Quantum states, groups and monotone metric tensors

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Abstract A novel link between monotone metric tensors and actions of suitable extensions of the unitary group on the manifold of faithful quantum states is presented here by means of three illustrative examples related with the Bures–Helstrom metric tensor, the Wigner–Yanase metric tensor, and the Bogoliubov–Kubo–Mori metric tensor.

1 Introduction

In this work, a preliminary analysis of the relation between monotone metric tensors on the manifold of faithful quantum states and group actions of suitable extensions of the unitary group is presented. In the context of Quantum Information Geometry of finite-dimensional systems with Hilbert space $\mathcal{H}$, the unitary group $\mathcal{U}(\mathcal{H})$ plays the role of universal symmetry group for the class of the so-called *monotone metric tensors* on the space $\mathscr{S}_+$ of faithful states (invertible density operators $\mathcal{H}$) providing the quantum counterpart of the classical Fisher–Rao metric tensor [35,40]. Specifically, the unitary group $\mathcal{U}(\mathcal{H})$ acts on quantum states (and more generally on the whole space $\mathscr{V}$ of Hermitian operators on $\mathcal{H}$) according to the standard action

$$\phi(\mathbf{U},\, \rho) \,\equiv\, \phi_{\mathbf{U}}(\rho) \,=\, \mathbf{U}\, \rho\, \mathbf{U}^{\dagger}, \tag{1}$$

where ρ is a quantum state and $\mathbf{U} \in \mathcal{U}(\mathcal{H})$, and $\phi_{\mathbf{U}}$ represents an isometry of every monotone metric tensor G because of the requirement of monotonicity under completely positive, trace-preserving maps representing the quantum version of classical coarse graining [35,40]. From the infinitesimal point of view, the action ϕ is described in terms of the fundamental vector fields on $\mathscr{S}_+$ providing an anti-representation of the Lie algebra $\mathfrak{u}(\mathcal{H})$ of the unitary group. These vector fields, denoted by $\mathbb{X}_{\mathbf{b}}$ with $\mathbf{b}$ an Hermitian operator on $\mathcal{H}$ (more on this in Sect. 2), are Killing vector fields for *all* the monotone metric tensors because $\mathcal{U}(\mathcal{H})$ acts by means of isometries.

Now, the Lie algebra $\mathfrak{u}(\mathcal{H})$ is a Lie-subalgebra of the space $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ endowed with the Lie product given by the commutator $[\cdot, \cdot]$ between linear operators. In particular, it turns out that $\mathcal{B}(\mathcal{H})$ (endowed with $[\cdot, \cdot]$) is isomorphic to the Lie algebra of the complexification of $\mathcal{U}(\mathcal{H})$, namely to the Lie algebra of the Lie group $\mathcal{GL}(\mathcal{H})$ consisting of invertible linear operators on $\mathcal{H}$. Furthermore, it is known [9,15,26,27] that $\mathcal{GL}(\mathcal{H})$ acts on the manifold $\mathscr{S}_+$, and more generally on the whole space of quantum states $\mathscr{S}$, according to

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$$\alpha(\mathrm{g}, \rho) \,=\, \frac{\mathrm{g}\,\rho\,\mathrm{g}^\dagger}{Tr(\mathrm{g}\rho\mathrm{g}^\dagger)}, \tag{2}$$

where $\mathrm{g} \in \mathcal{GL}(\mathcal{H})$. Clearly, if $\mathrm{g} \in \mathcal{U}(\mathcal{H}) \subset \mathcal{GL}(\mathcal{H})$, we obtain again the standard action ϕ of $\mathcal{U}(\mathcal{H})$ introduced above. The fundamental vector fields of α provide an anti-representation of the Lie algebra of $\mathcal{GL}(\mathcal{H})$ that "contains" the fundamental vector fields $\mathbb{X}_\mathbf{b}$ of the action ϕ of $\mathcal{U}(\mathcal{H})$ together with some *complementary* vector fields, written as $\mathbb{Y}_\mathbf{a}$ with $\mathbf{a}$ an Hermitian operator on $\mathcal{H}$ (more on this in Sect. 2), which do not close a Lie algebra on their own.

These complementary vector fields were shown to be essential ingredients in the geometric formulation of the Gorini–Kossakowski–Lindblad–Sudarshan equation governing the dynamical evolution of open quantum systems [10,13], while their role in the context of Quantum Information geometry is not completely clear. As will be noted in Sect. 2, the $\mathbb{Y}_\mathbf{a}$ are essentially the Symmetrized Logarithmic Derivative introduced by Helstrom [30–32]. Moreover, it will also be shown that the $\mathbb{Y}_\mathbf{a}$ are the gradient vector fields associated with the expectation value functions of quantum observables (Hermitian operators on $\mathcal{H}$) given by

$$f_\mathbf{a}(\rho) \,=\, Tr(\rho\,\mathbf{a}) \tag{3}$$

by means of the monotone metric tensor G_{BH} known as the Bures–Helstrom metric tensor [7,21,22,30,47].

Reading this instance backward, we can say that the gradient vector fields associated with the $f_\mathbf{a}$ by means of the Bures–Helstrom metric tensor, together with the $\mathbb{X}_\mathbf{b}$, provide a representation of a Lie algebra "enlargement" of $\mathfrak{u}(\mathcal{H})$, namely the Lie algebra of the complexification $\mathcal{GL}(\mathcal{H})$ of $\mathcal{U}(\mathcal{H})$, and this representation integrates to the group action α of $\mathcal{GL}(\mathcal{H})$ described above.

Since the Bures–Helstrom metric tensor G_{BH} is only one of the (infinitely many) possible quantum counterparts of the Fisher–Rao metric tensor, it is reasonable to ask if G_{BH} is the only metric tensor among the monotone ones for which such an instance is realized. Specifically, given a monotone metric tensor G (Quantum Fisher–Rao metric tensor), the question is if the gradient vector fields that it is possible to associated with the expectation value functions, together with the universal Killing vector fields $\mathbb{X}_\mathbf{b}$, provide a representation of a Lie algebra integrating to a group action. In the following sections, we will answer this question in the affirmative by providing explicit examples.

It will be proved in Sect. 3 that the $\mathbb{X}_\mathbf{b}$ together with the gradient vector fields, say $\mathbb{W}_\mathbf{a}$, associated with the functions $f_\mathbf{a}$ by means of the monotone metric tensor G_{WY} known as the Wigner–Yanase metric tensor [24,25,28,29] provide an anti-representation of the Lie algebra of $\mathcal{GL}(\mathcal{H})$ integrating to the action Θ of $\mathcal{GL}(\mathcal{H})$ given by

$$\Theta(\mathrm{g},\,\rho) \,=\, \frac{\left(\mathrm{g}\,\sqrt{\rho}\,\mathrm{g}^\dagger\right)^2}{Tr\left(\left(\mathrm{g}\,\sqrt{\rho}\,\mathrm{g}^\dagger\right)^2\right)}. \tag{4}$$

This action is clearly different from α, but reduces to ϕ if unitary elements are considered. The action Θ can be extended to the whole space of states $\mathscr{S}$, and it is easy to see that the orbits of this action are precisely the manifolds of quantum states with fixed rank, in analogy with what happens for the action α. In particular, Θ preserves the manifold $\mathscr{S}_1 \cong \mathbb{CP}(\mathcal{H})$ of pure (rank-1) quantum states, and, since every pure state ρ_p satisfies $\rho_p^2 = \rho_p$, it is easy to check that

$$\Theta(\mathrm{g}, \rho_p) \,=\, \alpha(\mathrm{g}, \rho_p) \tag{5}$$

for every $\mathrm{g} \in \mathcal{GL}(\mathcal{H})$, and every pure state ρ_p, while, in general, α is different from Θ on mixed states.

