operators on Hardy space H^2, where he proved that T_g is bounded on H^2 if and only if g is in BMOA, the space of holomorphic functions on $\mathbb{D}$ with bounded mean oscillation. This result later was extended to other Hardy spaces H^p, $1 < p < \infty$ (see [2]). Similar questions on weighted Bergman spaces were considered by Aleman and Siskakis in [3]: T_g is bounded on Bergman space A^2 if and only if g is in Bloch space. Henceforward, many papers have been published which discuss the action of Riemann-Stieltjes operators on distinct spaces of holomorphic functions, including Hardy spaces, weighted Bergman spaces, Dirichlet spaces, BMOA, VMOA, Bloch spaces, and so on; see, for example, [4–9] and the related references therein. Among the prominent results we mention the characterization of Riemann-Stieltjes operators on Bloch space in terms of the growth properties of the inducing symbols [8], where Yoneda proved that T_g is bounded on the Bloch space $\mathcal{B}$ if and only if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)\left|g'(z)\right| < \infty; \tag{1.3}$$

T_g is compact on the Bloch space $\mathcal{B}$ if and only if

$$\lim_{|z|\to 1^-}(1-|z|^2)\left(\log\frac{1}{1-|z|^2}\right)\left|g'(z)\right| = 0, \tag{1.4}$$

where the Bloch space $\mathcal{B} := \{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty\}$. Recently, several authors have published papers to extend this result from different angles. Some papers discussed a higher dimensional version Riemann-Stieltjes operator of (1.1) to the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ replacing $g'(z)$ by the radial derivative $\Re g$ of g. For example, Hu [10] gave the characterizations of bounded and compact Riemann-Stieltjes operators on the Bloch space of $\mathbb{B}_n$, Xiao [11] further studied the Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball, Zhang [9] studied the boundedness and compactness of Riemann-Stieltjes operators on Dirichlet-type spaces and Bloch-type spaces of $\mathbb{B}_n$, on general Bloch-type spaces, the Riemann-Stieltjes operators were studied in [5, 12]. From the main result of [9] (see also in [12, 13]), we know that $T_g : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded if and only if $g \in \mathcal{B}^\beta$ for $0 < \alpha < 1$; $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|g'(z)|\log(2/(1-|z|^2)) < \infty$ for $\alpha = 1$; and $\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta+1-\alpha}|g'(z)| < \infty$ for $\alpha > 1$, where $\alpha, \beta > 0$ and $B^\alpha := \{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty\}$. One can further refer to [10–12, 14, 15] for more study of Riemann-Stieltjes operators on Hardy spaces, Bergman spaces, and Bloch spaces of the unit ball $\mathbb{B}_n$.

It is worth remarking that all the above spaces which T_g targets are not beyond a spectrum of the scalar-valued holomorphic function spaces. The purpose of this paper is to initiate the study of Riemann-Stieltjes operators on spaces of vector-valued holomorphic functions. Let X be any complex Banach space and $\alpha > 0$, the vector-valued weighted Bloch space $\mathcal{B}^\alpha(X)$ consists of all X-valued holomorphic functions $f : \mathbb{D} \to X$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\left\|f'(z)\right\|_X < \infty. \tag{1.5}$$

The little weighted Bloch space $\mathcal{B}_0^\alpha(X)$ is the subspace of $\mathcal{B}^\alpha(X)$ consisting of the holomorphic functions $f : \mathbb{D} \to X$ for which $\lim_{|z|\to 1^-}(1-|z|^2)^\alpha\|f'(z)\|_X = 0$. For $f \in \mathcal{B}^\alpha(X)$, define

$$\|f\|_{\mathcal{B}^\alpha(X)} = \left\|f(0)\right\|_X + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\left\|f'(z)\right\|_X. \tag{1.6}$$

With this norm, both $\mathcal{B}^\alpha(X)$ and $\mathcal{B}_0^\alpha(X)$ are Banach spaces. These classes of vector-valued spaces have been studied quite extensively; see, for instance, [16, 17]. For simplification, we often write $\mathcal{B}^\alpha$ and $\mathcal{B}_0^\alpha$ instead of $\mathcal{B}^\alpha(\mathbb{C})$ and $\mathcal{B}_0^\alpha(\mathbb{C})$, respectively. For more information on the scalar-valued Bloch spaces, one can refer to [18, 19]. When $\alpha = 1$, we often omit the α from $\mathcal{B}^\alpha(X)$. Clearly, $f \in \mathcal{B}^\alpha(X)$ if and only if $x^* \circ f(\cdot) = x^*(f(\cdot)) \in \mathcal{B}^\alpha$ for all $x^* \in X^*$, the dual space of X. Moreover, $\|f\|_{\mathcal{B}^\alpha(X)} \approx \sup_{\|x^*\|_{X^*} \le 1} \|x^* \circ f\|_{\mathcal{B}^\alpha}$. Here and in the sequel, we write $a \lesssim b$ or $b \gtrsim a$ for any nonnegative quantities a and b if a is dominated by b times some inessential positive constant, and write $a \approx b$ for $a \lesssim b \lesssim a$.

Since

$$x^*(T_g f)(z) = x^*\left(\int_0^z f(\zeta)dg(\zeta)\right) = \int_0^z x^*(f(\zeta))dg(\zeta) = T_g(x^* f)(z) \tag{1.7}$$

for any $x^* \in X^*$ and $f \in \mathcal{B}^\alpha(X)$, T_g is bounded between $\mathcal{B}^\alpha(X)$ and $\mathcal{B}^\beta(X)$ if and only if it is bounded between the corresponding scalar-valued spaces $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$. In addition, in Section 3, we will see that when the Banach space X is infinite-dimensional, T_g is never compact between $\mathcal{B}^\alpha(X)$ and $\mathcal{B}^\beta(X)$ except for the trivial case that g is a constant function. In this paper, we will study some small property of Riemann-Stieltjes operators between X-valued Bloch spaces. The main goal is to generalize some characterizations of compact Riemann-Stieltjes operators on scalar-valued Bloch spaces to the vector-valued case.

