

# Branes as solutions of gauge theories in gravitational field

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Received: 12 June 2014 / Accepted: 26 August 2014 / Published online: 11 September 2014

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**Abstract** The idea of the Gauss map is unified with the concept of branes as hypersurfaces embedded into  $D$ -dimensional Minkowski space. The map introduces new generalized coordinates of branes alternative to their world vectors  $\mathbf{x}$  and identified with the gauge and other massless fields. In these coordinates the Dirac  $p$ -branes realize extremals of the Euler–Lagrange equations of motion of a  $(p + 1)$ -dimensional  $SO(D - p - 1)$  gauge-invariant action in a gravitational background.

## 1 Introduction

The paper is motivated by the problem of brane quantization which requires new tools for overcoming brane dynamics nonlinearities [1–22]. We assume that such a tool may be provided by the choice of special *generalized coordinates* for branes, similar to the angle and action variables. This can be realized by the use of the Gauss map, well known from the differential geometry of surfaces, to hypersurfaces in combination with the Cartan group theoretical approach.

The dynamics of the classic relativistic string is exactly linearizable and may be described in terms of *harmonic oscillators*. These properties were crucial for construction of quantum string theories [23]. The problem of quantization of  $p$ -branes [24] remains unsolved due to the nonlinearity of their dynamics. The latter is caused by the entangled *anharmonic* character [25, 26] of brane elastic forces which resembles the anharmonicity in liquid crystals (smectics), but it is much more complicated. Linearization of the brane dynamics in  $D$ -dimensional Minkowski space requires the construction of new *generalized coordinates* alternative to the generally used components of the brane world vector  $\mathbf{x}(\xi^\mu)$ . To preserve the number of classical physical degrees of freedom, new brane coordinates must be in one-to-one correspondence with  $\mathbf{x}(\xi^\mu)$ . Such coordinates are the coef-

ficients of the first and second quadratic differential forms of embedded (hyper)surfaces used in differential geometry [27]. Regge and Lund [28] proposed to use these coefficients for the description of a string worldsheet embedded in 4-dimensional Minkowski space (see also [29–31]). This choice encodes the string dynamics in the generalized sine-Gordon equation of which the solution reduces to the associated linear equations of the inverse scattering method [32].

A generalization of the Regge–Lund geometrical approach to branes provides a new technique for studying the brane dynamics. The final goal of the generalization is to verify whether this new geometrical approach helps to solve the problems of the brane integrability and quantization.

This paper is aimed at the development of the geometric technique for the branes embedded into  $D$ -dimensional Minkowski space. For this purpose we generalize the *gauge reformulation* of the Regge–Lund approach for strings developed in [33–36]. This reformulation has revealed an isomorphism between the string and a closed sector of states of the exactly integrable 2-dimensional  $SO(1, 1) \times SO(D - 2)$ -invariant model of interacting gauge and massless scalar fields. The isomorphism is analogous to that between the chiral and Yang–Mills field theories invariant under a Lie group  $G$  observed by Semenov-Tyan-Shansky and Faddeev [37]. A characteristic feature of their approach is the use of the Cartan  $G$ -invariant  $\omega$ -forms [38] introduced by Volkov in the physics of nonlinear sigma models [39–42]. The physical results presented in [39, 40] and in the papers by Callan et al. [42] and Coleman et al. [41] coincide. The difference is that Volkov used the Cartan method of the exterior differential forms in the geometry of symmetric spaces associated with spontaneously broken symmetry groups and their gauge fields.

The gauge approach considered here which presents the brane dynamics in terms of gauge field theories is *exact in all orders* in the tension  $T$ . The said exactness is one of the distinctions between this approach and the ones where some

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branes are considered as the *solitons* of supergravity theories describing the *low-energy* approximations of string theories (see e.g. [43–45]).

