Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 87650, 8 pages
doi:10.1155/2007/87650

Research Article

Extinction and Decay Estimates of Solutions for a Class of Porous Medium Equations

Wenjun Liu, Mingxin Wang, and Bin Wu

Received 3 April 2007; Accepted 6 September 2007

Recommended by Michel Chipot

The extinction phenomenon of solutions for the homogeneous Dirichlet boundary value problem of the porous medium equation $u_t = \Delta u^m + \lambda|u|^{p-1}u - \beta u, 0 < m < 1$, is studied. Sufficient conditions about the extinction and decay estimates of solutions are obtained by using L^p-integral model estimate methods and two crucial lemmas on differential inequality.

1. Introduction and main results

This paper is devoted to the extinction and decay estimates for the porous medium equation

$$u_t = \Delta u^m + \lambda|u|^{p-1}u - \beta u, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \overline{\Omega}, \tag{1.3}$$

with $0 < m < 1$ and $p,\lambda,\beta > 0$, where $\Omega \subset R^N (N > 2)$ is a bounded domain with smooth boundary.

The phenomenon of extinction is an important property of solutions for many evolutionary equations which have been studied extensively by many researchers. Especially, there are also some papers concerning the extinction for the porous medium equation. For instance, in [1–3], the authors studied the extinction and large-time behavior of solution of (1.1) for the case $\beta = 0$ and $\lambda < 0$; and in [4], the authors obtained conditions for the extinction of solutions of (1.1) without absorption by using sub- and supersolution methods and an eigenfunction argument. But as far as we know, few works are concerned with the decay estimates of solutions for the porous medium equation.

The existence and uniqueness of nonnegative solution for problem (1.1)–(1.3) have been studied in [5, 6]. The purpose of the present paper is to establish sufficient conditions about the extinction and decay estimations of solutions for problem (1.1)–(1.3). For the proof of our result, we employ L^p-integral model estimate methods and two crucial lemmas on differential inequality.

Our main results read as follows.

Theorem 1.1. *Assume that* $0 \leq u_0(x) \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$, $0 < m = p < 1$, *and* λ_1 *is the first eigenvalue of*

$$-\Delta\psi(x) = \lambda\psi(x), \quad \psi|_{\partial\Omega} = 0, \tag{1.4}$$

and $\varphi_1(x) \geq 0$ *with* $\|\varphi_1\|_\infty = 1$ *is the eigenfunction corresponding to the eigenvalue* λ_1.

(1) *If* $\lambda < 4m\lambda_1/(m+1)^2$, *then the weak solution of problem (1.1)–(1.3) vanishes in the sense of* $\|\cdot\|_2$ *as* $t \to \infty$.

(2) *If* $(N-2)/(N+2) \leq m < 1$ *with* $\lambda < \lambda_1$ *or* $0 < m < (N-2)/(N+2)$ *with* $\lambda < \lambda^*$, *then the weak solution of problem (1.1)–(1.3) vanishes in finite time, and*

$$\begin{aligned} ||u(\cdot,t)||_{m+1} &\leq \left[\left(||u_0||_{m+1}^{1-m} + \frac{C_1}{\beta}\right)e^{(m-1)\beta t} - \frac{C_1}{\beta}\right]^{1/(1-m)}, \quad \frac{N-2}{N+2} \leq m < 1, \\ ||u(\cdot,t)||_{r+1} &\leq \left[\left(||u_0||_{r+1}^{1-m} + \frac{C_2}{\beta}\right)e^{(m-1)\beta t} - \frac{C_2}{\beta}\right]^{1/(1-m)}, \quad 0 < m < \frac{N-2}{N+2}, \end{aligned} \tag{1.5}$$

for $t \in [0, T^*)$, *where*

$$0 < T^* \leq \begin{cases} T_1, & \dfrac{(N-2)}{(N+2)} \leq m < 1, \\ T_2, & 0 < m < \dfrac{(N-2)}{(N+2)}, \end{cases} \tag{1.6}$$

$$\begin{aligned} r &= \frac{N(1-m)}{2} - 1, \\ \lambda^* &= \frac{(r+m)^2\lambda}{4rm} < \lambda_1, \end{aligned} \tag{1.7}$$

and C_1, C_2, T_1, *and* T_2 *are given by (2.18), (2.24), (2.20), and (2.26),respectively.*