Our main result is for the weak compactness of T_g.

Theorem 1.1. *Let $\alpha, \beta > 0$, X be a complex Banach space and $g : \mathbb{D} \to \mathbb{C}$ a nonconstant holomorphic function. Then the following hold.*

(1) *For $0 < \alpha < 1$, $T_g : \mathcal{B}^\alpha(X) \to \mathcal{B}^\beta(X)$ (resp., $T_g : \mathcal{B}_0^\alpha(X) \to \mathcal{B}_0^\beta(X)$) is weakly compact if and only if X is reflexive and*

$$\sup_{z\in\mathbb{D}} \left(1 - |z|^2\right)^\beta \left|g'(z)\right| < \infty \quad \left(\text{resp., } \lim_{|z| \to 1^-} \left(1 - |z|^2\right)^\beta \left|g'(z)\right| = 0\right). \tag{1.8}$$

(2) *For $\alpha = 1$, $T_g : \mathcal{B}^\alpha(X) \to \mathcal{B}^\beta(X)$ (or $T_g : \mathcal{B}_0^\alpha(X) \to \mathcal{B}_0^\beta(X)$) is weakly compact if and only if X is reflexive and*

$$\lim_{|z| \to 1^-} \left(1 - |z|^2\right)^\beta \left(\ln \frac{2}{1 - |z|^2}\right) \left|g'(z)\right| = 0. \tag{1.9}$$

(3) *For $\alpha > 1$, $T_g : \mathcal{B}^\alpha(X) \to \mathcal{B}^\beta(X)$ (or $T_g : \mathcal{B}_0^\alpha(X) \to \mathcal{B}_0^\beta(X)$) is weakly compact if and only if X is reflexive and*

$$\lim_{|z| \to 1^-} \left(1 - |z|^2\right)^{\beta-\alpha+1} \left|g'(z)\right| = 0. \tag{1.10}$$

Theorem 1.1 illustrates that T_g is weakly compact between $\mathcal{B}^\alpha(X)$ and $\mathcal{B}^\beta(X)$ if and only if X is reflexive and T_g is compact between the corresponding scalar-valued spaces. It also illustrates that the weak compactness of T_g depends on α with $\alpha > 1$, but this is not the case when $\alpha \in (0,1)$, however, for the case $\alpha = 1$, the condition needs an additional logarithmic term.

The rest of the paper is organized as follows. We give some lemmas in Section 2, which are essentially needed for our proof of the main result. The proof of Theorem 1.1 and the reason why we do not consider the compactness of T_g are given in Section 3. Finally, we briefly consider the weakly conditional compactness of T_g between $\mathcal{B}^\alpha(X)$ and $\mathcal{B}^\beta(X)$ and obtain some counterpart of our main result.

In the sequel, we often use the same letter C, depending only on the allowed parameters, to denote various positive constants which may change at each occurrence.

2. Preliminaries

First, we need the following growth estimate of Bloch functions.

Lemma 2.1. *For $\alpha > 0$ and any complex Banach space X, if $f \in \mathcal{B}^\alpha(X)$, then*

(1) $\|f(z)\|_X \lesssim \|f\|_{\mathcal{B}^\alpha(X)}$ *for any $z \in \mathbb{D}$ and $0 < \alpha < 1$;*

(2) $\|f(z)\|_X \lesssim \ln(2/(1-|z|^2))\|f\|_{\mathcal{B}(X)}$ *for any $z \in \mathbb{D}$ and $\alpha = 1$;*

(3) $\|f(z)\|_X \lesssim (1/(1-|z|^2)^{\alpha-1})\|f\|_{\mathcal{B}^\alpha(X)}$ *for any $z \in \mathbb{D}$ and $\alpha > 1$.*

Proof. Since for any $x^* \in X^*$,

$$\left|(x^* \circ f)'(z)\right| \leq \frac{\|x^* \circ f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^\alpha}, \quad z \in \mathbb{D}, \tag{2.1}$$

so

$$\left|x^* \circ f(z) - x^* \circ f(0)\right| \leq \int_0^1 \left|(x^* \circ f)'(zt)\right| |z| dt \leq \|x^* \circ f\|_{\mathcal{B}^\alpha} \int_0^1 \frac{|z|}{(1-t^2|z|^2)^\alpha} dt. \tag{2.2}$$

Taking the supremum over x^* in the unit ball of X^* and estimating the last integral will give the desired results. □

For little Bloch spaces, we have the following improved behavior of f near the boundary $\partial\mathbb{D}$.

Lemma 2.2. *Let X be a Banach space.*

(1) *If $f \in \mathcal{B}_0(X)$, then*

$$\lim_{|z| \to 1^-} \frac{\|f(z)\|_X}{\ln(2/(1-|z|^2))} = 0. \tag{2.3}$$

(2) *If $f \in \mathcal{B}_0^\alpha(X)$ with $\alpha > 1$, then*

$$\lim_{|z| \to 1^-} (1-|z|^2)^{\alpha-1} \|f(z)\|_X = 0. \tag{2.4}$$

Proof. Since $f \in \mathcal{B}_0^\alpha(X)$, then $\lim_{|z|\to 1^-}(1-|z|^2)^\alpha \|f'(z)\|_X = 0$, that is,

$$\lim_{|z|\to 1^-}(1-|z|^2)^\alpha \sup_{\|x^*\|_{X^*}\le 1}\left|x^*\circ f'(z)\right| = 0. \tag{2.5}$$

