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Comment

A note on "Measuring propagation speed of Coulomb fields" by R. de Sangro, G. Finocchiaro, P. Patteri, M. Piccolo, G. Pizzella

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Abstract In connection with the discussion and the measurements fulfilled in Ref. (Eur Phys J C 75:137, 2015), the full identity is demonstrated between the Feynman formula for the field of a moving charge and the Liénard-Wiechert potentials.

In Ref. [1] measurements were performed to decide between two approaches to the field of a moving charge: one based on the Liénard-Wiechert potentials and the other on the Feynman interpretation. The aim of the present note is to demonstrate that although apparently different physical ideas are layed into the Feynman formula [5], it is as a matter of fact mathematically identical to that found in standard text books, e.g. [2–4], for the Liénard-Wiechert potentials both for accelerated motion of the charge and its motion with a constant speed. We believe that this observation should be taken into account and prove to be useful in the general discussion on the matter [6–8].

The Feynmann formula for the electric field $\mathbf{E}$ of a moving accelerated charge q is [5]

$$\mathbf{E}=\frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{e}_{r'}}{r'^2}+\frac{r'}{c}\frac{d}{dt}\left(\frac{\mathbf{e}_{r'}}{r'^2}\right)+\frac{1}{c^2}\frac{d^2}{dt^2}\mathbf{e}_{r'}\right], \tag{1}$$

We set $c = 1$ for the sake of simplicity and keep to the notations in [5]. In (1) the function

$$r'=\sqrt{x_1'^2+x_2'^2+x_3'^2} \tag{2}$$

is the modulus of vector $\mathbf{x}'$, directed from the position $\tilde{\mathbf{x}}$ of the moving charge to the observation point $\mathbf{x}$:

$$\mathbf{x}'=\mathbf{x}-\tilde{\mathbf{x}}, \tag{3}$$

and $\mathbf{e}_{r'}$ is the unit vector in the direction of $\mathbf{x}'$. The charge trajectory is given as $\tilde{\mathbf{x}}=\tilde{\mathbf{x}}(t')$, where t' is the time coordinate of the charge. Therefore, with the location of the observation point fixed, $\mathbf{x} = const.$, we see that $\mathbf{x}'$, as well as r', is a function solely of t'. Once the influence of the charge propagates exactly with the speed of light $c = 1$, the relation

$$r'(t')=t-t' \tag{4}$$

holds, where t is the time of observation. With Eqs. (3) and (4) the length (2) is just the distance between the position of the charge and the observation point at the moment of emission.

Relation (4) defines t as a function of t'.Then, according to the rule of differentiation of an inverse function, one has for any function $a(t')$

$$\frac{da\left(t'\right)}{dt}=\frac{da\left(t'\right)}{dt'}\frac{dt'}{dt}=\frac{da\left(t'\right)}{dt'}\left(\frac{dt}{dt'}\right)^{-1},$$

where $\frac{dt}{dt'}$ follows from (4) and (3) to be

$$\begin{aligned}\frac{dt}{dt'}&=1+\frac{dr'\left(t'\right)}{dt'}=1+\frac{d|\mathbf{x}-\tilde{\mathbf{x}}(t')|}{dt'}\\&=1-\frac{\left(x_i-\tilde{x}_i(t')\right)}{r'}\frac{d\tilde{x}_i\left(t'\right)}{dt'}=1-\frac{\left(\mathbf{v}\cdot\mathbf{x}'\right)}{r'}\\&=1-\left(\mathbf{v}\cdot\mathbf{e}_{r'}\right).\end{aligned}$$

We have used here that $\frac{d\tilde{\mathbf{x}}(t')}{dt'}=\mathbf{v}(t')$ is the instantaneous speed of the charge.

Referring to the designation $(\mathbf{e}_{r'}\cdot\mathbf{v})=\kappa$ used for brevity we can now rewrite Eq. (1) as

$$\begin{aligned}\mathbf{E}=\frac{q}{4\pi\epsilon_0}&\left[\frac{\mathbf{e}_{r'}}{r'^2}+\frac{r'}{(1-\kappa)}\frac{d}{dt'}\left(\frac{\mathbf{e}_{r'}}{r'^2}\right)\right.\\&\left.+\frac{1}{(1-\kappa)}\frac{d}{dt'}\left(\frac{1}{(1-\kappa)}\frac{d\mathbf{e}_{r'}}{dt'}\right)\right].\end{aligned} \tag{5}$$

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Taking into account that

$$\frac{d\mathbf{e}_{r'}}{dt'} = \frac{d}{dt'}\left(\frac{\mathbf{x}'}{r'}\right) = -\frac{\mathbf{v}}{r'} + \frac{\kappa\mathbf{x}'}{r'^2} = \frac{\kappa\mathbf{e}_{r'} - \mathbf{v}}{r'} \tag{6}$$

we calculate the second term in (5):

$$\frac{r'}{(1-\kappa)}\frac{d}{dt'}\left(\frac{\mathbf{e}_{r'}}{r'^2}\right) = \frac{3\kappa\mathbf{e}_{r'} - \mathbf{v}}{(1-\kappa)\,r'^2}. \tag{7}$$

The third term in (5) is

$$\frac{1}{(1-\kappa)}\frac{d}{dt'}\left(\frac{1}{(1-\kappa)}\frac{d\mathbf{e}_{r'}}{dt'}\right) = \frac{1}{(1-\kappa)}\frac{d}{dt'}\left(\frac{\kappa\mathbf{e}_{r'} - \mathbf{v}}{(1-\kappa)\,r'}\right). \tag{8}$$

