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Research Article

Oscillatory Solutions for Second-Order Difference Equations in Hilbert Spaces

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We consider the difference equation $\Delta^2 x_n + f(n, x_{n+\tau}) = 0$, $\tau = 0, 1, \ldots$, in the context of a Hilbert space. In this setting, we propose a concept of oscillation with respect to a direction and give sufficient conditions so that all its solutions be directionally oscillatory, as well as conditions which guarantee the existence of directionally positive monotone increasing solutions.

1. Introduction

The study of difference equations has experienced a significant interest in the past years, as they arise naturally in the modelling of real-world phenomena (see, e.g., [1–3] and the references therein). The qualitative properties of solutions of both differential and difference equations have been extensively studied, and some of the results obtained in the scalar case, for instance, the asymptotic behaviour are easily extended to an abstract setting (see, e.g., [4–10]). In this paper, we extend the concept of oscillation to the vector case. Hence, in the context of a real Hilbert space, we introduce the notion of oscillation with respect to a direction, and show that some known results in the scalar case have their analogues in this more general context.

The following two difference equations often appear in the literature in the study of oscillation and asymptotic behaviour:

$$\Delta^2 x_{n-1} + f(n, x_n) = 0, \tag{1.1}$$

$$\Delta^2 x_n + f(n, x_n) = 0, \tag{1.2}$$

where $\Delta x_n = x_{n+1} - x_n$ is the forward difference operator and $f(n,\cdot)$ is a continuous function. For the first one, it is also usual to assume, when dealing with oscillation and nonoscillation, that $f(n,\cdot)$ is "sign-preserving," that is, $f(n,x)\cdot x \geq 0$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. The simplest result about the nonexistence of eventually positive solutions of (1.1) is obtained under the stronger assumption $f(n,x) \geq 2x$ for all $n \in \mathbb{N}$ and all $x \geq 0$ for obvious reasons: the term $f(n,x_n)$ neutralises the action of the term $-2x_n$. If, in addition, we wish to guarantee the nonexistence of eventually negative solution, we may impose the condition $f(n,x) \leq 2x$ for all $n \in \mathbb{N}$ and all $x \leq 0$. The two conditions can be written together as $f(n,x)/x \geq 2$ for all $n \in \mathbb{N}$ and all $x \neq 0$. Observe that this argument is not valid for (1.2), which is used by some authors to give results on the existence of solutions with a prescribed asymptotic behaviour. In this paper, we give a unified treatment of both equations by considering the following one:

$$\Delta^2 x_n + f(n, x_{n+\tau}) = 0, \quad \tau = 0,1,2,\dots. \tag{1.3}$$

Recall that a sequence of real numbers is called nonoscillatory if there exists a positive integer N such that $x_{n+1}x_n > 0$ for all $n \geq N$; otherwise it is called oscillatory. We say that (1.3) is oscillatory if all of its solutions are oscillatory. As we mention above, the condition $f(n,x)/x \geq 2$ for all $n \in \mathbb{N}$ and all $x \neq 0$ is sufficient to guarantee that (1.1) is oscillatory.

From now on, we will assume that X is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. The unit sphere of X is the set $S_X = \{x \in X : \|x\| = 1\}$, and for any nonempty subset $A \subset X$, its orthogonal complement is $A^\perp = \{y \in X : \langle y,a\rangle = 0\ \forall a \in A\}$. In this context, the above notions and conditions may be emulated by replacing the product in $\mathbb{R}$ by the product in X. Hence, a possible definition for a sequence $\{x_n\}$ to be nonoscillatory is $\langle x_n, x_{n+1}\rangle > 0$, and the hypothesis $f(n,x)/x \geq 2$ may be replaced by

$$\frac{\langle f(n,x), y\rangle}{\langle x,y\rangle} \geq a_n > 0, \quad \forall x,y \in X, \text{ with } \langle x,y\rangle \neq 0. \tag{1.4}$$