So for any $\varepsilon > 0$, there is $r_0 \in (0,1)$ such that

$$(1-|z|^2)^\alpha \left|x^*\circ f'(z)\right| < \varepsilon \quad \text{for any } r_0 < |z| < 1,\ \left\|x^*\right\|_{X^*} \le 1. \tag{2.6}$$

Then for any $r_0 < |z| < 1$, $x^* \in X^*$ with $\|x^*\|_{X^*} \le 1$, we have

$$\begin{aligned}\left|x^*\circ f(z) - x^*\circ f(0)\right| &\le \int_0^{r_0/|z|}\left|(x^*\circ f)'(zt)\right||z|dt + \int_{r_0/|z|}^1 \left|(x^*\circ f)'(zt)\right||z|dt \\ &\lesssim \|f\|_{\mathcal{B}^\alpha(X)}\int_0^{r_0/|z|}\frac{|z|dt}{(1-|zt|^2)^\alpha} + \varepsilon\int_{r_0/|z|}^1\frac{|z|dt}{(1-|zt|^2)^\alpha} \\ &\lesssim \|f\|_{\mathcal{B}^\alpha(X)} + \varepsilon\int_{r_0/|z|}^1\frac{|z|dt}{(1-|zt|^2)^\alpha},\end{aligned} \tag{2.7}$$

the fact $(x^*\circ f)'(z) = x^*\circ f'(z)$ is used in the second inequality above. Since

$$\int_{r_0/|z|}^1\frac{|z|dt}{(1-|zt|^2)^\alpha} \lesssim \begin{cases}\ln\dfrac{2}{1-|z|^2}, & \alpha = 1,\\[2ex] \dfrac{1}{(1-|z|^2)^{\alpha-1}}, & \alpha > 1,\end{cases} \tag{2.8}$$

taking the supremum over all x^* with $\|x^*\|_{X^*} \le 1$ and a variance of (2.7) will complete the proof. $\square$

The following lemma is based on the well-known properties of the de la Vallée-Poussin summability kernel, which is used to approximate T_g in the operator norm by suitable weakly compact operators sequence.

Lemma 2.3. *For $\alpha > 0$ and any complex Banach space X, there are linear operators $\{V_n\}$ on $\mathcal{B}^\alpha(X)$ satisfying the following properties.*

(1) $\|V_n\| \le 3$ *for any* $n \ge 1$*. In addition,* $V_n(\mathcal{B}_0^\alpha(X)) \subset \mathcal{B}_0^\alpha(X)$.

(2) *For every* $r \in (0,1)$, $\lim_{n\to\infty}\sup_{\|f\|_{\mathcal{B}^\alpha(X)}\le 1}\sup_{|z|\le r}\|(f - V_n f)(z)\|_X = 0$.

(3) *If X is reflexive (resp., does not contain a copy of l^1), then V_n is weakly compact (resp., weakly conditionally compact) on $\mathcal{B}^\alpha(X)$ for all $n \ge 1$.*

Proof. We first define the operators $\widetilde{V_n}$ by setting

$$\widetilde{V_n}f(z) = \sum_{k=0}^{n} a_k z^k + \sum_{k=n+1}^{2n-1} \frac{2n-k}{n} a_k z^k \tag{2.9}$$

for any holomorphic function $f : \mathbb{D} \to X$ with the Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Note that

$$\widetilde{V_n}f(z) = (2K_{2n-1} - K_{n-1}) * f(z) = \frac{1}{2\pi}\int_0^{2\pi} \left[2K_{2n-1}(\theta) - K_{n-1}(\theta)\right] f\left(ze^{-i\theta}\right) d\theta, \tag{2.10}$$

where $K_n(\theta) = \sum_{k=-n}^{n}(1 - |k|/(n+1))e^{ik\theta}$ denotes the Fejér kernel, which is a summability kernel, that is, $(1/2\pi)\int_0^{2\pi} K_n(\theta)d\theta = 1$ (refer to [20]). Then we have

$$\left\|\widetilde{V_n}f\right\|_{H^\infty(X)} \leq 3\|f\|_{H^\infty(X)}, \tag{2.11}$$

where $\|\cdot\|_{H^\infty(X)}$ denotes the norm on the X-valued Hardy space $H^\infty(X)$ given by $\|f\|_{H^\infty(X)} = \sup_{z\in\mathbb{D}}\|f(z)\|_X$. For any $\varepsilon > 0$ and $r \in (0,1)$, there exists $n_0 > 0$ such that $r^n \leq \varepsilon/4$ for $n > n_0$. Given $f \in H^\infty(X)$, we write $f - \widetilde{V_n}f = z^n g$, then

$$\begin{aligned}
\|g\|_{H^\infty(X)} &= \sup_{z\in\mathbb{D}}\left\|g(z)\right\|_X \\
&= \sup_{z\in\partial\mathbb{D}}\left\|g(z)\right\|_X \\
&= \sup_{z\in\partial\mathbb{D}}\left\|z^n g(z)\right\|_X \\
&= \sup_{z\in\mathbb{D}}\left\|z^n g(z)\right\|_X \\
&= \sup_{z\in\mathbb{D}}\left\|f(z) - \widetilde{V_n}f(z)\right\|_X \\
&= \left\|f - \widetilde{V_n}f\right\|_{H^\infty(X)},
\end{aligned} \tag{2.12}$$

the second equality above is due to the subharmonicity of $\|g(z)\|_X$. So

$$\left\|f(z) - \widetilde{V_n}f(z)\right\|_X = |z|^n\left\|g(z)\right\|_X \leq r^n\|g\|_{H^\infty(X)} \leq \frac{\varepsilon}{4}\left\|f - \widetilde{V_n}f\right\|_{H^\infty(X)} \leq \varepsilon\|f\|_{H^\infty(X)} \tag{2.13}$$

for $n > n_0$ and all $|z| < r$, the last inequality comes from (2.11). Now we define the desired operators $\{V_n\}$ via $\widetilde{V_n}$ as follows:

$$V_n f(z) = f(0) + \int_0^z \widetilde{V_n} f'(\zeta) d\zeta \tag{2.14}$$

for any holomorphic function $f : \mathbb{D} \to X$. Clear $V_n f$ is holomorphic and actually

$$V_n f(z) = \sum_{k=0}^{n+1} a_k z^k + \sum_{k=n+2}^{2n} \frac{2n+1-k}{n} a_k z^k \tag{2.15}$$

for any holomorphic function $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Since

$$\begin{aligned}
\sup_{|z|=r} \left\| (V_n f)'(z) \right\|_X &= \sup_{|z|=r} \left\| \widetilde{V_n} f'(z) \right\|_X \\
&= \sup_{|z|=1} \left\| \widetilde{V_n} f'(rz) \right\|_X \\
&= \left\| \widetilde{V_n} f_r' \right\|_{H^\infty(X)} \\
&\leq 3 \left\| f_r' \right\|_{H^\infty(X)} \\
&= 3 \sup_{|z|=r} \left\| f'(z) \right\|_X,
\end{aligned} \tag{2.16}$$

where $f_r(\cdot) = f(r\cdot)$ for any $0 < r < 1$. Hence

$$\left\| V_n f \right\|_{\mathcal{B}^\alpha(X)} \leq 3 \|f\|_{\mathcal{B}^\alpha(X)}, \quad f \in \mathcal{B}^\alpha(X), \tag{2.17}$$

by the definition of the norm $\|\cdot\|_{\mathcal{B}^\alpha(X)}$ and $V_n f(0) = f(0)$. In addition, it is clear that $V_n(\mathcal{B}_0^\alpha(X)) \subset \mathcal{B}_0^\alpha(X)$, since $V_n f$ is always a polynomial by (2.15). This completes the proof of part (1).

Since

$$f(z) = f(0) + \int_0^z f'(\zeta) d\zeta, \qquad V_n f(z) = f(0) + \int_0^z \widetilde{V_n} f'(\zeta) d\zeta, \tag{2.18}$$

so

$$\left\| f(z) - V_n f(z) \right\|_X = \left\| \int_0^1 \left(f'(zt) - \widetilde{V_n} f'(zt) \right) z dt \right\|_X. \tag{2.19}$$

Hence for any $|z| \le r < 1$,

$$\begin{aligned}\|f(z) - V_n f(z)\|_X &\le \int_0^1 \sup_{|z|\le r} \|(f'(zt) - \widetilde{V_n} f'(zt))\|_X dt \\ &= \int_0^1 \sup_{|z|\le\sqrt{r}} \|(f'_{\sqrt{r}}(zt) - \widetilde{V_n} f'_{\sqrt{r}}(zt))\|_X dt \\ &\le \int_0^1 \varepsilon(1-r)^\alpha \|f'_{\sqrt{r}}\|_{H^\infty(X)} dt \\ &= \varepsilon(1-r)^\alpha \|f'_{\sqrt{r}}\|_{H^\infty(X)} \le \varepsilon\|f\|_{\mathcal{B}^\alpha(X)}\end{aligned} \tag{2.20}$$

for large enough n, the third inequality to the last is followed by applying (2.13) to the function $f'_{\sqrt{r}}$ and the constant $\varepsilon(1-r)^\alpha$. This completes the proof of part (2).

Finally, for any n, define

$$S_n f = (a_0, a_1, \ldots, a_{2n}) \tag{2.21}$$

for any holomorphic function $f : \mathbb{D} \to X$ with Taylor expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$, and define

$$(T_n \chi)(z) = \sum_{k=0}^{n+1} a_k z^k + \sum_{k=n+2}^{2n} \frac{2n+1-k}{n} a_k z^k \tag{2.22}$$

for any $\chi = (a_0, a_1, \ldots, a_{2n}) \in (\bigoplus_0^{2n} X)_{l^2}$. It is clear that $S_n : \mathcal{B}^\alpha(X) \to (\bigoplus_0^{2n} X)_{l^2}$ and $T_n : (\bigoplus_0^{2n} X)_{l^2} \to \mathcal{B}^\alpha(X)$ are well defined and bounded. Moreover, $V_n = T_n S_n$ by (2.15), that is, V_n has a factorization through $(\bigoplus_0^{2n} X)_{l^2}$. It follows from Alaoglu's theorem [21] and Rosenthal's l^1-criterion [22] that V_n is weakly compact (resp., weakly conditionally compact) if X is reflexive (resp., does not contain a copy of l^1). The proof is complete. □

3. Proof of the main results

Before proving Theorem 1.1, we first recall that a bounded linear operator $T : E \to F$ from the Banach space E to the Banach space F is compact (resp., weakly compact) if every bounded sequence $\{f_n\} \subset E$ has a subsequence $\{f_{n_k}\}$ such that $\{T f_{n_k}\}$ is norm convergent (resp., weakly convergent). A useful characterization for a bounded linear operators to be weakly compact is the Gantmacher's theorem [21]: T is weakly compact if and only if $T^{**}(E^{**}) \subset F$, where T^{**} is the second adjoint of T, and E^{**} is the second dual of E.