The fact that both actions α and Θ can be extended to (and agree on) the space $\mathscr{S}_1 \cong \mathbb{CP}(\mathcal{H})$ of pure quantum states is particularly interesting when it is noted that both the Bures–Helstrom metric tensor and the Wigner–Yanase metric tensor belong to the particular subset of monotone metric tensors admitting what is known as a "radial limit" to the space of pure quantum states [41] which is (proportional to) the metric tensor on $\mathscr{S}_1 \cong \mathbb{CP}(\mathcal{H})$ given by the Fubini–Study metric tensor. This may suggest that the "radial limit" of the monotone metric has something to do with the extension of the associated group action to the pure quantum states (if the group action exists, of course).

In support to this intuition, it will be shown in Sect. 4 that there is at least one monotone metric tensor which does not admit a "radial limit" and whose associated group action does not extend to the space of pure states. Curiously, this monotone metric tensor is associated with the action of an enlargement of $\mathcal{U}(\mathcal{H})$ different from $\mathcal{GL}(\mathcal{H})$, and connected with a structure of Lie algebra on $\mathcal{B}(\mathcal{H})$ which is different from the standard one given by the commutator. Specifically, the space $\mathcal{B}(\mathcal{H})$ may be endowed with a structure of Lie algebra which is isomorphic to the Lie algebra of the cotangent Lie group $T^*\mathcal{U}(\mathcal{H}) \cong \mathcal{U}(\mathcal{H}) \rtimes_{Ad^*} \mathscr{V}$ of the unitary group [2,3]. Here, $\mathscr{V}$ denote the space of Hermitian operators on $\mathcal{H}$, and Ad^* is the co-adjoint action of $\mathcal{U}(\mathcal{H})$ on $\mathscr{V}$ when the latter is identified with the dual of the Lie algebra $\mathfrak{u}(\mathcal{H})$ (more on this in Sect. 4). Up to now, and at the best of the author's knowledge, the role of $T^*\mathcal{U}(\mathcal{H})$ in Quantum Information Geometry is yet to be properly understood, and, as a preliminary step in this direction, in Sect. 4, it is proved that $T^*\mathcal{U}(\mathcal{H})$ acts on $\mathscr{S}_+$ by means of the action $\varXi$ given by

$$\varXi((\mathbf{U}, \mathbf{a}), \rho) \,=\, \frac{e^{\mathbf{U} \ln(\rho)\mathbf{U}^\dagger + \mathbf{a}}}{Tr(e^{\mathbf{U} \ln(\rho)\mathbf{U}^\dagger + \mathbf{a}})}. \tag{6}$$

This action reduces to the standard action ϕ when restricted to the unitary group $\mathcal{U}(\mathcal{H})$. Moreover, it will be proved that the $\mathbb{X}_\mathbf{b}$ together with the gradient vector fields associated with the $f_\mathbf{a}$ by means of the monotone metric tensor G_{BKM} known as the Bogoliubov–Kubo–Mori metric tensor [23,36,37,39,42], and denoted by $\mathbb{Z}_\mathbf{a}$, are the fundamental vector fields for the action $\varXi$. The explicit expression of $\varXi$ given above cannot be extended outside the space $\mathscr{S}_+$ of faithful quantum states because the logarithm of a non-faithful (non-invertible) quantum state is not defined. One may argue that this is not enough to conclude that $\varXi$ does not extend to the space of pure states because the expression of $\varXi$ given above may suitably change when a non-faithful state is considered. However, since the tangent vector at ρ associated with the vector field $\mathbb{Z}_\mathbf{a}$ is given by

$$\mathbb{Z}_\mathbf{a}(\rho) \,=\, \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{\ln(\rho)+t\mathbf{a}}}{Tr\left(e^{\ln(\rho)+t\mathbf{a}}\right)}\right)_{t=0} = \int_0^1 \mathrm{d}\lambda \,\left(\rho^\lambda \, \mathbf{a} \, \rho^{1-\lambda}\right) - Tr(\rho \, \mathbf{a}) \, \rho\,, \tag{7}$$

if we take ρ to be a pure state, then $\rho = |\psi\rangle\langle\psi|$ for some normalized vector in $\mathcal{H}$, $\rho^\lambda = \rho^{1-\lambda}$, and it is easy to check that $\mathbb{Z}_\mathbf{a}(\rho) = \mathbf{0}$. This means that the infinitesimal version of the action $\varXi$ does not extend properly to the space of pure quantum states.

In Sect. 5, a brief discussion elucidates the future developments that may be followed starting from the results presented here. In particular, it will be argued that the unitary invariance satisfied by every monotone metric tensor G is powerful enough to impose a particular constraint on the possible Lie algebra enlargement of $\mathfrak{u}(\mathcal{H})$ which could be linked to G in the way described above.

Finally, a note on the methodology followed in the rest of the work. For the sake of simplicity, the case of not-normalized quantum states will be considered first, and the case of normalized states will be discussed after. This will reduce to the minimum the complexity

of the computations involved because the normalized case always requires the introduction of a suitable denominator factor that brings in some additional computational complexity without, however, changing the conceptual scheme of things.

2 Bures–Helstrom metric tensor

Let $\mathscr{P}_+ \subset \mathcal{B}(\mathcal{H})$ be the set of strictly positive operators on $\mathcal{H}$. Elements in $\mathscr{P}_+$ can be interpreted as not-normalized quantum states, and they form an homogeneous space of the general linear group $\mathcal{GL}(\mathcal{H})$ with respect to the action [15,26,27]

$$\widetilde{\alpha}(g, \rho) := g \rho g^\dagger. \tag{8}$$

The Lie algebra of $\mathcal{GL}(\mathcal{H})$ is the whole algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$ endowed with the canonical commutator operator. Since every element in $\mathcal{B}(\mathcal{H})$ can be uniquely written as the sum of an Hermitian element $\mathbf{a}$ and a skew-Hermitian element $i\mathbf{b}$, where $\mathbf{b}$ is self-adjoint, the fundamental vector fields of α are indexed by couples $(\mathbf{a}, \mathbf{b})$. These vector fields are written as $\widetilde{\Gamma}_{\mathbf{ab}}$. Considering the smooth curve

$$g_t = e^{\frac{t}{2}(\mathbf{a}+i\mathbf{b})}, \tag{9}$$

the tangent vector $\widetilde{\Gamma}_{\mathbf{ab}}(\rho)$ at ρ can easily be computed to be

$$\widetilde{\Gamma}_{\mathbf{ab}}(\rho) = \frac{\mathrm{d}}{\mathrm{d}t} (\widetilde{\alpha}(g_t, \rho))_{t=0} = \frac{1}{2} (\rho\,\mathbf{a} + \mathbf{a}\,\rho) + \frac{1}{2i} (\rho\,\mathbf{b} - \mathbf{b}\,\rho) . \tag{10}$$

We may decompose $\Gamma_{\mathbf{ab}}$ according to

$$\Gamma_{\mathbf{ab}} = \mathcal{Y}_{\mathbf{a}} + \mathbb{X}_{\mathbf{b}}, \tag{11}$$

where the vector fields $\mathcal{Y}_{\mathbf{a}}$ and $\mathbb{X}_{\mathbf{b}}$ are such that

$$\begin{aligned}
\mathcal{Y}_{\mathbf{a}}(\rho) &:= \frac{1}{2} (\rho\,\mathbf{a} + \mathbf{a}\,\rho) \equiv \{\rho,\, \mathbf{a}\} \\
\mathbb{X}_{\mathbf{b}}(\rho) &:= \frac{1}{2i} (\rho\,\mathbf{b} - \mathbf{b}\,\rho) \equiv [[\rho,\, \mathbf{b}]] .
\end{aligned} \tag{12}$$