To explain the idea of the gauge approach [33–36] let us consider the simplest example of the Nambu string embedded into 3-dimensional Euclidean space-time. The string worldsheet  $\Sigma_2$ , described by its world vector  $\mathbf{x}(\tau, \sigma)$ , may be supplied with the local vectors  $\mathbf{n}(\tau, \sigma)$  normal to  $\Sigma_2$  at each point parametrized by the internal coordinates  $\xi^\mu := (\tau, \sigma)$ . The vector field  $\mathbf{n}(\xi^\mu)$  defines a family of 2-dimensional planes tangent to  $\Sigma_2$  and fixed by the equation  $\mathbf{n}d\mathbf{x} = 0$ . A solution of this equation allows one *to restore* the worldsheet  $\Sigma_2$  up to its translations and rotations as a whole. Therefore, the vector  $\mathbf{n}$  may be chosen as a new dynamical variable coupled with  $\mathbf{x}$ . Moreover, without loss of generality one can put  $\mathbf{n}^2 = 1$ , which creates the map of  $\Sigma_2$  in the points of a 2-dimensional sphere  $S^2$  invariant under the group  $SO(3)$ . The map was discovered by Gauss in the differential geometry of surfaces. We assume that the Gauss map makes it possible to consider an alternative description of the Nambu string for  $D = 3$  in terms of the  $SO(3)$  gauge field attached to its worldsheet. This group theoretical consideration can be naturally extended to the case of Minkowski spaces with higher dimensions.

For the Minkowski spaces with  $D > 3$  and the metric  $\eta_{mn} = (1, -1, \dots, -1)$  the number of local unit vectors  $\mathbf{n}_\perp(\xi^\mu)$  normal to  $\Sigma_2$  increases up to  $(D - 2)$ , where the index  $\perp = (2, 3, \dots, D - 1)$ . One can choose these vectors to be mutually orthogonal:  $\mathbf{n}_\perp \mathbf{n}_{\perp'} = -\delta_{\perp \perp'}$ . They form an orthonormal basis in the local (hyper)planes orthogonal to  $\Sigma_2$  and span the vector space invariant under the gauge group  $SO(D - 2)$ . Then the string dynamics is presented by a chain of  $(D - 2)$  exactly integrable equations [33–36] generalizing the Liouville equation which encodes the dynamics for  $D = 3$ . These PDEs select a closed sector of the solutions of 2-dimensional  $SO(1, 1) \times SO(D - 2)$ -invariant gauge model of the Y-M fields interacting with a massless scalar multiplet.

In this paper the above-described gauge approach to strings is generalized to  $p$ -branes embedded into  $D$ -dimensional Minkowski space. We construct a  $(p + 1)$ -dimensional  $SO(D - p - 1)$ -invariant gauge model in a gravitational background formulated in terms of the dynamical variables associated with the generalized Gauss map for  $(p + 1)$ -dimensional hypersurfaces. The equations of motion of the model are shown to have the exact solution which describes the minimal world volumes of the fundamental  $p$ -branes presented by the Dirac action in terms of derivatives of  $\mathbf{x}(\xi^\mu)$ . These world vectors  $\mathbf{x}$  are treated as *collective coordinates* of the gauge model. So, the generalized Gauss map provides a clear mechanism for the creation of “*macroscopic*” fundamental branes by gluing together the “*microscopic*” field degrees of freedom of the initial gauge model in gravitational background. The exact solution of

the EOM of the constructed gauge model has the form of the first-order Gauss–Codazzi (G-C) differential equations. Therefore, their role is mathematically analogous to the self-duality conditions for instantons in *pure* gauge 4-dimensional theory [46]. This qualitative analogy helps to clear up the mathematics of the gauge approach for branes. Unification of the approach with the quantization methods for gauge theories in curved backgrounds [47–49] may give a new clue to a better understanding of the brane quantization problem. Below we generalize the gauge formalism and construct new gauge-invariant models of *hypersurfaces* and *branes* embedded into the Minkowski spaces. This formalism yields a new brane  $\leftrightarrow$  gauge field theory (GFT) map non-perturbative in  $T$ , independent of both the brane and target space dimensions.