Theorem 1.2. *Let $0 < m < 1$, $m < p$. Then the weak solution of problem (1.1)–(1.3) vanishes in finite time, and*

$$\begin{aligned}
&||u(\cdot,t)||_{m+1} \leq B_1 e^{-\alpha_1 t}, \quad t \in [0, T_{01}),\\
&||u(\cdot,t)||_{m+1} \leq \left[\left(||u(\cdot,T_{01})||_{m+1}^{1-m} + \frac{C_3}{\beta}\right)e^{(m-1)\beta(t-T_{01})} - \frac{C_3}{\beta}\right]^{1/(1-m)}, \quad t \in [T_{01}, T_3),\\
&||u(\cdot,t)||_{m+1} \equiv 0, \quad t \in [T_3, +\infty),
\end{aligned} \tag{1.8}$$

for $(N-2)/(N+2) \leq m < 1$,

$$\begin{aligned}
&||u(\cdot,t)||_{r+1} \leq B_2 e^{-\alpha_2 t}, \quad t \in [0, T_{02}),\\
&||u(\cdot,t)||_{r+1} \leq \left[\left(||u(\cdot,T_{02})||_{r+1}^{1-m} + \frac{C_4}{\beta}\right)e^{(m-1)\beta(t-T_{02})} - \frac{C_4}{\beta}\right]^{1/(1-m)}, \quad t \in [T_{02}, T_4),\\
&||u(\cdot,t)||_{r+1} \equiv 0, \quad t \in [T_4, +\infty),
\end{aligned} \tag{1.9}$$

for $0 < m < (N-2)/(N+2)$, where C_3, C_4, T_3, and T_4 are given by (2.29), (2.34), (2.31), and (2.36), respectively.

To obtain the above results, we will use the following lemmas which are of crucial importance in the proofs of decay estimates.

Lemma 1.3 [7]. *Let $y(t) \geq 0$ be a solution of the differential inequality*

$$\frac{dy}{dt} + Cy^k + \beta y \leq 0 \quad (t \geq 0), \quad y(T_0) \geq 0, \tag{1.10}$$

where $C > 0$ is a constant and $k \in (0,1)$. Then one has the decay estimate

$$\begin{gathered}
y(t) \leq \left[\left(y(T_0)^{1-k} + \frac{C}{\beta}\right)e^{(k-1)\beta(t-T_0)} - \frac{C}{\beta}\right]^{1/(1-k)}, \quad t \in [T_0, T_*),\\
y(t) \equiv 0, \quad t \in [T_*, +\infty),
\end{gathered} \tag{1.11}$$

where $T_* = (1/(1-k)\beta)\ln(1 + (\beta/C)y(T_0)^{1-k})$.

Lemma 1.4 [8]. *Let $0 < k < p$, and let $y(t) \geq 0$ be a solution of the differential inequality*

$$\frac{dy}{dt} + Cy^k + \beta y \leq \gamma y^p \quad (t \geq 0),\ y(0) \geq 0, \tag{1.12}$$

where $C, \gamma > 0$ and $k \in (0,1)$. Then there exist $\alpha > \beta$, $B > 0$, such that

$$0 \leq y(t) \leq Be^{-\alpha t}, \quad t \geq 0. \tag{1.13}$$

2. Proofs of theorems

In this section, we will give detailed proofs for our result. Let $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ denote $L^p(\Omega)$ and $W^{1,p}(\Omega)$ norms, respectively, $1 \le p \le \infty$.

2.1. Proof of Theorem 1.1. (1) First of all, we show that

$$||u(\cdot,t)||_\infty \le ||u_0(\cdot)||_\infty := M. \tag{2.1}$$

Multiplying (1.1) by $(u-M)_+$ and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (u-M)_+^2 dx + \int_{A_M(t)} \nabla u^m \cdot \nabla u dx \\
&\quad \le \lambda \int_\Omega u^m (u-M)_+ dx - \beta \int_\Omega u(u-M)_+ dx \le \lambda \int_{A_M(t)} u^{m+1} dx,
\end{aligned} \tag{2.2}$$

where $A_M(t) = \{x \in \Omega \mid u(x,t) > M\}$. Since λ_1 is the first eigenvalue, then we have

$$\int_\Omega \nabla u^m \cdot \nabla u dx \ge \frac{4m}{(m+1)^2}\lambda_1 \int_\Omega u^{m+1} dx, \tag{2.3}$$

for any $u \in W_0^{1,2}(\Omega)$. We further have

$$\int_{A_M(t)} \nabla u^m \cdot \nabla u dx \ge \frac{4m}{(m+1)^2}\lambda_1 \int_{A_M(t)} u^{m+1} dx. \tag{2.4}$$