The vector fields $\mathbb{X}_{\mathbf{b}}$ provide an anti-representation of the Lie algebra $\mathfrak{u}(\mathcal{H})$ of the unitary group $\mathcal{U}(\mathcal{H})$ [15,26,27] integrating to the action ϕ of $\mathcal{U}(\mathcal{H})$ mentioned in the introduction and given by Eq. (1). On the other hand, the vector fields $\mathcal{Y}_{\mathbf{a}}$ do not close a Lie algebra, and may be thought of as complementary vector fields that are needed in order to enlarge the anti-representation of $\mathfrak{u}(\mathcal{H})$ to an anti-representation of the Lie algebra of the complexification $\mathcal{GL}(\mathcal{H})$ of $\mathcal{U}(\mathcal{H})$. Moreover, it is worth noting that the tangent vector $\mathcal{Y}_{\mathbf{a}}(\rho)$ provides a geometrical version of the Symmetric Logarithmic Derivative at ρ widely used in quantum estimation theory and metrology [30–32,38,44,46].

It will now be proved that the vector field $\mathcal{Y}_{\mathbf{a}}$ is the gradient vector field associated with the function $f_{\mathbf{a}}$ given by

$$f_{\mathbf{a}}(\rho) = \mathrm{Tr}(\rho\,\mathbf{a}) \tag{13}$$

by means of the Bures–Helstrom metric tensor G_{BH} on $\mathscr{P}_+$. The Bures–Helstrom metric tensor G_{BH} [7,21,22,30,47] may be extracted from the Bures distance

$$d_B^2(\rho, \sigma) = 2\left(Tr\rho + Tr\sigma - 2Tr\left(\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}\right)\right) \tag{14}$$

according to the general procedure used in (Classical and Quantum) Information Geometry and given by

$$(G_{BH}(X, Y))(\rho) = -\frac{d}{dt}\frac{d}{ds}\left(d_B^2\left(\gamma_\rho^X(t), \gamma_\rho^Y(s)\right)\right)_{t,s=0}, \tag{15}$$

where X and Y are arbitrary vector fields, and $\gamma_\rho^X(t)$ and $\gamma_\rho^Y(s)$ their integral curves starting at ρ.

Remark 1 Note that, in the literature, the Bures–Helstrom metric tensor G_{BH} is also called "Quantum Fisher Metric" [8,33,43,50,51], while the Bures metric tensor G_B is usually defined to be the metric tensor associated with half the distance function in Eq. (14), and it holds $G_{BH} = 4G_B$. Moreover, what will be proved below for the Bures–Helstrom metric tensor G_{BH} holds also for the Bures metric tensor G_B provided we replace d_B^2 with $\frac{1}{2}d_B^2$, and **a** with 2**a** in all the expressions. The choice made here of using G_{BH} instead of G_B is essentially due to the fact that G_{BH} may be thought of as the "natural" counterpart of the Fisher–Rao metric tensor in the context of (finite-dimensional) von Neumann algebras as explained in [20].

To actually prove that $\mathcal{Y}_\mathbf{a}$ is the gradient vector field associated with $f_\mathbf{a}$ by means of G_{BH}, we have to prove that

$$G_{BH}(X, \mathcal{Y}_\mathbf{a}) = \mathcal{L}_X f_\mathbf{a}, \tag{16}$$

where X is an arbitrary vector field and $\mathcal{L}_X$ denote the Lie derivative with respect to X. We start computing

$$(G_{BH}(X, \mathcal{Y}_\mathbf{a}))(\rho) = 4\frac{d}{dt}\frac{d}{ds}Tr\left(\sqrt{\sqrt{\gamma_\rho^X(t)}\gamma_\rho^{\mathcal{Y}_\mathbf{a}}(s)\sqrt{\gamma_\rho^X(t)}}\right)_{s,t=0}. \tag{17}$$

To perform the computation, we define the strictly positive operator $\mathbf{C}_{s,t}$, depending parametrically on s and t, given by

$$\mathbf{C}_{s,t} := \sqrt{\gamma_\rho^X(t)}\gamma_\rho^{\mathcal{Y}_\mathbf{a}}(s)\sqrt{\gamma_\rho^X(t)}. \tag{18}$$

Then, we perform a series expansion for $\sqrt{\mathbf{C}_{s,t}}$ around the identity matrix $\mathbb{I}$ obtaining

$$\sqrt{\mathbf{C}_{s,t}} = \sum_{k=0}^{\infty} c_k\left(\mathbf{C}_{s,t} - \mathbb{I}\right)^k$$

$$= \sum_{k=0}^{\infty} c_k\left(\sqrt{\gamma_\rho^X(t)}\gamma_\rho^{\mathcal{Y}_\mathbf{a}}(s)\sqrt{\gamma_\rho^X(t)} - \mathbb{I}\right)^k. \tag{19}$$

Using the Leibniz rule and the cyclicity of the trace, the expression for the series expansion of the derivative of a function, and the Leibniz rule again, we obtain

$$(G_{BH}(X, \mathcal{Y}_\mathbf{a}))(\rho) = 4\frac{d}{dt}\frac{d}{ds}\sum_{k=0}^{\infty} c_k\ Tr\left(\mathbf{C}_{s,t} - \mathbb{I}\right)^k\Big|_{t,s=0}$$

$$
\begin{aligned}
&= 4 \frac{\mathrm{d}}{\mathrm{d}s} \left(\sum_{k=1}^{\infty} c_k \, k \, Tr \left((\mathbf{C}_{s,0})^{k-1} \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{C}_{s,t})_{t=0} \right) \right)_{s=0} \\
&= 2 \frac{\mathrm{d}}{\mathrm{d}s} \left(Tr \left((\mathbf{C}_{s,0})^{-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{C}_{s,t})_{t=0} \right) \right)_{s=0} \\
&= 2 Tr \left(\rho^{-1} \frac{\mathrm{d}}{\mathrm{d}s} \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{C}_{s,t})_{s,t=0} \right) \\
&\quad + 2 Tr \left(\frac{\mathrm{d}}{\mathrm{d}s} \left((\mathbf{C}_{s,0})^{-\frac{1}{2}} \right)_{s=0} \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{C}_{0,t})_{t=0} \right)
\end{aligned}
\tag{20}
$$

Now, we recall the equality [45]

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left(\sqrt{\gamma_\rho^X(t)} \right)_{t=0} = \mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho)),
\tag{21}
$$

where we introduced the superoperator $\mathcal{A}_{\sqrt{\rho}}(\mathbf{b}) = \sqrt{\rho}\, \mathbf{b} + \mathbf{b} \sqrt{\rho}$. Accordingly, using the Leibniz rule and Eq. (21), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{C}_{s,t})_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\sqrt{\gamma_\rho^X(t)} \gamma_\rho^{\mathcal{Y}_\mathbf{a}}(s) \sqrt{\gamma_\rho^X(t)} \right)_{t=0} \\
&= \mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho)) \gamma_\rho^{\mathcal{Y}_\mathbf{a}}(s) \sqrt{\rho} + \sqrt{\rho} \gamma_\rho^{\mathcal{Y}_\mathbf{a}}(s) \mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho)),
\end{aligned}
\tag{22}
$$

which implies

$$
\frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{C}_{0,t})_{t=0} = \mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho)) \rho^{\frac{3}{2}} + \rho^{\frac{3}{2}} \mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho)),
\tag{23}
$$

and we also have

$$
\frac{\mathrm{d}}{\mathrm{d}s} \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{C}_{s,t})_{s,t=0} = \mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho)) \mathcal{Y}_\mathbf{a}(\rho) \sqrt{\rho} + \sqrt{\rho} \mathcal{Y}_\mathbf{a}(\rho) \mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho)).
\tag{24}
$$