## 2 Hypersurfaces in the Minkowski space

The above-discussed correspondence between the world vector  $\mathbf{x}(\xi^\mu)$  of the hypersurface  $\Sigma_{p+1}$  and its normal vectors  $\mathbf{n}_\perp(\xi^\mu)$  is realized in the Cartan formalism of the orthonormal moving frames  $\mathbf{n}_A(\xi^\mu)$ ,

$$\mathbf{n}_A(\xi^\mu)\mathbf{n}_B(\xi^\mu) = \eta_{AB}, \quad A, B = (0, 1, \dots, D - 1), \quad (1)$$

where  $\xi^\mu = (\tau, \sigma^r)$  ( $r = 1, 2, \dots, p$ ) parametrize  $\Sigma_{p+1}$  embedded into the  $D$ -dimensional Minkowski space  $\mathbf{R}^{1, D-1}$ . The vectors  $\mathbf{n}_A$  form the fundamental linear representation of the Lorentz group of  $\mathbf{R}^{1, D-1}$ ,

$$\mathbf{n}'_A = L_A{}^B \mathbf{n}_B, \quad L_A{}^B L^C{}_B = \delta^C_A, \quad (2)$$

where  $\frac{1}{2}D(D - 1)$  pseudoorthogonal matrices  $L_A{}^B(\xi^\mu)$  in the fundamental representation describe the local Lorentz transformations in the planes spanned by the  $\frac{1}{2}D(D - 1)$  pairs  $(\mathbf{n}_A(\xi), \mathbf{n}_B(\xi))$ . To distinguish the frame vectors  $\mathbf{n}_i$  ( $i, k = 0, 1, \dots, p$ ) tangent to the hypersurface  $\Sigma_{p+1}$  from the vectors  $\mathbf{n}_a$  ( $a, b = p + 1, p + 2, \dots, D - p - 1$ ) normal to this hypersurface, we split the capital index  $A$  in a pair  $A = (i, a)$ . This divides the set of the vectors  $\mathbf{n}_A$  into two subsets  $\mathbf{n}_A = (\mathbf{n}_i, \mathbf{n}_a)$ . The Latin indices  $a, b$  are used here instead of the condensed index  $\perp$  running in the orthogonal directions. One can expand any small displacement  $d\mathbf{x}(\xi^\mu)$  and  $d\mathbf{n}_A(\xi^\mu)$  in the local orths  $\mathbf{n}_A(\xi^\mu)$  attached to the point  $P(\xi^\mu)$

$$d\mathbf{x} = \omega^i \mathbf{n}_i, \quad \omega^a = 0, \quad (3)$$

$$d\mathbf{n}_A = -\omega^B{}_A \mathbf{n}_B. \quad (4)$$

Here we partially fix the gauge for the Lorentz group  $SO(1, D - 1)$  by the conditions  $\omega^a = 0$  for the local displacements of  $\mathbf{x}$  orthogonal to  $\Sigma_{p+1}$ , taking into account that the vector  $d\mathbf{x}$  is tangent to the hypersurface. This signifies a special choice of orientation of the moving frame which

breaks the local  $SO(1, D - 1)$  symmetry up to its subgroup  $SO(1, p) \times SO(D - p - 1)$ . Then the matrices  $L_A^B$  of the Lorentz group split into the block submatrices  $l_i^k, l_a^b$  and  $l_i^a$ . As a result, the antisymmetric matrices  $\omega_{AB} = -\omega_{BA}$  generating the infinitesimal Lorentz transformations take the form

$$\omega_A^B := \omega_{\mu A}^B d\xi^\mu = \begin{pmatrix} A_{\mu i}^k & W_{\mu i}^b \\ W_{\mu a}^k & B_{\mu a}^b \end{pmatrix} d\xi^\mu, \tag{5}$$

with the submatrices  $A_{\mu i}^k$  and  $B_{\mu a}^b$  considered as gauge fields in the fundamental representations of  $SO(1, p)$  and  $SO(D - p - 1)$  subgroups, respectively. The off-diagonal matrices  $W_{\mu i}^b$  form a charged vector multiplet in the fundamental representation of the local subgroups  $SO(1, p)$  and  $SO(D - p - 1)$ . The strengths for  $\hat{A}_\mu, \hat{B}_\mu$  denoted as  $F_{\mu\nu i}^k$  and  $H_{\mu\nu a}^b$  are

$$F_{\mu\nu i}^k \equiv [D_\mu^\parallel, D_\nu^\parallel]_i^k = (\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]})_i^k, \tag{6}$$

$$H_{\mu\nu a}^b \equiv [D_\mu^\perp, D_\nu^\perp]_a^b = (\partial_{[\mu} B_{\nu]} + B_{[\mu} B_{\nu]})_a^b. \tag{7}$$