Notice that if g is a nonconstant holomorphic function such that $T_g : \mathcal{B}^\alpha(X) \to \mathcal{B}^\beta(X)$ (or $T_g : \mathcal{B}_0^\alpha(X) \to \mathcal{B}_0^\beta(X)$) is compact, then for any bounded sequence $\{x_n\}$ in X and $f_n(z) \equiv x_n$, $\{f_n\}$ is a bounded sequence of $\mathcal{B}_0^\alpha(X)$ since $\|f_n\|_{\mathcal{B}^\alpha(X)} = \|x_n\|_X$, and then there exists a subsequence $\{f_{n_k}\}$ by the definition of compact operators such that $\{T_g f_{n_k}\}$ is norm convergent in $\mathcal{B}^\beta(X)$. On the other hand,

$$F_n(z) := T_g f_n(z) = \int_0^z x_n g'(\zeta) d\zeta = x_n (g(z) - g(0)). \tag{3.1}$$

So $\{F_{n_k}\}$ is norm convergent in $\mathcal{B}^\beta(X)$. It follows from Lemma 2.1 that F_{n_k} converges uniformly on any compact subset of $\mathbb{D}$, especially it is pointwise convergent. That is, for any bounded sequence $\{x_n\} \subset X$, there is a subsequence $\{x_{n_k}\}$ such that it is norm convergence in X since g is nonconstant, so X must be finite-dimensional Banach space by Bolzano-Weierstrass theorem [21]. Namely, for any infinite-dimensional Banach space X, T_g is never compact between X-valued weighted Bloch spaces except for the trivial case that $g$which is a constant function.

From here on, we always assume that X is an infinite-dimensional Banach space, similar analysis as above shows that if the Riemann-Stieltjes operator T_g is weakly compact from $\mathcal{B}^\alpha(X)$ to $\mathcal{B}^\beta(X)$ (or from $\mathcal{B}_0^\alpha(X)$ to $\mathcal{B}_0^\beta(X)$), then for $f_n(z) \equiv x_n$, a bounded sequence in X, there exists a subsequence $\{f_{n_k}\}$ such that $\{T_g f_{n_k}\}$ is weakly convergent. Without loss of generality, we may assume that $\{T_g f_{n_k}\}$ converges weakly to 0. Fix any $z \in \mathbb{D}$, let δ_z be the point evaluation function at z, that is, $\delta_z(f) = f(z)$, $f \in \mathcal{B}^\beta(X)$. Then for any $x^* \in X^*$, the functional

$$x^* \circ \delta_z(f) = x^*\big(f(z)\big), \quad f \in \mathcal{B}^\beta(X) \tag{3.2}$$

satisfies

$$\left|x^* \circ \delta_z(f)\right| \le \left\|x^*\right\|_{X^*}\left\|f(z)\right\|_X \lesssim C_z\left\|x^*\right\|_{X^*}\|f\|_{\mathcal{B}^\beta(X)}, \quad f \in \mathcal{B}^\beta(X) \tag{3.3}$$

for some constant $C_z > 0$ by Lemma 2.1. That is, $x^* \circ \delta_z \in (\mathcal{B}^\beta(X))^*$, so $x^* \circ \delta_z(T_g f_{n_k}) \to 0$ as $k \to \infty$, that is,

$$x^*(x_{n_k})\big(g(z) - g(0)\big) \longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \tag{3.4}$$

for any $x^* \in X^*$ and $z \in \mathbb{D}$. Since g is nonconstant, so $x^*(x_{n_k}) \to 0$ as $k \to \infty$ for any $x^* \in X^*$. That is, for any bounded sequence $\{x_n\} \subset X$, there is a subsequence $\{x_{n_k}\}$ such that it is weakly convergent, then X must be reflexive (refer to [23]). Namely, the reflexivity of X is a necessary condition for the weak compactness of T_g between X-valued Bloch spaces. Under this assumption, Theorem 1.1 states that $T_g : \mathcal{B}^\alpha(X) \to \mathcal{B}^\beta(X)$ (or $T_g : \mathcal{B}_0^\alpha(X) \to \mathcal{B}_0^\beta(X)$) is weakly compact if and only if the corresponding scalar-valued operator T_g is compact by the main result in [9].

We are now going to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We first assume that X is reflexive and define the operator V_n as in Lemma 2.3, that is,

$$V_n f(z) = \sum_{k=0}^{n+1} a_k z^k + \sum_{k=n+2}^{2n} \frac{2n+1-k}{n} a_k z^k \tag{3.5}$$

for any holomorphic function $f(z) = \sum_{k=0}^\infty a_k z^k$. From Lemma 2.3, we know that the operators $\{V_n\}$ are all weakly compact on $\mathcal{B}^\alpha(X)$ and uniform bounded. So it suffices to prove that the norm $\|T_g - T_g V_n\| \to 0$ as $n \to \infty$ under the conditions (1.8), (1.9), and (1.10), respectively, because the weakly compact operators form a closed operator ideal. For any $f \in \mathcal{B}^\alpha(X)$, we know $f - V_n f \in \mathcal{B}^\alpha(X)$ and

$$(1-|z|^2)^\beta \left\| \left[(T_g - T_g V_n) f\right]'(z) \right\|_X = (1-|z|^2)^\beta \left|g'(z)\right| \left\| f(z) - V_n f(z) \right\|_X =: A(z). \tag{3.6}$$

For $\alpha = 1$, if $\lim_{|z| \to 1^-}(1-|z|^2)^\beta \ln(2/(1-|z|^2))|g'(z)| = 0$, then for arbitrary $\varepsilon > 0$, there is $r \in (0,1)$ such that $(1-|z|^2)^\beta \ln(2/(1-|z|^2))|g'(z)| < \varepsilon$ for $|z| > r$. So

$$\begin{aligned} A(z) &= (1-|z|^2)^\beta \left|g'(z)\right| \left\| (I - V_n) f(z) \right\|_X \\ &= (1-|z|^2)^\beta \left(\ln \frac{2}{1-|z|^2} \right) \left|g'(z)\right| \frac{\left\| (I - V_n) f(z) \right\|_X}{\ln\left(2/(1-|z|^2)\right)} \\ &\lesssim \varepsilon \|f\|_{\mathcal{B}^\alpha(X)} \end{aligned} \tag{3.7}$$

for $|z| > r$ by Lemmas 2.1(2) and 2.3(1). And for $|z| \le r$,

$$A(z) = (1-|z|^2)^\beta \left|g'(z)\right| \left\| (I - V_n) f(z) \right\|_X \lesssim \left\| (I - V_n) f(z) \right\|_X \lesssim \varepsilon \|f\|_{\mathcal{B}(X)} \tag{3.8}$$

for large enough n by Lemma 2.3(2). Hence $\|T_g - T_g V_n\| < \varepsilon$ for $n$sufficiently large. This completes the sufficiency for the case $\alpha = 1$ since at this time we again have $T_g(\mathcal{B}_0(X)) \subset \mathcal{B}_0^\beta(X)$.