Unfortunately, this condition is extremely strong, since it implies that $f(n,x)$ is in the ray $\{ax : a \geq 0\}$. A more convenient version of oscillation and condition (1.4) is obtained using directional notions: if $u \in S_X$, we say that a point $x \in X$ is positive with respect to u if $\langle x,u\rangle > 0$ (x is negative if $-x$ is positive) and we say that a sequence $\{x_n\}$ in X is increasing with respect to u if $\langle x_{n+1} - x_n, u\rangle > 0$ for all $n \in \mathbb{N}$ ($\{x_n\}$ is decreasing if $\{-x_n\}$ is increasing). A sequence $\{x_n\}$ in X is oscillatory with respect to u if it is neither eventually positive nor eventually negative with respect to u; and we say that (1.3) is oscillatory with respect to u if all of its solutions are oscillatory with respect to u. Now, instead of condition (1.4), we will impose that $\langle f(n,x),u\rangle/\langle x,u\rangle \geq 0$ for all $x \in X \setminus \{u\}^\perp$, which can be interpreted as that $f(n,\cdot)$ preserves the sign with respect to u. When $X = \mathbb{R}$, this latter condition really implies that $f(n,\cdot)$ preserves signs, but in the vector case, we will need to add the extra hypothesis

$$\langle f(n,x), x\rangle \geq 0. \tag{H0}$$

Our final comment concerns oscillation of systems. Jiang and Li considered in [11] the system

$$\Delta x_n = a_n g(y_n), \qquad \Delta y_{n-1} = -f(n,x_n), \tag{1.5}$$

and studied the oscillation of its solutions in the following sense: a solution $(\{x_n\},\{y_n\})$ is oscillatory if both components are oscillatory. With our definitions, the system is interpreted as a vector equation ($X=\mathbb{R}^2$), and the oscillation in the sense of Jiang and Li is interpreted as the oscillation of that vector equation with respect to both directions $u_1=(1,0)$ and $u_2=(0,1)$.

2. The results

Our first theorem is devoted to the existence of asymptotically constant solutions which are positive and monotone increasing, hence nonoscillatory. It is a discrete counterpart of recent results by Dubé and Mingarelli [12], Ehrnström [13], and Wahlén [14]. The proof relies on the Schauder fixed point theorem applied to certain operator defined on a subset of $\ell_\infty(X)$, that is, the Banach space of all bounded sequences $x=\{x_n\}$ in X with the norm $\|x\|_\infty = \sup_n \|x_n\|$.

In this paper, we use the following terminology: by a compact operator we mean a continuous operator which maps bounded sets onto relatively compact sets, so that Schauder fixed point theorem asserts that any compact operator $T: C \to C$ defined on a nonempty, bounded, closed, and convex subset C of a Banach space has a fixed point in C.

At some moment, we will also use the following version of the Leray-Schauder fixed point theorem: if $B(0,R)$ denotes the closed ball of centre 0 and radius R in X, and if $T: B(0,R) \to X$ is compact and satisfies the so-called Leray-Schauder condition, that is, $Tx \neq \lambda x$ whenever $\|x\| = R$ and $\lambda > 1$, then T has a fixed point. (See, cf. [15].)

Theorem 2.1. *Consider the second-order difference equation (1.3), in the real Hilbert space X, together with the following assumptions:*

(H1) *for each positive integer n, the function $f(n,\cdot): X \to X$ is compact and satisfies (H0);*

(H2) *there exist $\mu > 0$ and $u \in S_X$, such that*

$$\sum_{k=0}^{\infty} k \sup_{0 \le \langle x,u\rangle \le \mu} \|f(k,x)\| < \infty, \tag{H2.1}$$

$$\frac{\langle f(n,x),u\rangle}{\langle x,u\rangle} \ge 0, \quad \forall n \in \mathbb{N} \text{ and all } x \in X \setminus \{u\}^{\perp}. \tag{H2.2}$$

Then, for each $M \in X$ with $\langle M,u\rangle = \mu$, there exists a solution $\{x_k\}$ to (1.3), with $x_k \to M$ as $k \to \infty$, which is eventually positive and nondecreasing with respect to u.