Inserting Eq. (24) into Eq. (20), recalling that

$$
\mathcal{Y}_\mathbf{a}(\rho) = \{\rho, \mathbf{a}\} = \frac{1}{2} (\rho \mathbf{a} + \mathbf{a} \rho),
\tag{25}
$$

and exploiting again the cyclicity of the trace, we obtain

$$
\begin{aligned}
(G_{\mathrm{BH}}(X, \mathcal{Y}_\mathbf{a}))(\rho) &= Tr(\mathbf{a}\, X(\rho)) + Tr \left(\rho^{-\frac{1}{2}} \left(\mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho)) \rho \mathbf{a} + \mathbf{a} \rho \mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho)) \right) \right) \\
&\quad + 2 Tr \left(\frac{\mathrm{d}}{\mathrm{d}s} \left((\mathbf{C}_{s,0})^{-\frac{1}{2}} \right)_{s=0} \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{C}_{0,t})_{t=0} \right).
\end{aligned}
\tag{26}
$$

To perform the last derivative with respect to s, we exploit the identity [45]

$$
\frac{\mathrm{d}}{\mathrm{d}s} \left((\sqrt{\rho} \, \gamma_\rho^{\mathcal{Y}_\mathbf{a}}(s) \sqrt{\rho})^{-\frac{1}{2}} \right)_{s=0} = -\rho^{-\frac{1}{2}} \mathcal{A}_\rho^{-1}(\mathcal{Y}_\mathbf{a}(\rho)) \rho^{-\frac{1}{2}},
\tag{27}
$$

which becomes

$$
\frac{\mathrm{d}}{\mathrm{d}s} \left((\sqrt{\rho} \, \gamma_\rho^{\mathcal{Y}_\mathbf{a}}(s) \sqrt{\rho})^{-\frac{1}{2}} \right)_{s=0} = -\frac{1}{2} \rho^{-\frac{1}{2}} \mathbf{a} \, \rho^{-\frac{1}{2}}
\tag{28}
$$

because of Eq. (25). Inserting Eqs. (28) and (23) into Eq. (26), and exploiting once again the cyclicity of the trace, we arrive at the final expression

$$
(G_{\mathrm{BH}}(X, \mathcal{Y}_\mathbf{a}))(\rho) = Tr(\mathbf{a}\, X(\rho))
\tag{29}
$$

On the other hand, the Lie derivative $\mathcal{L}_X f_{\mathbf{a}}$ is easily seen to be

$$
\begin{aligned}
(\mathcal{L}_X f_{\mathbf{a}})(\rho) &= \frac{\mathrm{d}}{\mathrm{d}t} f_{\mathbf{a}}\left(\gamma_\rho^X(t)\right)_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} Tr\left(\mathbf{a}\,\gamma_\rho^X(t)\right)_{t=0} \\
&= Tr\left(\mathbf{a}\,X(\rho)\right),
\end{aligned}
\tag{30}
$$

so that we have

$$
\mathrm{G}_{BH}\left(X,\,\mathcal{Y}_{\mathbf{a}}\right) = \mathcal{L}_X f_{\mathbf{a}},
\tag{31}
$$

and thus $\mathcal{Y}_{\mathbf{a}}$ is actually the gradient vector field associated with $f_{\mathbf{a}}$ by means of the Bures–Helstrom metric tensor G_{BH}.

Normalized states

The normalized case is easily obtained from the not-normalized one. Indeed, let $\mathscr{S}_+$ denote the manifold of faithful, normalized quantum states. This is the submanifold of $\mathscr{P}_+$ determined by the (affine) constraint $Tr\rho = 1$ for every $\rho \in \mathscr{S}_+$. It is easily seen that the action α given in Eq. (8) does not preserve $\mathscr{S}_+$ unless we restrict to the unitary group, which means that the $\mathbb{X}_{\mathbf{b}}$'s are tangent to $\mathscr{S}_+$. However, it is possible to deform the action $\widetilde{\alpha}$ to an action α preserving $\mathscr{S}_+$ by setting [9,10,15,27]

$$
\alpha(\mathrm{g},\rho) := \frac{\widetilde{\alpha}(\mathrm{g},\rho)}{Tr(\widetilde{\alpha}(\mathrm{g},\rho))} = \frac{\mathrm{g}\,\rho\,\mathrm{g}^\dagger}{Tr(\mathrm{g}\,\rho\,\mathrm{g}^\dagger)}.
\tag{32}
$$

Then, we may proceed following the steps outlined before, thus obtaining the fundamental vector fields

$$
\Gamma_{\mathbf{ab}} = \mathbb{Y}_{\mathbf{a}} + \mathbb{X}_{\mathbf{b}},
\tag{33}
$$

where the $\mathbb{X}_{\mathbf{b}}$'s are the vector fields generating the standard action of the unitary group, and the $\mathbb{Y}_{\mathbf{a}}$'s are given by

$$
\mathbb{Y}_{\mathbf{a}}(\rho) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{\frac{t}{2}\mathbf{a}}\,\rho\,e^{\frac{t}{2}\mathbf{a}}}{Tr\left(e^{\frac{t}{2}\mathbf{a}}\,\rho\,e^{\frac{t}{2}\mathbf{a}}\right)}\right)_{t=0} = \{\rho,\,\mathbf{a}\} - Tr(\rho\mathbf{a})\,\rho.
\tag{34}
$$

The Bures–Helstrom metric tensor on $\mathscr{S}_+$ is the metric tensor associated with the pullback to $\mathscr{S}_+$ (with respect to the canonical immersion) of the Bures distance in Eq. (14). With an evident abuse of notation, we denote by d_B^2 the pullback to $\mathscr{S}_+$ of the Bures distance given by

$$
\mathrm{d}_B^2(\rho,\,\sigma) = 4\left(1 - Tr\left(\sqrt{\sqrt{\rho}\,\sigma\,\sqrt{\rho}}\right)\right)
\tag{35}
$$

and by G_{BH} the associated metric tensor on $\mathscr{S}_+$.

Then, the computations performed in the case of not-normalized states may be easily adapted to prove that

$$
\mathrm{G}_{\mathrm{BH}}(X,\,\mathbb{Y}_{\mathbf{a}}) = \mathcal{L}_X f_{\mathbf{a}},
\tag{36}
$$

for every vector field X on $\mathscr{S}_+$, and where $f_{\mathbf{a}}$ is the pullback to $\mathscr{S}_+$ of the smooth function $f_{\mathbf{a}}$ on $\mathscr{P}_+$, with another evident abuse of notation. From this, we conclude that every vector field $\mathbb{Y}_{\mathbf{a}}$ is the gradient vector field associated with the smooth function $f_{\mathbf{a}}$ by means of the Bures–Helstrom metric tensor G_{BH}.