Therefore, we have

$$\frac{d}{dt}\int_\Omega (u-M)_+^2 dx \le 0. \tag{2.5}$$

Since $\int_\Omega (u_0 - M)_+^2 dx = 0$, it follows that

$$\int_\Omega (u-M)_+^2 dx \equiv 0, \quad \forall t \ge 0, \tag{2.6}$$

which implies that $\|u(\cdot,t)\|_\infty \le \|u_0(\cdot)\|_\infty$.

Multiplying (1.1) by u and integrating over Ω, we conclude that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega \nabla u^m \cdot \nabla u dx \le \lambda \int_\Omega u^{m+1} dx - \beta \int_\Omega u^2 dx. \tag{2.7}$$

We further have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + (m\lambda_1 - \lambda)\int_\Omega u^{m+1} dx + \beta \int_\Omega u^2 dx \le 0. \tag{2.8}$$

Let $v = u/M$. Then, we have

$$\frac{d}{dt}\int_\Omega v^2 dx + 2M^{m-1}(m\lambda_1 - \lambda)\int_\Omega v^{m+1} dx + 2\beta M^{m-1}\int_\Omega v^2 dx \le 0. \tag{2.9}$$

Since $0 < m < 1$, we have

$$\frac{d}{dt}\int_\Omega v^2 dx + 2M^{m-1}(m\lambda_1 - \lambda + \beta)\int_\Omega v^2 dx \le 0, \tag{2.10}$$

which implies that

$$\int_\Omega v^2 dx \le e^{-2M^{m-1}\left(m\lambda_1-\lambda+\beta\right)t}\int_\Omega v_0^2 dx, \tag{2.11}$$

that is,

$$\int_\Omega u^2 dx \le e^{-2\|u_0\|_\infty^{m-1}\left(m\lambda_1-\lambda+\beta\right)t}\int_\Omega u_0^2 dx. \tag{2.12}$$

Therefore, we conclude that $\|u(\cdot,t)\|_2 \to 0$ as $t \to \infty$.

(2) We consider first the case $(N-2)/(N+2) \le m < 1$.

Multiplying (1.1) by u^m and integrating over Ω, we have [9]

$$\frac{1}{m+1}\frac{d}{dt}\|u\|_{m+1}^{m+1} + \left\|u^m\right\|_{1,2}^2 = \lambda\|u\|_{2m}^{2m} - \beta\|u\|_{m+1}^{m+1}. \tag{2.13}$$

Noticing that $\lambda_1 = \inf_{v\in W_0^{1,2}(\Omega),v\neq 0}\int_\Omega|\nabla v|^2 dx/\int_\Omega v^2 dx$, we obtain

$$\frac{1}{m+1}\frac{d}{dt}\|u\|_{m+1}^{m+1} + \left(1-\frac{\lambda}{\lambda_1}\right)\left\|u^m\right\|_{1,2}^2 + \beta\|u\|_{m+1}^{m+1} \le 0. \tag{2.14}$$

By the Hölder inequality, we have

$$\|u\|_{m+1}^{m+1} = \int_\Omega 1\cdot u^m \cdot u dx \le |\Omega|^{m/(m+1)-(N-2)/2N}\left\|u^m\right\|_{2N/(N-2)}\|u\|_{m+1}. \tag{2.15}$$