The covariant derivatives  $D_\mu^\parallel$  and  $D_\mu^\perp$  defined as

$$D_\mu^\parallel \Phi^i = \partial_\mu \Phi^i + A_{\mu i}^k \Phi^k, \tag{8}$$

$$D_\mu^\perp \Psi^a = \partial_\mu \Psi^a + B_{\mu a}^b \Psi^b \tag{9}$$

are associated either with the local Lorentz subgroup  $SO(1, p)$ , operating in the local planes tangent to  $\Sigma_{p+1}$ , or with the rotation subgroup  $SO(D - p - 1)$  operating in the planes orthogonal to  $\Sigma_{p+1}$ . The covariant derivative for  $W_{\mu i}^b$ ,

$$(D_\mu W_\nu)_i^a = \partial_\mu W_{\nu i}^a + A_{\mu i}^k W_{\nu k}^a + B_{\mu a}^b W_{\nu i}^b, \tag{10}$$

includes both the  $\hat{A}_\mu$  and the  $\hat{B}_\mu$  gauge fields.

The Cartan differential forms  $\omega_A := \omega_{\mu A} d\xi^\mu$  and  $\omega_A^B := \omega_{\mu A}^B d\xi^\mu$  are linear in the independent differentials  $d\xi^\mu$ . This shows that the PDEs (3–4) define  $\mathbf{x}(\xi^\mu)$  and  $\mathbf{n}_A(\xi^\mu)$  as functions of the parameters  $\xi^\mu$  when the functions  $\omega_{\mu A}$  and  $\omega_{\mu A}^B$  are known. If Eqs. (3–4) are integrable one can find  $\mathbf{x}, \mathbf{n}_A$  and restore the hypersurface  $\Sigma_{p+1}$  up to translations and rotations of it as a whole. Therefore, we will use the Cartan  $\omega$ -forms as new generalized coordinates alternative to the world vectors  $\mathbf{x}(\xi^\mu)$  of the embedded hypersurfaces. The integrability conditions for the PDEs (3–4)

$$d \wedge \omega_A + \omega_A^B \wedge \omega_B = 0, \tag{11}$$

$$d \wedge \omega_A^B + \omega_A^C \wedge \omega_C^B = 0 \tag{12}$$

are the well-known Maurer–Cartan (M-C) equations of the Minkowski space [38]. We use here the symbols  $\wedge$  and  $d \wedge$  for the exterior product and exterior differential of the differential one-forms  $\Phi$  and  $\Psi$ ,

$$\Phi \wedge \Psi := \frac{1}{2} \Phi_{[\mu} \Psi_{\nu]} d\xi^\mu \wedge d\xi^\nu,$$

$$d \wedge \Phi := \frac{1}{2} (\partial_\nu \Phi_\mu - \partial_\mu \Phi_\nu) d\xi^\nu \wedge d\xi^\mu = \frac{1}{2} \partial_{[\nu} \Phi_{\mu]} d\xi^\nu \wedge d\xi^\mu,$$

where  $\Phi_{[\mu} \Psi_{\nu]} := \Phi_\mu \Psi_\nu - \Phi_\nu \Psi_\mu$  and  $d\xi^\mu \wedge d\xi^\nu := d\xi^\mu \delta\xi^\nu - \delta\xi^\mu d\xi^\nu$ .

Equations (11–12) are the key input which allows one to construct the promised gauge model of the  $p$ -branes. We carry out in two steps the construction of the model.

In the first step in Sect. 3 we shall construct the gauge model compatible only with Eq. (12) because they form a closed system of PDEs for the spin connection one-forms  $\omega_A^B$ . In the second step, realized in Sect. 4, we shall take into account the remaining M-C equations (11), which establish relations between the hypersurface metric and its spin connection.