3 Wigner–Yanase metric tensor

Let us consider the diffeomorphism $\varphi \colon \mathscr{P}_+ \longrightarrow \mathscr{P}_+$ given by

$$\varphi(\rho) := \sqrt{\rho}, \tag{37}$$

and its inverse

$$\varphi^{-1}(\rho) = \rho^2. \tag{38}$$

By means of this diffeomorphism, we may define another action $\widetilde{\Theta}$ of $\mathcal{GL}(\mathcal{H})$ on $\mathscr{P}_+$ given by

$$\widetilde{\Theta}(g, \rho) := \varphi^{-1} \circ \widetilde{\alpha}_g \circ \varphi(\rho) = \left(g \sqrt{\rho} \, g^\dagger\right)^2, \tag{39}$$

where we have set $\widetilde{\alpha}_g(\rho) = \widetilde{\alpha}(g, \rho)$ with $\widetilde{\alpha}$ the action given in Eq. (8). Clearly, $\widetilde{\Theta}$ is different from $\widetilde{\alpha}$ in general, but, if we take $g = U$ in the unitary group, we have

$$\widetilde{\Theta}(U, \rho) = \left(U \sqrt{\rho} \, U^\dagger\right)^2 = U \rho \, U^\dagger = \phi(U, \rho). \tag{40}$$

Following the steps outlined in the previous section, it is easy to show that the fundamental vector fields $\widetilde{\Psi}_{\mathbf{ab}}$ of $\widetilde{\Theta}$ decompose as

$$\widetilde{\Psi}_{\mathbf{ab}} = \mathcal{W}_{\mathbf{a}} + \mathbb{X}_{\mathbf{b}} \tag{41}$$

where the $\mathbb{X}_{\mathbf{b}}$'s are the vector fields generating the standard action of the unitary group, and the $\mathcal{W}_{\mathbf{a}}$'s are given by

$$\mathcal{W}_{\mathbf{a}}(\rho) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\left(e^{\frac{t}{2}\mathbf{a}} \sqrt{\rho} \, e^{\frac{t}{2}\mathbf{a}} \right)^2 \right)_{t=0} = \{\rho, \mathbf{a}\} + \sqrt{\rho} \, \mathbf{a} \, \sqrt{\rho}. \tag{42}$$

Furthermore, note that $\mathcal{W}_{\mathbf{a}} = \varphi_*^{-1} \mathcal{Y}_{\mathbf{a}}$, that is, $\mathcal{W}_{\mathbf{a}}$ is the pushforward of $\mathcal{Y}_{\mathbf{a}}$ by means of φ^{-1} (see proposition 4.2.4 in [1]).

We will now prove that every $\mathcal{W}_{\mathbf{a}}$ is the gradient vector field associated with the smooth function $f_{\mathbf{a}}$ by means of the Wigner–Yanase metric tensor G_{WY} [24,25,28,29]. The Wigner–Yanase metric tensor G_{WY} is the metric tensor associated with the Wigner–Yanase skew information

$$S_{\mathrm{WY}}(\rho, \sigma) = Tr\rho + Tr\sigma - 2Tr\left(\sqrt{\rho} \, \sqrt{\sigma}\right). \tag{43}$$

Remark 2 Note that, in the literature, the Wigner–Yanase metric tensor G_{WY} is usually defined as the metric tensor associated with twice the divergence function in Eq. (43), and what will be proved below can be easily adapted provided we replace S_{WY} with $2S_{\mathrm{WY}}$, and $\mathbf{a}$ with $\frac{1}{2}\mathbf{a}$ in all the expressions. The choice made here leads to the fact that the vector fields $\mathcal{W}_{\mathbf{a}}$ are the pushforward of the vector fields $\mathcal{Y}_{\mathbf{a}}$ by means of φ^{-1} (and similarly for the normalized ones in the following subsection).

Just as we did for the Bures–Helstrom metric tensor in Sect. 2, we have to show that

$$G_{\mathrm{WY}}(X, \mathcal{W}_{\mathbf{a}}) = \mathcal{L}_X f_{\mathbf{a}}, \tag{44}$$

where X is an arbitrary vector field and $\mathcal{L}_X$ denote the Lie derivative with respect to X. Therefore, we must compute

$$
\begin{aligned}
(G_{\mathrm{WY}}(X, \mathcal{W}_{\mathbf{a}}))(\rho) &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} \left(D_{\mathrm{WY}}(\gamma_\rho^X(t), \gamma_\rho^{\mathcal{W}_{\mathbf{a}}}(s)) \right)_{t,s=0} \\
&= 2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} \left(Tr\sqrt{\gamma_\rho^X(t)} \sqrt{\gamma_\rho^{\mathcal{W}_{\mathbf{a}}}(s)} \right)_{t,s=0}
\end{aligned}
\tag{45}
$$

From Eq. (42), we immediately conclude that

$$\sqrt{\gamma_\rho^{\mathcal{W}_{\mathbf{a}}}(s)} = e^{\frac{s}{2}\mathbf{a}} \sqrt{\rho} \, e^{\frac{s}{2}\mathbf{a}}, \tag{46}$$

and recalling Eq. (21) we obtain

$$\frac{d}{dt}\left(\sqrt{\gamma_\rho^X(t)}\right)_{t=0} = \mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho)), \tag{47}$$

where we introduced the superoperator $\mathcal{A}_{\sqrt{\rho}}(\mathbf{b}) = \sqrt{\rho}\,\mathbf{b} + \mathbf{b}\sqrt{\rho}$. Consequently, we get

$$\begin{aligned}
(\mathrm{G}_{\mathrm{WY}}(X, \mathcal{W}_{\mathbf{a}}))(\rho) &= 2\frac{d}{dt}\frac{d}{ds}\left(\mathrm{Tr}\sqrt{\gamma_\rho^X(t)}\sqrt{\gamma_\rho^{\mathcal{W}_{\mathbf{a}}}(s)}\right)_{s,t=0}\\
&= 2\frac{d}{ds}\left(Tr\left(\mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho))\sqrt{\gamma_\rho^{\mathcal{W}_{\mathbf{a}}}(s)}\right)\right)_{s=0}\\
&= Tr\left(\mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho))\left(\mathbf{a}\sqrt{\rho} + \sqrt{\rho}\,\mathbf{a}\right)\right)\\
&= Tr\left(\mathbf{a}\,\mathcal{A}_{\sqrt{\rho}}\left(\mathcal{A}_{\sqrt{\rho}}^{-1}(X(\rho))\right)\right)\\
&= Tr(\mathbf{a}\,X(\rho)).
\end{aligned} \tag{48}$$

Therefore, recalling Eq. (30), we conclude that

$$\mathrm{G}_{\mathrm{WY}}(X, \mathcal{W}_{\mathbf{a}}) = \mathcal{L}_X f_{\mathbf{a}} \tag{49}$$

holds for all vector fields X as desired.