As an illustrative example for this theorem, we can consider the following difference equation in $\mathbb{R}^2$:

$$\Delta^2 x_n + \frac{\operatorname{sign}(x_n)\,|x_n|^{\gamma}}{n^3(1+y_n^2)} = 0, \qquad \Delta^2 y_n - \frac{x_n + y_n}{1+y_n^2} = 0, \tag{2.1}$$

where γ is any real number. In this case,

$$f(n,x,y) = \left(\frac{\operatorname{sign}(x)|x|^{\gamma}}{n^3(1+y^2)}, -\frac{x+y}{1+y^2}\right), \quad n \in \mathbb{N},\ (x,y) \in \mathbb{R}^2. \tag{2.2}$$

Observe that the first component of f keeps the sign of x, which means that (H2.2) is fulfilled for $u = (1,0)$. On the other hand, if $\mu > 0$ and $(x,y) \in \mathbb{R}^2$ is such that $0 \le \langle (x,y),(1,0)\rangle \le \mu$, that is, $0 \le x \le \mu$, then $\|f(n,x,y)\| \le (1+y^2)^{-1}(\mu^{\gamma}/n^3 + \mu + |y|) \le c/n^3$ for certain constant $c > 0$. This means that (H2.1) is also fulfilled. Therefore, our theorem asserts for any $(x_0,y_0) \in \mathbb{R}^2$ with $x_0 > 0$, there exists a solution (x_n,y_n) to the system (2.1) converging to (x_0,y_0), and for which the sequence of its first components, $\{x_n\}$, is eventually positive and nondecreasing.

Proof of Theorem 2.1. Fix $M \in X$ such that $\langle M,u\rangle = \mu$. We will reduce our problem to a fixed point problem for certain operator $T : C \to \ell_\infty(X)$, where C is a subset of $\ell_\infty(X)$. The set C is defined as $C = \{x = \{x_j\} \in \ell_\infty(X) : 0 \le \langle x_j,u\rangle \le \mu \text{ and } \|x_j\| \le \|M\| + A\}$, where

$$A = \sum_{k=0}^{\infty} k \sup_{0 \le \langle x,u\rangle \le \mu} ||f(k,x)||. \tag{2.3}$$

The operator $T = (T_0,T_1,\dots)$ is defined, for $x = \{x_j\} \in C$, as

$$T_n(x) = \begin{cases} M & \text{if } n \le n_0, \\ M - \sum_{j=n}^{\infty}(j-n+1)f(j,x_{j+\tau}) & \text{if } n > n_0, \end{cases} \tag{2.4}$$

where n_0 is a previously chosen positive integer with the following property

$$\sum_{n=n_0}^{\infty} n \sup_{0 \le \langle x,u\rangle \le \mu} \langle f(n,x),u\rangle < \mu. \tag{2.5}$$

Observe that by (H2.1), $T_n(x) \to M$ as $n \to \infty$, and that for $n > n_0$,

$$\Delta T_n(x) = \sum_{j=n}^{\infty} f(j,x_{j+\tau}), \qquad \Delta^2 T_n(x) = -f(n,x_{n+\tau}), \tag{2.6}$$

so that any fixed point of T, $x = \{x_n\}$, is a solution to (1.3) for $n > n_0$, with the desired limit M. Moreover, we obtain that, for $n > n_0$, $\langle x_n,u\rangle \ge 0$, and by (H2.2),

$$\Delta\langle x_n,u\rangle = \Delta\langle T_n(x),u\rangle = \sum_{j=n}^{\infty}\langle f(j,x_{j+\tau}),u\rangle \ge 0, \tag{2.7}$$

that is, x is nonnegative and nondecreasing with respect to u for $n > n_0$. With more precision, x is eventually positive with respect to u since, by (H2.1),

$$\langle x_n,u\rangle = \langle T_n(x),u\rangle \longrightarrow \langle M,u\rangle = \mu > 0. \tag{2.8}$$