Let us calculate the derivative of $(1-\kappa)^{-1}$:

$$\begin{aligned}\frac{d\,(1-\kappa)^{-1}}{dt'} &= (1-\kappa)^{-2}\left(\frac{d\mathbf{e}_{r'}}{dt'}\cdot\mathbf{v} + \mathbf{e}_{r'}\cdot\frac{d\mathbf{v}}{dt'}\right)\\ &= (1-\kappa)^{-2}\left[\frac{\kappa^2 - v^2}{r'} + (\mathbf{e}_{r'}\cdot\dot{\mathbf{v}})\right],\end{aligned} \tag{9}$$

where $\dot{\mathbf{v}}$ is the acceleration of the charge, and v is the modulus of the vector $\mathbf{v}$. Then the third term in (5) becomes

$$\begin{aligned}&\frac{1}{(1-\kappa)}\frac{d}{dt'}\left(\frac{1}{(1-\kappa)}\frac{d\mathbf{e}_{r'}}{dt'}\right)\\ &= \frac{\left(\kappa^2 - v^2\right)(\kappa\mathbf{e}_{r'} - \mathbf{v}) + (1-\kappa)\left[2\kappa\left(\kappa\mathbf{e}_{r'} - \mathbf{v}\right) + \mathbf{e}_{r'}\left(\kappa^2 - v^2\right)\right]}{(1-\kappa)^3\,r'^2}\\ &+ \frac{(\kappa\mathbf{e}_{r'} - \mathbf{v})\,(\mathbf{e}_{r'}\cdot\mathbf{v}) + \left[\mathbf{e}_{r'}\left(\mathbf{e}_{r'}\cdot\dot{\mathbf{v}}\right) - \dot{\mathbf{v}}\right](1-\kappa)}{(1-\kappa)^3\,r'}.\end{aligned} \tag{10}$$

Finally, substituting (7) and (10) in (5) and separating the factor $(1-\kappa)^3\,r'^2$, we get

$$\begin{aligned}\mathbf{E} = \frac{q}{4\pi\epsilon_0\,(1-\kappa)^3\,r'^2}&\Big[\mathbf{e}_{r'}\,(1-\kappa)^3 + (3\kappa\mathbf{e}_{r'} - \mathbf{v})\,(1-\kappa)^2\\ &+ \left(\kappa^2 - v^2\right)(\kappa\mathbf{e}_{r'} - \mathbf{v}) + 2\kappa\,(\kappa\mathbf{e}_{r'} - \mathbf{v})\,(1-\kappa)\\ &+ \mathbf{e}_{r'}\left(\kappa^2 - v^2\right)(1-\kappa) + r'\,(\kappa\mathbf{e}_{r'} - \mathbf{v})\left(\mathbf{e}_{r'}\cdot\dot{\mathbf{v}}\right)\\ &+ r'\left[\mathbf{e}_{r'}\left(\mathbf{e}_{r'}\cdot\dot{\mathbf{v}}\right) - \dot{\mathbf{v}}\right](1-\kappa)\Big].\end{aligned} \tag{11}$$

It is easy to show that this is reduced to

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi\epsilon_0\,(1-\kappa)^3\,r'^2}\Big[\mathbf{e}_{r'}\left(1 - v^2\right) - \mathbf{v}\left(1 - v^2\right)\\ &\quad + r'\,(\kappa\mathbf{e}_{r'} - \mathbf{v})\left(\mathbf{e}_{r'}\cdot\dot{\mathbf{v}}\right) + r'\left(\mathbf{e}_{r'}\left(\mathbf{e}_{r'}\cdot\dot{\mathbf{v}}\right) - \dot{\mathbf{v}}\right)(1-\kappa)\Big]\\ &= \frac{q}{4\pi\epsilon_0}\frac{\left(1 - v^2\right)(\mathbf{x}' - \mathbf{v}r')}{(r' - \mathbf{x}'\cdot\mathbf{v})^3}\\ &+ \frac{q}{4\pi\epsilon_0}\frac{(\mathbf{x}' - \mathbf{v}r')\left(\mathbf{x}'\cdot\dot{\mathbf{v}}\right) - \dot{\mathbf{v}}r'\,(r' - \mathbf{x}'\cdot\mathbf{v})}{(r' - \mathbf{x}'\cdot\mathbf{v})^3}.\end{aligned} \tag{12}$$

Taking into account that the numerator in the second term in the latter expression can be rewritten as the double vector product

$$\begin{aligned}&(\mathbf{x}' - \mathbf{v}r')\left(\mathbf{x}'\cdot\dot{\mathbf{v}}\right) - \dot{\mathbf{v}}r'\,(r' - \mathbf{x}'\cdot\mathbf{v})\\ &= \left[\mathbf{x}'\times\left[(\mathbf{x}' - r'\mathbf{v})\times\dot{\mathbf{v}}\right]\right],\end{aligned} \tag{13}$$

expression (12) can be recognized (with the identification $q/4\pi\epsilon_0 = e$, $\mathbf{x}' = \mathbf{R}$, $r' = R$) as the expression (63.8) for electric field in Ref. [2]. Thus, expressions in Refs. [2,5] are the same.

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