Normalized states

To tackle the normalized case, we mimic what has been done for the Bures–Helstrom metric tensor in Sect. 2. Specifically, we define a normalized version of the action $\widetilde{\Theta}$, given by

$$\Theta(\mathrm{g}, \rho) := \frac{\widetilde{\Theta}(\mathrm{g}, \rho)}{Tr(\widetilde{\Theta}(\mathrm{g}, \rho))} = \frac{\left(\mathrm{g}\sqrt{\rho}\,\mathrm{g}^\dagger\right)^2}{Tr\left(\left(\mathrm{g}\sqrt{\rho}\,\mathrm{g}^\dagger\right)^2\right)}. \tag{50}$$

Then, the fundamental vector fields $\Psi_{\mathbf{ab}}$ of Θ are easily computed to be

$$\Psi_{\mathbf{ab}} = \mathbb{W}_{\mathbf{a}} + \mathbb{X}_{\mathbf{b}} \tag{51}$$

where the $\mathbb{X}_{\mathbf{b}}$'s are the vector fields generating the standard action of the unitary group, and the $\mathbb{W}_{\mathbf{a}}$'s are given by

$$\mathbb{W}_{\mathbf{a}}(\rho) = \frac{d}{dt}\left(\frac{\left(e^{\frac{t}{2}\mathbf{a}}\sqrt{\rho}\,e^{\frac{t}{2}\mathbf{a}}\right)^2}{Tr\left(\left(e^{\frac{t}{2}\mathbf{a}}\sqrt{\rho}\,e^{\frac{t}{2}\mathbf{a}}\right)^2\right)}\right)_{t=0} = \{\rho, \mathbf{a}\} + \sqrt{\rho}\,\mathbf{a}\sqrt{\rho} - 2Tr(\rho\mathbf{a})\,\rho. \tag{52}$$

Then, the Wigner–Yanase metric tensor on $\mathscr{S}_+$ is associated with the pullback to $\mathscr{S}_+$ (with respect to the canonical immersion) of the Wigner–Yanase skew information given in Eq. (43). With an evident abuse of notation, we denote by S_{WY} the pullback to $\mathscr{S}_+$ of the Wigner–Yanase skew information given by

$$S_{\mathrm{WY}}(\rho, \sigma) = 2\left(1 - Tr\left(\sqrt{\rho}\,\sqrt{\sigma}\right)\right) \tag{53}$$

and by G_{WY} the associated metric tensor on $\mathscr{S}_+$.

Just as for the Bures–Helstrom metric tensor, the computations performed in the not-normalized case may be easily adapted to prove that

$$G_{WY}(X, \mathbb{W}_\mathbf{a}) = \mathcal{L}_X f_\mathbf{a},$$
(54)

for every vector field X on $\mathscr{S}_+$, and where, with an evident abuse of notation, $f_\mathbf{a}$ is the pullback to $\mathscr{S}_+$ of the smooth function $f_\mathbf{a}$ on $\mathscr{P}_+$. From this, we conclude that every vector field $\mathbb{W}_\mathbf{a}$ is the gradient vector field associated with the smooth function $f_\mathbf{a}$ by means of the Wigner–Yanase metric tensor G_{WY}.

4 Bogoliubov–Kubo–Mori metric tensor

The manifold $\mathscr{P}_+$ is an open subset of the vector space $\mathscr{V}$ of Hermitian (self-adjoint) linear operators on $\mathcal{H}$. Moreover, every $\mathbf{h} \in \mathscr{V}$ gives rise to an element in $\mathscr{P}_+$ by means of $e^\mathbf{h}$, and every $\rho \in \mathscr{P}_+$ gives rise to an element in $\mathscr{V}$ by means of $\ln(\rho)$. Essentially, the map $\psi : \mathscr{P}_+ \to \mathscr{V}$ given by

$$\psi(\rho) := \ln(\rho)$$
(55)

is a diffeomorphism with inverse

$$\psi^{-1}(\mathbf{h}) = e^\mathbf{h}.$$
(56)

Following what we have done for the Wigner–Yanase metric tensor, we can use ψ to transport every group action on $\mathscr{V}$ to a group action on $\mathscr{P}_+$. In particular, $\mathscr{V}$ is a real Euclidean space with respect to the (restriction of) the Hilbert–Schmidt product

$$\langle \mathbf{h}, \mathbf{k} \rangle = \mathrm{Tr}(\mathbf{h}\,\mathbf{k}).$$
(57)

Therefore, the Euclidean group acts on $\mathscr{V}$ as

$$A_{R,\mathbf{a}}(\mathbf{h}) = R(\mathbf{h}) + \mathbf{a},$$
(58)

where R is an element of the orthogonal group and $\mathbf{a} \in \mathscr{V}$.

Quite interestingly, the unitary group may be realized as a subgroup of the orthogonal group of $\mathscr{V}$ according to

$$R_\mathbf{U}(\mathbf{h}) := \mathbf{U}\mathbf{h}\mathbf{U}^\dagger.$$
(59)

Indeed, it is easy to check that $R_\mathbf{U}$ preserves the Euclidean product

$$\langle R_U(\mathbf{h}), R_U(\mathbf{k}) \rangle = \langle \mathbf{U}\mathbf{h}\mathbf{U}^\dagger, \mathbf{U}\mathbf{k}\mathbf{U}^\dagger \rangle = \langle \mathbf{h}, \mathbf{k} \rangle.$$
(60)

Consequently, the group $\mathcal{U}(\mathcal{H}) \rtimes_R \mathscr{V}$ acts on $\mathscr{P}_+$ according to

$$\widetilde{\Xi}((\mathbf{U}, \mathbf{a}), \rho) := \varphi^{-1} \circ A_{R_\mathbf{U},\mathbf{a}} \circ \varphi(\rho) = e^{\mathbf{U}\ln(\rho)\mathbf{U}^\dagger + \mathbf{a}}$$
(61)

Now, if we consider $\mathbf{a} = \mathbf{0}$, we have

$$\Xi((\mathbf{U}, \mathbf{0}), \rho) = e^{\mathbf{U}\ln(\rho)\mathbf{U}^\dagger} = \sum_{k=0}^\infty \frac{\left(\mathbf{U}\ln(\rho)\mathbf{U}^\dagger\right)^k}{k!} = \mathbf{U}\,\rho\,\mathbf{U}^\dagger = \phi(\mathbf{U}, \rho),$$
(62)

and thus we obtain again the standard action ϕ of the unitary group on $\mathscr{P}_+$. The fundamental vector fields $\widetilde{\Upsilon}_{ab}$ of $\widetilde{\Xi}$ decompose as

$$\widetilde{\Upsilon}_\mathbf{ab} = \mathcal{Z}_\mathbf{a} + \mathbb{X}_\mathbf{b}$$
(63)

where the $\mathbb{X}_\mathbf{b}$'s are the vector fields generating the standard action of the unitary group, and the $\mathcal{Z}_\mathbf{a}$'s are given by

$$\mathcal{Z}_\mathbf{a}(\rho) = \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\ln(\rho)+t\mathbf{a}}\right)_{t=0} = \int_0^1 \mathrm{d}\lambda \left(\rho^\lambda \mathbf{a} \rho^{1-\lambda}\right), \tag{64}$$

where we used the well-known equality [45]

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{A(t)} = \int_0^1 \mathrm{d}\lambda \left(e^{\lambda A(t)} \frac{\mathrm{d}}{\mathrm{d}t}(A(t)) e^{(1-\lambda)A(t)}\right), \tag{65}$$

which is valid for every smooth curve $A(t)$ inside $\mathcal{B}(\mathcal{H})$ (remember that the canonical immersion of $\mathscr{P}_+$ inside $\mathcal{B}(\mathcal{H})$ is smooth).