Similarly, for $\alpha > 1$, if $\lim_{|z| \to 1^-}(1-|z|^2)^{\beta-\alpha+1}|g'(z)| = 0$, then for arbitrary $\varepsilon > 0$, there is $r \in (0,1)$ such that $(1-|z|^2)^{\beta-\alpha+1}|g'(z)| < \varepsilon$ for $|z| > r$. So

$$A(z) = (1-|z|^2)^\beta \left|g'(z)\right| \left\| (I - V_n) f(z) \right\|_X \lesssim (1-|z|^2)^{\beta-\alpha+1} \left|g'(z)\right| \left\| (I - V_n) f \right\|_{\mathcal{B}^\alpha(X)} \lesssim \varepsilon \|f\|_{\mathcal{B}^\alpha(X)} \tag{3.9}$$

for $|z| > r$ by Lemmas 2.1(3) and 2.3(1). Hence $\|T_g - T_g V_n\| < \varepsilon$ for n sufficiently large by (3.8). This completes the sufficiency for the case $\alpha > 1$ since again $T_g(\mathcal{B}_0^\alpha(X)) \subset \mathcal{B}_0^\beta(X)$.

For $\alpha \in (0,1)$, the method above does not work. We complete the proof by the definition of weak compactness of T_g. Since g satisfies (1.8), it is obvious that $T_g : \mathcal{B}^\alpha(X) \to \mathcal{B}^\beta(X)$ (resp., $T_g : \mathcal{B}_0^\alpha(X) \to \mathcal{B}_0^\beta(X)$) is bounded. For any bounded sequence $\{f_n\} \subset \mathcal{B}^\alpha(X)$, we have

$$\|f_n(z)\|_X \lesssim \|f_n\|_{\mathcal{B}^\alpha(X)} \lesssim 1, \quad z \in \mathbb{D}, \tag{3.10}$$

by Lemma 2.1. From Montel's theorem, since X is reflexive, there are a subsequence $\{f_{n_k}\}$ and a holomorphic function $h : \mathbb{D} \to X$ such that $\{f_{n_k}\}$ converges uniformly to h on compact subsets of $\mathbb{D}$. It is clear that $h \in \mathcal{B}^{\alpha}(X)$. We now claim that $\{f_{n_k}\}$ actually converges uniformly to h on $\mathbb{D}$. In fact, for any $\varepsilon > 0$, there is $r \in (0,1)$ such that $(1-r)^{1-\alpha} < \varepsilon$. So for any $r < |z| < 1$,

$$\begin{aligned}\left\|(f_{n_k}(z)-h(z))-\left(f_{n_k}\left(\frac{r}{|z|}z\right)-h\left(\frac{r}{|z|}z\right)\right)\right\|_X &= \left\|\int_{r/|z|}^{1}(f'_{n_k}(tz)-h'(tz))z\,dt\right\|_X \\ &\lesssim \int_{r/|z|}^{1}\frac{|z|}{(1-|tz|^2)^{\alpha}}dt \\ &\lesssim (1-r)^{1-\alpha} < \varepsilon,\end{aligned} \tag{3.11}$$

then

$$\|f_{n_k}(z)-h(z)\|_X \lesssim \varepsilon + \left\|f_{n_k}\left(\frac{r}{|z|}z\right)-h\left(\frac{r}{|z|}z\right)\right\|_X. \tag{3.12}$$

Since $\{f_{n_k}\}$ converges uniformly to h on any compact subset of $\mathbb{D}$, so it follows from (3.12) that $\{f_{n_k}\}$ actually converges uniformly to h on $\mathbb{D}$, that is,

$$\sup_{z\in\mathbb{D}}\|f_{n_k}(z)-h(z)\|_X \longrightarrow 0 \quad (\text{as } k \longrightarrow \infty). \tag{3.13}$$

So for any $x^* \in X^*$ and $z \in \mathbb{D}$,

$$\left|x^* \circ \delta_z(f_{n_k}-h)\right| = \left|x^*(f_{n_k}(z)-h(z))\right| \longrightarrow 0 \quad (\text{as } k \longrightarrow \infty), \tag{3.14}$$

that is,

$$x^* \circ \delta_z(f_{n_k}-h) \longrightarrow 0 \quad (\text{as } k \longrightarrow \infty). \tag{3.15}$$

Since $\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}|g'(z)| < \infty$ by our hypothesis,

$$\begin{aligned}\left|x^* \circ \delta_z(T_g(f_{n_k}-h))\right| &= \left|x^* \circ \int_0^1 (f_{n_k}-h)(zt)g'(zt)z\,dt\right| \\ &\lesssim \int_0^1 \left|x^*(f_{n_k}-h)(zt)\right|\frac{1}{(1-|zt|^2)^{\beta}}dt \\ &\longrightarrow 0 \quad (\text{as } k \longrightarrow \infty),\end{aligned} \tag{3.16}$$

the last limit follows from (3.15) and the Lebesgue's dominated convergence theorem. Now, we claim that $\operatorname{Span}\{x^* \circ \delta_z : x^* \in X^*, z \in \mathbb{D}\}$ is ω^*-dense in $(\mathcal{B}^{\beta}(X))^*$. In fact, if $\operatorname{Span}\{x^* \circ \delta_z : x^* \in X^*, z \in \mathbb{D}\}$ is not ω^*-dense in $(\mathcal{B}^{\beta}(X))^*$, then by Hahn-Banach theorem [23] there are $f \in \mathcal{B}^{\beta}(X)$ and $\delta \in (\mathcal{B}^{\beta}(X))^*$ such that $x^* \circ \delta_z(f) = 0$, for all $x^* \in X^*$, $z \in \mathbb{D}$ and $\delta(f) \neq 0$. That is, $f \neq 0$ and $x^* \circ f(z) = 0$, for all $x^* \in X^*$, $z \in \mathbb{D}$, then applying Hahn-Banach theorem again we have $f(z) = 0$, for all $z \in \mathbb{D}$, so $f \equiv 0$. This contradiction proves our claim.