Observe also that, although a fixed point of T, $x = \{x_n\}$, needs not to be a solution of (1.3), we can always obtain a solution of this equation with the same tail as x. We can replace the first n_0 terms of x by appropriate elements of X using a backward recursive process in order to obtain a solution to (1.3). In the first step of the method, we want to find x_{n_0} such that $\Delta^2 x_{n_0} = -f(n_0, x_{n_0+\tau})$, where x_{n_0+1} and x_{n_0+2} are known. If $\tau > 0$, then $x_{n_0+\tau}$ is also known and the solution is easy. However, if $\tau = 0$, the situation is different, $x_{n_0+\tau}$ is not known, and we need to solve an equation of the type $z + f(n_0, z) = b$, where the unknown is z. In other words, we need to be sure that the operator $g(z) = b - f(n_0, z)$ has a fixed point, and this is true because g is compact and satisfies the Leray-Schauder condition in any ball $B(0,R)$, with $R > \|b\|$: if $g(z) = \lambda z$ for some $\lambda > 1$ and $\|z\| = R$, then we would have

$$\lambda R^2 = \langle g(z), z\rangle = \langle b, z\rangle - \langle f(n_0, z), z\rangle \tag{2.9}$$

from which, using the Cauchy-Schwarz inequality,

$$\langle f(n_0, z), z\rangle \leq R(\|b\| - \lambda R) < 0, \tag{2.10}$$

which contradicts hypothesis (H0).

To end the proof, we will show that T has a fixed point in C. In the first place, C is nonempty, bounded, closed, convex, and invariant under T by (H2) and the properties of linearity and continuity of the inner product, so that, by the Schauder fixed point theorem, T will have a fixed point in C if it is a compact operator, that is, if it is continuous and sends bounded sets onto relatively compact sets.

First, we prove that T is continuous. Assuming that $x = \{x_n\} \in C$ and that $\varepsilon > 0$ is given, use (H2.1) to select $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} n \sup_{0 \leq \langle x, u\rangle \leq \mu} \|f(n,x)\| < \frac{\varepsilon}{4}. \tag{2.11}$$

Next, use that $f(j,\cdot)$ is continuous at $x_{j+\tau}$, to obtain $\delta > 0$ such that, for each $j = 0, 1, \ldots, N$, $\|f(j, x_{j+\tau}) - f(j,z)\| < \varepsilon/N(N+1)$ whenever $z \in X$ with $0 \leq \langle z, u\rangle \leq \mu$ and $\|z - x_{j+\tau}\| < \delta$. This supplies all the necessary ingredients in order to obtain that $\|T(x) - T(y)\|_\infty < \varepsilon$ whenever $y \in C$ satisfies that $\|y - x\|_\infty < \delta$. We let the reader fill in the details.

Finally, we proceed to prove that $T(C)$ is a relatively compact subset in $\ell_\infty(X)$. To do it, suppose that $\{x^n\}$ is a sequence in C and let us see that $\{T(x^n)\}$ has a convergent subsequence. We follow a diagonal process: if $x^n = \{x_j^n\}_{j\in\mathbb{N}}$, use that $f(0,\cdot)$ is compact to obtain a subsequence $\{x^{0,n}\}_{n\in\mathbb{N}}$ of $\{x^n\}_{n\in\mathbb{N}}$ such that $\{f(0, x_\tau^{0,n})\}_{n\in\mathbb{N}}$ converges.

Again, use that $f(1,\cdot)$ is compact to obtain a subsequence $\{x^{1,n}\}_{n\in\mathbb{N}}$ of $\{x^{0,n}\}_{n\in\mathbb{N}}$ such that $\{f(1, x_{1+\tau}^{1,n})\}_{n\in\mathbb{N}}$ converges, and observe that $\{f(0, x_\tau^{1,n})\}_{n\in\mathbb{N}}$ is also convergent. Arguing in this way, we find for each $k \in \mathbb{N}$ a subsequence $\{x^{k+1,n}\}_{n\in\mathbb{N}}$ of $\{x^{k,n}\}_{n\in\mathbb{N}}$ such that the sequences $\{f(0, x_\tau^{k+1,n})\}_{n\in\mathbb{N}}, \{f(1, x_{1+\tau}^{k+1,n})\}_{n\in\mathbb{N}}, \ldots, \{f(k+1, x_{k+1+\tau}^{k+1,n})\}_{n\in\mathbb{N}}$ are convergent.