Remark 3 The Lie group $\mathcal{U}(\mathcal{H}) \rtimes_R \mathscr{V}$ is diffeomorphic to the cotangent bundle of the unitary group. Indeed, if G is any Lie group, the cotangent space $T^*G \cong G \times \mathfrak{g}^*$ is endowed with the structure of Lie group [2,3] according to

$$(g_1, a_1) \cdot (g_2, a_2) := (g_1 g_2, Ad^*_{g_1}(a_2) + a_1), \tag{66}$$

where Ad^* is the dual of the adjoint action of G on its Lie algebra $\mathfrak{g}$. The resulting Lie group is also denoted by $G \rtimes_{Ad^*} \mathfrak{g}^*$ to emphasize the fact that the group structure is associated with a semidirect product. Now, when $G = \mathcal{U}(\mathcal{H})$, its Lie algebra $\mathfrak{g}$ is given by skew-adjoint operators on $\mathcal{H}$ according to

$$i\mathbf{b} \mapsto \mathbf{U} = e^{i\mathbf{b}}, \tag{67}$$

where $\mathbf{b}$ is an Hermitian operator. Then, we can identify the dual space $\mathfrak{g}^*$ with the vector space $\mathscr{V}$ of Hermitian operators by means of the pairing

$$\langle \mathbf{a}, i\mathbf{b} \rangle := \mathrm{Tr}(\mathbf{a}\,\mathbf{b}). \tag{68}$$

Consequently, the coadjoint action reads

$$Ad^*_\mathbf{U}(\mathbf{a}) = \mathbf{U}\,\mathbf{a}\,\mathbf{U}^\dagger = R_\mathbf{U}(\mathbf{a}), \tag{69}$$

where we used Eq. (59) in the last equality, and we conclude that $\mathcal{U}(\mathcal{H}) \rtimes_R \mathscr{V}$ is actually diffeomorphic to the Lie group $T^*\mathcal{U}(\mathcal{H}) \equiv \mathcal{U}(\mathcal{H}) \rtimes_{Ad^*} \mathfrak{g}^*$ as claimed.

We will now prove that every $\mathcal{Z}_\mathbf{a}$ is the gradient vector field associated with the smooth function $f_\mathbf{a}$ by means of the Bogoliubov–Kubo–Mori metric tensor G_{BKM} [23,36,37,39,42]. The Bogoliubov–Kubo–Mori metric tensor G_{BKM} is the metric tensor associated with the von Neumann–Umegaki relative entropy [5,48,49]

$$S_{vNU}(\rho, \sigma) = \mathrm{Tr}\left(\rho \ln \rho - \rho \ln \sigma\right) \tag{70}$$

Just as we did for the Bures–Helstrom metric tensor and for the Wigner–Yanase metric tensor, we have to show that

$$G_{\mathrm{BKM}}(X, \mathcal{Z}_\mathbf{a}) = \mathcal{L}_X f_\mathbf{a}, \tag{71}$$

where X is an arbitrary vector field and $\mathcal{L}_X$ denote the Lie derivative with respect to X. Therefore, we compute

$$\begin{aligned}
(G_{\mathrm{BKM}}(X, \mathcal{Z}_\mathbf{a}))(\rho) &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\left(S_{vNU}(\gamma_\rho^X(t), \gamma_\rho^{\mathcal{Z}_\mathbf{a}}(s))\right)_{t,s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{Tr}\gamma_\rho^X(t)\,(\ln(\rho)+s\mathbf{a})\right)_{t,s=0} \\
&= \mathrm{Tr}\left(X(\rho)\,\mathbf{a}\right),
\end{aligned} \tag{72}$$

where we used the definition of S_{vNU} given in Eq. (70), and the equality

$$\gamma_\rho^{\mathcal{Z}_\mathbf{a}}(s) = e^{\ln(\rho) + s\mathbf{a}} \tag{73}$$

stemming from Eq. (64). Recalling Eq. (30), we conclude that

$$G_{BKM}(X, \mathcal{Z}_\mathbf{a}) = \mathcal{L}_X f_\mathbf{a} \tag{74}$$

holds for all vector fields X as desired.

Normalized states

Once again, the normalized case follows from the not-normalized one. Indeed, we just need to define the normalized action $\varXi$ of $T^*\mathcal{U}(\mathcal{H})$ on $\mathscr{S}_+$ given by

$$\varXi((\mathbf{U}, \mathbf{a}), \rho) := \frac{\widetilde{\varXi}(g, \rho)}{Tr(\widetilde{\varXi}(g, \rho))} = \frac{e^{\mathbf{U}\ln(\rho)\mathbf{U}^\dagger + \mathbf{a}}}{Tr(e^{\mathbf{U}\ln(\rho)\mathbf{U}^\dagger + \mathbf{a}})}. \tag{75}$$

Then, the fundamental vector fields $\varUpsilon_\mathbf{ab}$ of $\varXi$ are easily computed to be

$$\varUpsilon_\mathbf{ab} = \mathbb{Z}_\mathbf{a} + \mathbb{X}_\mathbf{b} \tag{76}$$

where the $\mathbb{X}_\mathbf{b}$'s are the vector fields generating the standard action of the unitary group, and the $\mathbb{Z}_\mathbf{a}$'s are given by

$$\mathbb{Z}_\mathbf{a}(\rho) = \frac{d}{dt} \left(\frac{e^{\ln(\rho) + t\mathbf{a}}}{Tr\left(e^{\ln(\rho) + t\mathbf{a}}\right)} \right)_{t=0} = \int_0^1 d\lambda \left(\rho^\lambda \, \mathbf{a} \, \rho^{1-\lambda} \right) - Tr(\rho \, \mathbf{a}) \, \rho \,. \tag{77}$$

Remark 4 It is worth mentioning the recent work [4] where the finite transformations associated with the vector fields $\mathbb{Z}_\mathbf{a}$ are exploited in the definition of a Hilbert space structure on $\mathscr{S}_+$.

Then, the Bogoliubov–Kubo–Mori metric tensor on $\mathscr{S}_+$ is associated with the pullback to $\mathscr{S}_+$ (with respect to the canonical immersion) of the von Neumann–Umegaki relative entropy given in Eq. (70). With an evident abuse of notation, we denote by S_{vNU} the pullback to $\mathscr{S}_+$ of the von Neumann–Umegaki relative entropy given by

$$S_{vNU}(\rho, \sigma) = Tr(\rho \ln \rho - \rho \ln \sigma) \tag{78}$$

and by G_{BKM} the associated metric tensor on $\mathscr{S}_+$.

Once again, the computations performed in the not-normalized case may be easily adapted to prove that

$$G_{BKM}(X, \mathbb{Z}_\mathbf{a}) = \mathcal{L}_X f_\mathbf{a} \,, \tag{79}$$

for every vector field X on $\mathscr{S}_+$, and where, with an evident abuse of notation, $f_\mathbf{a}$ is the pullback to $\mathscr{S}_+$ of the smooth function $f_\mathbf{a}$ on $\mathscr{P}_+$. From this, we conclude that every vector field $\mathbb{Z}_\mathbf{a}$ is the gradient vector field associated with the smooth function $f_\mathbf{a}$ by means of the Bogoliubov–Kubo–Mori metric tensor G_{BKM}.

5 Conclusions

The results of this work should be interpreted as a preliminary step toward a more general analysis aimed at characterizing those monotone metric tensors on the manifold of faithful

quantum states that are linked with group actions in the sense explained in the previous sections. Indeed, the fact that three of the most used metric tensors in Quantum Information Geometry like the Bures–Helstrom metric tensor, the Wigner–Yanase metric tensor, and the Bogoliubov–Kubo–Mori metric tensor are related with group actions seems to point at a more profound connection which is yet to be discovered. However, it is not yet known if this link is just a mathematical curiosity, or if it has to do with some geometrical aspects of the monotonicity property usually required in Quantum Information Theory.