Therefore, $\{T_g f_{n_k}\}$ converges weakly to $T_g h$, so T_g is weakly compact from $\mathcal{B}^{\alpha}(X)$ to $\mathcal{B}^{\beta}(X)$. If, in addition, $\lim_{|z|\to 1^-}(1-|z|^2)^{\beta}|g'(z)| = 0$, then it is easy to see that $T_g(\mathcal{B}_0^{\alpha}(X)) \subset \mathcal{B}_0^{\beta}(X)$, so $T_g : \mathcal{B}_0^{\alpha}(X) \to \mathcal{B}_0^{\beta}(X)$ is also weakly compact.

Towards the converse direction, if $T_g : \mathcal{B}^{\alpha}(X) \to \mathcal{B}^{\beta}(X)$ is weakly compact, we have shown that X is reflexive by the remarks before the proof. Now fix $x_0 \in X$ with $\|x_0\|_X = 1$ and consider the closed subspace $\mathcal{M}_{\alpha} := \{x_0 f : f \in \mathcal{B}^{\alpha}\} \subset \mathcal{B}^{\alpha}(X)$. Clearly $T_g x_0 f(z) = \int_0^z x_0 f(\zeta)g'(\zeta)d\zeta = x_0\int_0^z f(\zeta)g'(\zeta)d\zeta = x_0\widetilde{T_g}f(z)$, $f \in \mathcal{B}^{\alpha}$, where we denote by $\widetilde{T_g}$ the corresponding scalar-valued Riemann-Stieltjes operator from $\mathcal{B}^{\alpha}$ to $\mathcal{B}^{\beta}$. So $T_g(\mathcal{M}_{\alpha}) \subset \mathcal{M}_{\beta}$ and $T_g : \mathcal{M}_{\alpha} \to \mathcal{M}_{\beta}$ is "isomorphic" to the scalar-valued operator $\widetilde{T_g} : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$. Then $\widetilde{T_g} : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$ is weakly compact.

For $\alpha = 1$, we know $f_a(z) = \ln(2/(1-\overline{a}z)) \in \mathcal{B}$ and $\|f_a\|_{\mathcal{B}} \le 3$ for any $a \in \mathbb{D}$. Then we have $\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta}\ln(2/(1-|z|^2))|g'(z)| < \infty$. In addition, if $f \in \mathcal{B}_0$, then $(1-|z|^2)^{\beta}|(\widetilde{T_g}f)'(z)| = (1-|z|^2)^{\beta}\ln(2/(1-|z|^2))|g'(z)|(|f(z)|/\ln(2/(1-|z|^2))) \lesssim |f(z)|/\ln(2/(1-|z|^2)) \to 0$ (as $|z| \to 1$), by Lemma 2.2. That is, $\widetilde{T_g}f \in \mathcal{B}_0^{\beta}$. Similarly, for $\alpha > 1$, we know $f_a(z) = (1-|a|^2)/(1-\overline{a}z)^{\alpha} \in \mathcal{B}^{\alpha}$ and $\|f_a\|_{\mathcal{B}^{\alpha}} \lesssim 1$ for any $a \in \mathbb{D}$, then $\sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta-\alpha+1}|g'(z)| \lesssim 1$. Hence for any $f \in \mathcal{B}_0^{\alpha}$, we have $(1-|z|^2)^{\beta}|(\widetilde{T_g}f)'(z)| = (1-|z|^2)^{\beta-\alpha+1}|g'(z)|(1-|z|^2)^{\alpha-1}|f(z)| \to 0$ (as $|z| \to 1$) by Lemma 2.2. That is, $\widetilde{T_g}(\mathcal{B}_0^{\alpha}) \subset \mathcal{B}_0^{\beta}$ for $\alpha \ge 1$. So $\widetilde{T_g} : \mathcal{B}_0^{\alpha} \to \mathcal{B}_0^{\beta}$ is weakly compact too, which is obvious for the case that $T_g : \mathcal{B}_0^{\alpha}(X) \to \mathcal{B}_0^{\beta}(X)$ is weakly compact. By Grantmacher's theorem [21],

$$\widetilde{T_g}(\mathcal{B}^{\alpha}) \subset \mathcal{B}_0^{\beta}. \tag{3.17}$$

For $\alpha = 1$, if (1.9) does not hold, then there exist some $\varepsilon_0 > 0$ and a sequence $\{z_n\} \subset \mathbb{D}$ such that $|z_n| \to 1$ as $n \to \infty$ and

$$(1-|z_n|^2)^{\beta}\ln\frac{2}{1-|z_n|^2}\left|g'(z_n)\right| \ge \varepsilon_0, \quad \text{for each } n \ge 1. \tag{3.18}$$

By the interpolation result in [24], there are a function $h \in \mathcal{B}$ and a subsequence $\{z_{n_k}\}$ of $\{z_n\}$, which is R-separated such that

$$h(z_{n_k}) = \ln\frac{2}{1-\left|z_{n_k}\right|^2}, \quad \text{for any } k \ge 1. \tag{3.19}$$

Since $\widetilde{T_g}h \in \mathcal{B}_0^{\beta}$ by (3.17), then

$$\lim_{|z|\to 1^-}(1-|z|^2)^{\beta}\left|h(z)\right|\left|g'(z)\right| = 0. \tag{3.20}$$

But by (3.18), we know

$$(1-|z_{n_k}|^2)^{\beta}|h(z_{n_k})||g'(z_{n_k})| = (1-|z_{n_k}|^2)^{\beta}\ln\frac{2}{1-|z_{n_k}|^2}|g'(z_{n_k})| \gtrsim \varepsilon_0, \quad \text{for each } k\geq 1, \tag{3.21}$$

which contradicts (3.20).