The field content of (12) becomes clear after the splitting of the matrix indices into the components tangent and normal to  $\Sigma_{p+1}$ , as prescribed by (5). Then Eq. (12) take the form of the field constraints

$$F_{\mu\nu i}^k = -(W_{[\mu} W_{\nu]})_i^k, \tag{13}$$

$$H_{\mu\nu a}^b = -(W_{[\mu} W_{\nu]})_a^b, \tag{14}$$

$$(D_{[\mu} W_{\nu]})_i^a = 0 \tag{15}$$

which yield the desired reformulation of the Gauss–Codazzi equations in terms of the gauge and massless vector fields  $W_{\mu i}^a$  associated with the embedded hypersurfaces. They generalize the gauge constraints [33–36] associated with the string worldsheets ( $p = 1$ ) to the constraints for the fields describing  $(p + 1)$ -dimensional hypersurfaces embedded into the Minkowski space.

For the case of a string these constraints together with (11) and the string EOM select the exactly solvable sector of states of the two-dimensional  $SO(1, 1) \times SO(D - 2)$  gauge-invariant model [33–36]. The model includes a massless scalar multiplet interacting with the Yang–Mills fields. This hints at the existence of a  $(p + 1)$ -dimensional  $SO(1, p) \times SO(D - p - 1)$ -invariant gauge model with the extremals of its EOM compatible with the constraints (13–15).

In accordance with our two-step procedure, the  $(p + 1)$ -dimensional  $SO(1, p) \times SO(D - p - 1)$  gauge model of the first step [compatible with (13–15)] will not still fix the world hypersurfaces of the  $p$ -branes, because Eq. (5) have not been taken into account. Thus, this gauge model has to be qualified as a gauge model in a fiber bundle space with a  $(p + 1)$ -dimensional curved space-time as a base manifold, and the  $SO(1, p) \times SO(D - p - 1)$  group forming the fibers and treated as an internal gauge symmetry. This treatment has strong intersections with the approach proposed in [50, 51] to describe gravitation as a gauge theory, which will be discussed in the next section.

An attempt to treat  $(p + 1)$ -dimensional gravity as a dynamical system in  $D$ -dimensional Minkowski space described by a  $p$ -brane interacting with the Kalb–Ramond field [52–54] was made in [55]. The authors applied the gauge technique of the embedding approach used in our papers [33–36] for the string theory changing its group  $SO(1, 1) \times SO(D - 2)$  into the  $p$ -brane group  $SO(1, p) \times SO(D - p - 1)$  in a way similar to the one considered in this section. In the capacity of the gravity action they intended to use the new  $p$ -brane action generalizing the string/brane actions earlier constructed by Volkov and Zheltukhin [56,57] and Bandos and Zheltukhin [58,59]. A characteristic feature of these actions is the fact that they use *new constituents* presented by the split components of the moving frame  $\mathbf{n}_A = (\mathbf{n}_i, \mathbf{n}_a)$  (1), identical to the Lorentz harmonics  $(u_m^a, u_m^i)$ , in addition to the  $p$ -brane vielbein, as well as the pull-back of the Minkowski space vielbein. Variation of the  $p$ -brane action [55] in these *harmonic variables* reproduced Eqs. (3–4) and their integrability conditions (11–12), respectively. It is just the point from which we *start our approach* in this section following Cartan’s description of embedded hypersurfaces. The remaining variation [55] in the Minkowski world vector added an expression for the extrinsic curvature in terms of the Kalb–Ramond field and the  $u$ -harmonics. These data combined with the well-known theorem [27] on the embedding of an arbitrary  $(p + 1)$ -dimensional manifold into  $D$ -dimensional Minkowski space with  $D \geq \frac{(p+1)(p+2)}{2}$  connect  $p$ -branes with  $(p + 1)$ -dimensional gravity, and they may be considered complementary to the observations presented below.

### 3 $SO(1, p) \times SO(D - p - 1)$ -invariant gauge model

The *first-step*  $SO(1, p) \times SO(D - p - 1)$  gauge-invariant model in curved  $(p + 1)$ -dimensional space-time with the coordinates  $\xi^\mu$  is defined by the action

$$\begin{aligned}
 S &= \gamma \int d^{p+1} \xi \sqrt{|g|} \mathcal{L} \\
 &= \gamma \int d^{p+1} \xi \sqrt{|g|} \left[ \frac{1}{4} Sp(F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} Sp(H_{\mu\nu} H^{\mu\nu}) \right. \\
 &\quad \left. + \frac{1}{2} \hat{\nabla}_\mu W_\nu^{ia} \hat{\nabla}^{[\mu} W_{ia}^{\nu]} - \hat{\nabla}_\mu W^{\mu ia} \hat{\nabla}_\nu W_{ia}^\nu + V \right], \quad (16)
 \end{aligned}$$

where  $g_{\mu\nu}$  is a given pseudo-Riemannian metric,  $V$  is an unknown potential term describing generally the gauge-invariant self-interaction of  $W_\mu^{ia}$ .