The embedding theorem gives that

$$\|u\|_{m+1}^{m} \le |\Omega|^{m/(m+1)-(N-2)/2N}\left\|u^m\right\|_{2N/(N-2)} \le C_0|\Omega|^{m/(m+1)-(N-2)/2N}\left\|u^m\right\|_{1,2}, \tag{2.16}$$

where C_0 is the embedding constant. By (2.14)–(2.16), we obtain the differential inequality

$$\frac{d}{dt}\|u\|_{m+1} + C_1\|u\|_{m+1}^m + \beta\|u\|_{m+1} \le 0, \tag{2.17}$$

where

$$C_1 = C_0^{-2}|\Omega|^{(N-2)/N-2m/(m+1)}\left(1-\frac{\lambda}{\lambda_1}\right). \tag{2.18}$$

Setting $y(t) = \|u(\cdot,t)\|_{m+1}$, $y(0) = \|u_0(\cdot)\|_{m+1}$, by Lemma 1.3, we obtain

$$\begin{gathered}\|u\|_{m+1} \le \left[\left(\left\|u_0\right\|_{m+1}^{1-m} + \frac{C_1}{\beta}\right)e^{(m-1)\beta t} - \frac{C_1}{\beta}\right]^{1/(1-m)}, \quad t\in[0,T_1), \\ \|u\|_{m+1} \equiv 0, \quad t\in[T_1,+\infty),\end{gathered} \tag{2.19}$$

where

$$T_1 = \frac{1}{(1-m)\beta}\ln\left(1+\frac{\beta}{C_1}||u_0||_{m+1}^{1-m}\right). \tag{2.20}$$

We now turn to the case $0 < m < (N-2)/(N+2)$ with $\lambda < \lambda^* = (r+m)^2\lambda/4rm < \lambda_1$. Multiplying (1.1) by u^r $(r = N(1-m)/2-1)$ and integrating over Ω, we have

$$\frac{1}{r+1}\frac{d}{dt}\|u\|_{r+1}^{r+1} + \left(\frac{4rm}{(r+m)^2} - \frac{\lambda}{\lambda_1}\right)||u^{(r+m)/2}||_{1,2}^2 + \beta\|u\|_{r+1}^{r+1} \le 0. \tag{2.21}$$

By the embedding theorem and the specific choice of r, we obtain

$$\|u\|_{r+1}^{(r+m)/2} = \left(\int_\Omega u^{((r+m)/2)\cdot(2N/(N-2))}dx\right)^{(N-2)/2N} \le C_0||u^{(r+m)/2}||_{1,2}. \tag{2.22}$$

Therefore,

$$\frac{d}{dt}\|u\|_{r+1} + C_2\|u\|_{r+1}^m + \beta\|u\|_{r+1} \le 0, \tag{2.23}$$

where

$$C_2 = C_0^{-2}\left(\frac{4rm}{(r+m)^2} - \frac{\lambda}{\lambda_1}\right) > 0. \tag{2.24}$$

Setting $y(t) = \|u(\cdot,t)\|_{r+1}$, $y(0) = \|u_0(\cdot)\|_{r+1}$, by Lemma 1.3, we obtain

$$\begin{gathered} \|u\|_{r+1} \le \left[\left(||u_0||_{r+1}^{1-m} + \frac{C_2}{\beta}\right)e^{(m-1)\beta t} - \frac{C_2}{\beta}\right]^{1/(1-m)}, \quad t \in [0,T_2), \\ \|u\|_{r+1} \equiv 0, \quad t \in [T_2,+\infty), \end{gathered} \tag{2.25}$$

where

$$T_2 = \frac{1}{(1-m)\beta}\ln\left(1+\frac{\beta}{C_2}||u_0||_{m+1}^{1-m}\right). \tag{2.26}$$

2.2. Proof of Theorem 1.2. We consider first the case $p \le 1$. When $(N-2)/(N+2) \le m < 1$, multiplying (1.1) by u^m, and by the embedding theorem and the Hölder inequality, we can easily obtain

$$\frac{d}{dt}\|u\|_{m+1} + C_0^{-2}\,|\,\Omega|^{(N-2)/N-2m/(m+1)}\|u\|_{m+1}^m + \beta\|u\|_{m+1} \le \lambda|\Omega|^{1-(m+p)/(m+1)}\|u\|_{m+1}^p. \tag{2.27}$$