Observe now that the subsequence $\{x^{n,n}\}$ of $\{x^n\}$ satisfies that $\{f(j, x_{j+\tau}^{n,n})\}_{n\in\mathbb{N}}$ is convergent for all $j \in \mathbb{N}$ and use it, together with (H2.1), to prove that $\{T(x^{n,n})\}$ is convergent, that is, that it is a Cauchy sequence in $\ell_\infty(X)$ (the details are left to the reader). □

The next theorem is an oscillation result for (1.3) based on the ideas outlined in the introduction.

Theorem 2.2. *Consider the second-order difference equation (1.3), in the real Hilbert space X, together with the following assumption:*

(H3) *there exist $u \in S_X$ and a sequence of positive real numbers $\{a_n\}$ such that*

$$\sum_{j=0}^{\infty} a_j = \infty, \tag{H3.1}$$

$$\frac{\langle f(n,x),u\rangle}{\langle x,u\rangle} \geq a_n, \quad \forall n \in \mathbb{N} \text{ and all } x \in X \setminus \{u\}^{\perp}; \tag{H3.2}$$

then (1.3) is oscillatory with respect to u.

Again, we give an example for this situation in $X = \mathbb{R}^2$,

$$\begin{aligned} \Delta^2 x_n + \frac{x_n(1+y_n^2)}{n} &= 0, \\ \Delta^2 y_n - y_n(1+x_n^2) &= 0. \end{aligned} \tag{2.12}$$

In this case, $f(n,x,y) = (x(1+y^2)/n, y(1+x^2))$. Observe that for $u = (1,0)$, condition (H3) is satisfied with $a_n = 1/n$; and for $v = (0,1)$, condition (H3) is satisfied with $a_n = 1$. Therefore, all the solutions to the system (2.12) are oscillatory with respect to $u = (1,0)$ and with respect to $v = (0,1)$, that is, all the solutions to (2.12) have oscillating components. We wonder whether all the solutions to this system are oscillatory with respect to all possible directions.

Proof of Theorem 2.2. We argue by contradiction: suppose that (1.3) is not oscillatory with respect to u. Then, there exists a solution $\{x_n\}$ of (1.3) which is eventually positive or eventually negative with respect to u. Since we follow a similar argument in both possibilities, we only consider the case of a solution $\{x_n\}$ which is eventually positive. This means that there exists a positive integer $N \in \mathbb{N}$ such that $\langle x_n,u\rangle > 0$ for all $n \geq N$. As a consequence, using that $\{x_n\}$ is a solution of (1.3), we obtain that $\{x_n\}$ is eventually nondecreasing with respect to u: suppose not and choose a positive integer $K \geq N$ such that $\Delta\langle x_K,u\rangle < 0$. Since $\langle x_{j+\tau},u\rangle \geq 0$ for $j \geq K$, the hypothesis (H3.2) implies that $\langle f(j,x_{j+\tau}),u\rangle \geq 0$ for $j \geq K$ and then

$$\Delta^2\langle x_j,u\rangle \leq 0, \quad \text{for } j \geq K. \tag{2.13}$$

If n is any positive integer with $n > K$, summing both sides of the above inequalities from $j = K$ to $j = n-1$, we obtain that $\Delta\langle x_n,u\rangle - \Delta\langle x_K,u\rangle \leq 0$, that is, we have that

$$\Delta\langle x_n,u\rangle \leq \Delta\langle x_K,u\rangle < 0, \quad \text{for } n > K. \tag{2.14}$$

This implies that $\langle x_n,u\rangle \to -\infty$, which enters in contradiction with the fact that $\langle x_n,u\rangle > 0$ for $n \geq N$. Hence, we may assume that

$$\langle x_j,u\rangle > 0, \quad \Delta\langle x_j,u\rangle \geq 0, \quad \text{for } j \geq N. \tag{2.15}$$

Next, for any $k > N$, consider the relations

$$\Delta^2 x_j + f(j, x_{j+\tau}) = 0, \quad j = N, \ldots, k, \tag{2.16}$$

and sum in both sides to obtain the following relation:

$$\Delta x_{k+1} - \Delta x_N + \sum_{j=N}^{k} f(j, x_{j+\tau}) = 0. \tag{2.17}$$