At this purpose, it is not hard to see that the requirement of unitary invariance encoded in the monotonicity property has a direct effect on the possible group actions considered as will be now explained. It is clear that the Lie algebra of the suitable extension of $\mathcal{U}(\mathcal{H})$ we can hope to link to a given monotone metric tensor G must be isomorphic, as a vector space, to $\mathfrak{u}(\mathcal{H}) \times \mathcal{V}$. Indeed, besides the fundamental vector fields $\mathbb{X}_\mathbf{b}$ of the action of $\mathcal{U}(\mathcal{H})$ and associated with elements in $\mathfrak{u}(\mathcal{H})$, we have the *complementary* vector fields that must be the gradient vector fields associated with the expectation value functions $f_\mathbf{a}$ by means of G, and the latters are clearly labelled by elements in the space $\mathcal{V}$ of Hermitian operators on $\mathcal{H}$. Let us write these vector fields generically as $\mathbb{V}_\mathbf{a}$. By assumption, we must have

$$G(X, \mathbb{V}_\mathbf{a}) = \mathcal{L}_X f_\mathbf{a} \tag{80}$$

for every vector field X on $\mathscr{S}_+$. Then, we evaluate the Lie derivative of the function $f_\mathbf{c}$ with respect to the commutator $[\mathbb{X}_\mathbf{b}, \mathbb{V}_\mathbf{a}]$ to obtain

$$
\begin{aligned}
\mathcal{L}_{[\mathbb{X}_\mathbf{b}, \mathbb{V}_\mathbf{a}]} f_\mathbf{c} &= \mathcal{L}_{\mathbb{X}_\mathbf{b}} \left(\mathcal{L}_{\mathbb{V}_\mathbf{a}} f_\mathbf{c} \right) - \mathcal{L}_{\mathbb{V}_\mathbf{a}} \left(\mathcal{L}_{\mathbb{X}_\mathbf{b}} f_\mathbf{c} \right) \\
&= \mathcal{L}_{\mathbb{X}_\mathbf{b}} \left(G(\mathbb{V}_\mathbf{a}, \mathbb{V}_\mathbf{c}) \right) - \mathcal{L}_{\mathbb{V}_\mathbf{a}} \left(f_{\frac{i}{2}[\mathbf{b}, \mathbf{c}]} \right) \\
&= \mathcal{L}_{\mathbb{X}_\mathbf{b}} \left(G(\mathbb{V}_\mathbf{a}, \mathbb{V}_\mathbf{c}) \right) - G(\mathbb{V}_\mathbf{a}, \mathbb{V}_{\frac{i}{2}[\mathbf{b}, \mathbf{c}]}),
\end{aligned} \tag{81}
$$

where, in the second equality, we used the fact that

$$
\begin{aligned}
\left(\mathcal{L}_{\mathbb{X}_\mathbf{b}} f_\mathbf{c} \right)(\rho) &= \frac{\mathrm{d}}{\mathrm{d}t} \left(Tr(e^{i\frac{t}{2}\mathbf{b}} \rho \, e^{-i\frac{t}{2}\mathbf{b}} \, \mathbf{c}) \right)_{t=0} \\
&= \frac{i}{2} Tr([\mathbf{b}, \rho], \mathbf{c}) = \frac{i}{2} Tr(\rho[\mathbf{b}, \mathbf{c}]) = f_{\frac{i}{2}[\mathbf{b}, \mathbf{c}]}(\rho).
\end{aligned} \tag{82}
$$

The unitary invariance of G implies that $\mathcal{L}_{\mathbb{X}_\mathbf{b}} G = 0$, so that

$$
\begin{aligned}
\mathcal{L}_{[\mathbb{X}_\mathbf{b}, \mathbb{V}_\mathbf{a}]} f_\mathbf{c} &= G([\mathbb{X}_\mathbf{b}, \mathbb{V}_\mathbf{a}], \mathbb{V}_\mathbf{c}) + G(\mathbb{V}_\mathbf{a}, [\mathbb{X}_\mathbf{b}, \mathbb{V}_\mathbf{c}]) - G(\mathbb{V}_\mathbf{a}, \mathbb{V}_{\frac{i}{2}[\mathbf{b}, \mathbf{c}]}) \\
&= \mathcal{L}_{[\mathbb{X}_\mathbf{b}, \mathbb{V}_\mathbf{a}]} f_\mathbf{c} + G(\mathbb{V}_\mathbf{a}, [\mathbb{X}_\mathbf{b}, \mathbb{V}_\mathbf{c}]) - G(\mathbb{V}_\mathbf{a}, \mathbb{V}_{\frac{i}{2}[\mathbf{b}, \mathbf{c}]}),
\end{aligned} \tag{83}
$$

from which we conclude that

$$G(\mathbb{V}_\mathbf{a}, [\mathbb{X}_\mathbf{b}, \mathbb{V}_\mathbf{c}] - \mathbb{V}_{\frac{i}{2}[\mathbf{b}, \mathbf{c}]}) = 0. \tag{84}$$

Since the differentials of the $f_\mathbf{a}$ form a basis of the module of one-forms on $\mathscr{S}_+$, and since the $\mathbb{V}_\mathbf{a}$ are the gradient vector fields associated with the $f_\mathbf{a}$, we have that the $\mathbb{V}_\mathbf{a}$ form a basis of the module of vector fields on $\mathscr{S}_+$, and thus the previous equation implies

$$[\mathbb{X}_\mathbf{b}, \mathbb{V}_\mathbf{c}] = \mathbb{V}_{\frac{i}{2}[\mathbf{b}, \mathbf{c}]}. \tag{85}$$

This means that the commutator between the $\mathbb{X}_\mathbf{b}$ and the *complementary* vector fields is fixed by the requirement of unitary invariance. Therefore, the Lie algebra structure on $\mathfrak{u}(\mathcal{H}) \times \mathcal{V}$ must fulfill this additional constraint. Taking into account that $\mathfrak{u}(\mathcal{H})$ must be realized as a Lie subalgebra of $\mathfrak{u}(\mathcal{H}) \times \mathcal{V}$, we conclude that the freedom we have in choosing the enlargement

of $\mathfrak{u}(\mathcal{H})$ is only in the Lie product between elements in the complementary space $\mathcal{V}$. In particular, for the Lie algebra of $\mathcal{GL}(\mathcal{H})$ considered in Sects. 2 and 3, this bracket gives back an element in $\mathfrak{u}(\mathcal{H})$, while, for the Lie algebra of $T^*\mathcal{U}(\mathcal{H})$ considered in Sect. 4, this bracket vanishes identically. A classification of the possible extensions of the Lie algebra $\mathfrak{u}(\mathcal{H})$ satisfying the constraints found above will certainly give a hint on what type of group actions one can hope to link to monotone metric tensors. According to [2,3], the groups $\mathcal{GL}(\mathcal{H})$ and $T^*\mathcal{U}(\mathcal{H})$ are symplectomorphic and Morita-equivalent, and each of them is a so-called group double of $\mathcal{U}(\mathcal{H})$. This leads to conjecture that the appropriate enlargements may be looked for in the context of the double metric-Lie algebras containing $\mathfrak{u}(\mathcal{H})$ as a subalgebra. The emergence of a group double of $\mathcal{U}(\mathcal{H})$ in both situations we discussed seems to suggest that we may deal with the Non-Commutative Extension of Information Geometry by using the whole machinery introduced by Drinfeld to deal with Poisson-Lie groups (see again [2,3] and references therein).

From another point of view, the case of monotone metric tensors obtained from the quantum Tsallis relative entropies [34] and from the $\alpha - z$-relative Rény relative entropies [6,14] is currently being investigated using the methodology developed in Sect. 3.

Finally, it should be noted that it is possible to reformulate the results contained in this work no more in terms of quantum states on $\mathcal{H}$, but in terms of the more general notion of states on an arbitrary finite-dimensional von Neumann algebra. This would help to give a unifying picture for the classical and quantum case, and would be necessary in order to extend this information geometrical considerations also to the recent groupoidal approach to quantum theories developed in [11,12,16–19].

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