By Lemma 1.4, there exist $\alpha_1 > \beta$, $B_1 > 0$, such that

$$0 \le \|u\|_{m+1} \le B_1 e^{-\alpha_1 t}, \quad t \ge 0. \tag{2.28}$$

Furthermore, there exist T_{01}, such that

$$\begin{aligned} & C_0^{-2}|\Omega|^{(N-2)/N-2m/(m+1)} - \lambda|\Omega|^{1-(m+p)/(m+1)}\|u\|_{m+1}^{p-m} \\ & \quad \geq C_0^{-2}|\Omega|^{(N-2)/N-2m/(m+1)} - \lambda|\Omega|^{1-(m+p)/(m+1)}\left(B_1 e^{-\alpha_1 T_{01}}\right)^{p-m} := C_3 > 0 \end{aligned} \tag{2.29}$$

holds for $t \in [T_{01}, +\infty)$. Therefore, (2.27) turns to

$$\frac{d}{dt}\|u\|_{m+1} + C_3\|u\|_{m+1}^{m} + \beta\|u\|_{m+1} \leq 0. \tag{2.30}$$

By Lemma 1.3, we can obtain the desire decay estimate for

$$T_3 = \frac{1}{(1-m)\beta}\ln\left(1 + \frac{\beta}{C_3}||u(\cdot, T_{01})||_{m+1}^{1-m}\right). \tag{2.31}$$

For the case $0 < m < (N-2)/(N+2)$, we multiply (1.1) by u^r and obtain

$$\frac{d}{dt}\|u\|_{r+1} + C_0^{-2}\frac{4rm}{(r+m)^2}\|u\|_{r+1}^{m} + \beta\|u\|_{r+1} \leq \lambda|\Omega|^{1-(r+p)/(r+1)}\|u\|_{r+1}^{p}. \tag{2.32}$$

By Lemma 1.4, there exist $\alpha_2 > \beta$, $B_2 > 0$, such that

$$0 \leq \|u\|_{r+1} \leq B_2 e^{-\alpha_2 t}, \quad t \geq 0. \tag{2.33}$$

Furthermore, there exist T_{02}, such that

$$\begin{aligned} & C_0^{-2}\frac{4rm}{(r+m)^2} - \lambda|\Omega|^{1-(r+p)/(r+1)}\|u\|_{r+1}^{p-m} \\ & \quad \geq C_0^{-2}\frac{4rm}{(r+m)^2} - \lambda|\Omega|^{1-(r+p)/(r+1)}\left(B_2 e^{-\alpha_2 T_{02}}\right)^{p-m} := C_4 > 0 \end{aligned} \tag{2.34}$$

holds for $t \in [T_{02}, +\infty)$. Therefore, (2.32) turns to

$$\frac{d}{dt}\|u\|_{r+1} + C_4\|u\|_{r+1}^{m} + \beta\|u\|_{r+1} \leq 0. \tag{2.35}$$

By Lemma 1.3, we can obtain the desire decay estimate for

$$T_4 = \frac{1}{(1-m)\beta}\ln\left(1 + \frac{\beta}{C_4}||u(\cdot, T_{02})||_{r+1}^{1-m}\right). \tag{2.36}$$

For the case $p > 1$, we can rewrite (2.27) and (2.32) as (e.g., (2.27))

$$\frac{d}{dt}\|u\|_{m+1} + C_0^{-2}|\Omega|^{(N-2)/N-2m/(m+1)}\|u\|_{m+1}^{m} + \beta\|u\|_{m+1} \leq \lambda k^{p-1}\|u\|_{m+1}^{m+1} \tag{2.37}$$

since $k\varphi_1^{1/m}(x)$ is a supersolution of problem (1.1)–(1.3), where $\varphi_1(x)$ is given in Theorem 1.1. The above argument can also be applied, and hence we omit it.

Acknowledgments

This work was supported by the National NSF of China (10471022), the NSF of Jiangsu Province (BK2006088), the NSF of Jiangsu Province Education Department (07KJD510133), and the Science Research Foundation of NUIST.

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Wenjun Liu: Department of Mathematics, Southeast University, Nanjing 210096, China;
College of Mathematics and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China
Email address: lwjboy@126.com

Mingxin Wang: Department of Mathematics, Southeast University, Nanjing 210096, China
Email address: mxwang@seu.edu.cn

Bin Wu: College of Mathematics and Physics, Nanjing University of Information Science and Technology, Nanjing 210044, China
Email address: wubing790831@126.com