For $\alpha > 1$, if (1.10) does not hold, we can find $\varepsilon_0 > 0$ and a sequence $\{z_n\} \subset \mathbb{D}$ such that $|z_n| \to 1$ as $n \to \infty$ and

$$(1-|z_n|^2)^{\beta-\alpha+1}|g'(z_n)| \geq \varepsilon_0, \quad \text{for each } n \geq 1. \tag{3.22}$$

By [11, Lemma 3.1], there are functions $f_1, f_2 \in \mathcal{B}^{\alpha}$ such that

$$\inf_{z\in\mathbb{D}}(1-|z_1|)^{\alpha-1}(|f_1(z)|+|f_2(z)|) = c > 0. \tag{3.23}$$

It follows from (3.17) that $\widetilde{T_g}f_1, \widetilde{T_g}f_2 \in \mathcal{B}_0^{\beta}$, that is,

$$\lim_{|z|\to 1^-}(1-|z|^2)^{\beta}(|f_1(z)|+|f_2(z)|)|g'(z)| = 0. \tag{3.24}$$

But it follows from (3.23) and (3.22) that

$$\begin{aligned}
&(1-|z_n|^2)^{\beta}(|f_1(z_n)|+|f_2(z_n)|)|g'(z_n)| \\
&\quad = (1-|z_n|^2)^{\beta-\alpha+1}(1-|z_n|^2)^{\alpha-1}(|f_1(z_n)|+|f_2(z_n)|)|g'(z_n)| \\
&\quad \gtrsim (1-|z_n|^2)^{\beta-\alpha+1}|g'(z_n)| \geq \varepsilon_0, \quad \text{for each } n \geq 1,
\end{aligned} \tag{3.25}$$

which contradicts (3.24). So for $\alpha \geq 1$, the weak compactness of T_g must imply (1.9) and (1.10), respectively.

As $\alpha \in (0,1)$ and $T_g : \mathcal{B}^{\alpha}(X) \to \mathcal{B}^{\beta}(X)$ (resp., $T_g : \mathcal{B}_0^{\alpha}(X) \to \mathcal{B}_0^{\beta}(X)$) is weakly compact, we can easy check that $g \in \mathcal{B}^{\beta}$ (resp., $g \in \mathcal{B}_0^{\beta}$), since constant function $f(z) \equiv x_0 \in X$ belongs to $\mathcal{B}_0^{\alpha}(X)$. Then the proof is complete. □

Finally, we briefly consider the weakly conditional compactness of T_g. Given Banach spaces E and F, recall that a bounded linear operator $T : E \to F$ is weakly conditionally compact if any bounded sequence (f_n) in E has a subsequence (f_{n_k}) such that (Tf_{n_k}) is weakly Cauchy. Rosenthal's l^1-criterion [22] implies that T is not weakly conditionally compact if and only if T fixes a copy of l^1 in E. Hence if l^1 does not embed into E, then every bounded operator $T : E \to F$ is weakly conditionally compact, so from our Theorem 1.1, we know that the set of weakly conditionally compact operators is, in general, strictly larger than the set of weakly compact operators. The following theorem is a counterpart of our previous result.

Theorem 3.1. *Let $\alpha, \beta > 0$, X be a complex Banach space and $g : \mathbb{D} \to \mathbb{C}$ a nonconstant holomorphic function. Then $T_g : \mathcal{B}^\alpha(X) \to \mathcal{B}^\beta(X)$ (or $T_g : \mathcal{B}_0^\alpha(X) \to \mathcal{B}_0^\beta(X)$) is weakly conditionally compact if and only if X does not contain any copy of l^1 and the conditions* (1.8), (1.9), *and* (1.10) *hold, respectively, for* $0 < \alpha < 1$, $\alpha = 1$, $\alpha > 1$.

Proof. The sufficiency is established exactly as in the proof of Theorem 1.1. Since X does not contain a copy of l^1, V_n defined in Lemma 2.3 are, in fact, weakly conditionally compact, so is $T_g = \lim_{n \to \infty} T_g V_n$ for $\alpha \geq 1$ by the fact that weakly conditionally compact operators also form a closed operator ideal. For $\alpha \in (0, 1)$, the weakly conditional compactness of T_g follows easily from the definition by modifying the corresponding part of the proof of Theorem 1.1.

For the necessity, assume that T_g is weakly conditionally compact, a similar analysis as the remarks before the proof of Theorem 1.1 shows that for any bounded sequence (x_n) in X, there exists a subsequence (x_{n_k}) such that it is weakly Cauchy, so l^1 can not embed in X by Rosenthal's l^1-theorem. Similar to the proof of Theorem 1.1, T_g is "isomorphic" to the corresponding scalar-valued operator $\widetilde{T_g}$, then $\widetilde{T_g} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ (or $\widetilde{T_g} : \mathcal{B}_0^\alpha \to \mathcal{B}_0^\beta$) is weakly conditionally compact. Because every nonweakly compact operator on l^∞ acts isomorphically on a copy of l^∞, and hence on a copy of l^1 (refer to [22]). Notice that $\mathcal{B}^\alpha$ is isomorphic to l^∞ [19], so $\widetilde{T_g}$ is actually weakly compact. Then the desired conditions (1.8), (1.9), and (1.10) hold by the proof of Theorem 1.1. □

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