Again, summing both sides of the above from $k = N$ to $k = n$, obtain that

$$x_{n+2} - x_{N+1} - (n+1-N)\Delta x_N + \sum_{j=N}^{n} (n+1-j) f(j, x_{j+\tau}) = 0, \tag{2.18}$$

and then, using the properties of $\langle \cdot, \cdot \rangle$,

$$\begin{aligned} &\langle x_{n+2} - x_{N+1}, u \rangle - (n+1-N)\langle x_{N+1}, u \rangle + (n+1-N)\langle x_N, u \rangle \\ &\quad + \sum_{j=N}^{n} (n+1-j)\langle f(j, x_{j+\tau}), u \rangle = 0. \end{aligned} \tag{2.19}$$

Now, use (2.15) and (H3.2) to obtain that $\langle x_{n+2} - x_{N+1}, u \rangle \geq 0$, $\langle x_N, u \rangle > 0$, and also that

$$\langle f(j, x_{j+\tau}), u \rangle \geq a_j \langle x_{j+\tau}, u \rangle, \quad j \geq N. \tag{2.20}$$

Then, combine these inequalities with (2.19) to obtain

$$-(n+1-N)\langle x_{N+1}, u \rangle + \sum_{j=N+1}^{n} (n+1-j) a_j \langle x_{j+\tau}, u \rangle < 0. \tag{2.21}$$

Since the sequence $\{\langle x_j, u \rangle\}_{j \geq N}$ is nondecreasing and $\tau \geq 0$, we can continue the above relation with a chain of inequalities to obtain this other one,

$$-(n+1-N)\langle x_{N+1}, u \rangle + \sum_{j=N+1}^{n} (n+1-j) a_j \langle x_{N+1}, u \rangle < 0, \tag{2.22}$$

by which, after cancelling out the term $\langle x_{N+1}, u \rangle$, we get

$$\sum_{j=N+1}^{n} (n+1-j) a_j < n+1-N. \tag{2.23}$$

We obtain from this that

$$1 > \sum_{j=N+1}^{n} \frac{n+1-j}{n+1-N} a_j \geq \frac{1}{2}\left(a_{N+1} + a_{N+2} + \cdots + a_{r(n)}\right), \tag{2.24}$$

where $r(n) = \mathrm{E}[(n+1+N)/2]$, the biggest integer smaller than or equal to $(n+1+N)/2$. Since $r(n) \to \infty$, this contradicts (H3.1), and the proof is completed. $\square$

Remark 2.3. Grace and El-Morshedy considered in [16] the following second-order difference equation on the real line:

$$\Delta^2 x_{n-1} + a_n f(x_n) = 0, \quad n = 1,2,\ldots, \tag{2.25}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies that $xf(x) > 0$ for $x \neq 0$. Using the Riccati technique, they were able to prove that this last equation (i.e., (1.1) with $f(n,x) = a_n f(x)$) is oscillatory under the following additional assumptions: the function g defined by $f(x) - f(y) = g(x,y)(x-y)$ for $x, y \neq 0$ satisfies that

$$g(x,y) \geq \lambda > 0, \quad \text{for } x, y \neq 0, \tag{GM1}$$

$$\liminf_{n\to\infty} \sum_{i=N}^{n} a_i \geq \phi_N, \quad \text{for large } N \text{ with } \sum^{\infty} \frac{(\phi_i^+)^2}{1+\lambda\phi_i^+} = \infty, \tag{GM2}$$

where $\phi_n^+ = \max\{\phi_n, 0\}$.

The hypothesis (H3) for $X = \mathbb{R}$ becomes

$$\sum_{j=0}^{\infty} a_j = \infty, \qquad \frac{f(n,x)}{x} \geq a_n, \quad n \in \mathbb{N},\ x \neq 0, \tag{2.26}$$

where a_n is positive. Observe that (GM2) is a weaker assumption than ours on $\{a_n\}$, because (GM2) allows changing of sign. On the other hand, for $a_n \geq 0$, our hypotheses are more general than those in [16] since (GM1) implies that f is strictly increasing, while (H3) does not.

We wonder whether the mentioned result by Grace and El-Morshedy may be adapted to the context of Hilbert spaces under the assumption of f being strongly monotone, that is, satisfying $\langle f(x) - f(y), x - y\rangle \geq a\|x-y\|^2$, perhaps in a directional sense.

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