The general and gauge-covariant derivative  $\hat{\nabla}_\mu$  in (16),

$$\hat{\nabla}_\mu W_{via} = \partial_\mu W_{via} - \Gamma_{\mu\nu}^\rho W_{\rho ia} + A_{\mu i}^k W_{vka} + B_{\mu a}^b W_{vib}, \quad (17)$$

differs from the conventionally used general covariant derivative

$$\nabla_\mu W_{via} = \partial_\mu W_{via} - \Gamma_{\mu\nu}^\rho W_{\rho ia}, \quad \nabla_\mu g_{\nu\rho} = 0, \quad (18)$$

which includes only the Levi-Civita connection  $\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho = \frac{1}{2} g^{\rho\gamma} (\partial_\mu g_{\nu\gamma} + \partial_\nu g_{\mu\gamma} - \partial_\gamma g_{\mu\nu})$ . While constructing the kinetic part of the vector matrix field  $\hat{W}_\mu$  in (16) we kept in mind the analogy with the Ginzburg–Landau Lagrangian [60]. This analogy presupposed the replacement of the massive G-L scalar field by the massless vector field  $\hat{W}_\mu$  and the transition from the Minkowski space to a curved space-time. Such a generalization has an arbitrariness connected with the possibility to build three different invariants from the first-order derivatives of  $\hat{W}_\mu$ :  $Sp(\hat{\nabla}_\mu W_\nu \hat{\nabla}^\mu W^\nu)$ ,  $Sp(\hat{\nabla}_\mu W_\nu \hat{\nabla}^\nu W^\mu)$ , and  $Sp(\hat{\nabla}_\mu W^\mu \hat{\nabla}_\nu W^\nu)$ . We included these invariants into a generalized G-L Lagrangian with three arbitrary phenomenological constants, and we compared the corresponding EOM with the constraints (13–15) requiring their compatibility. This procedure uniquely fixed the phenomenological constants and resulted in the kinetic terms for  $\hat{W}_\mu$  in (16), where  $\hat{\nabla}^{[\mu} W^{\nu]}$  :=  $\hat{\nabla}^\mu W^\nu + \hat{\nabla}^\nu W^\mu$ , and the gauge fields, respectively. The corresponding Euler–Lagrange equations for the gauge and vector fields following from (16) with the kinetic part we found are

$$\hat{\nabla}_\mu F_{ik}^{\mu\nu} = -\hat{\nabla}_\mu (W_{ia}^{[\mu} W^{\nu]a}{}_k) - \frac{1}{2} W_{\mu[ia} \hat{\nabla}^{[v} W^{\mu]a}{}_{|k]}, \quad (19)$$

$$\hat{\nabla}_\mu H_{ab}^{\mu\nu} = -\hat{\nabla}_\mu (W_{ai}^{[\mu} W^{\nu]i}{}_b) - \frac{1}{2} W_{\mu[a|i} \hat{\nabla}^{[v} W^{\mu]i}{}_{|b]}, \quad (20)$$

$$\hat{\nabla}_\mu \hat{\nabla}^{[\mu} W^{\nu]ia} = 2\hat{\nabla}^\nu \hat{\nabla}_\mu W^{\mu ia} + \frac{\partial V}{\partial W_{via}}. \quad (21)$$

The potential  $V$  can also be found from the compatibility of these equations with the constraints (13–15). To simplify these calculations we introduce the shifted field strengths  $\mathcal{F}_{ik}^{\mu\nu}$  and  $\mathcal{H}_{ab}^{\mu\nu}$ ,

$$\mathcal{F}_{\mu\nu}^{ik} := (F_{\mu\nu} + W_{[\mu} W_{\nu]})^{ik}, \quad (22)$$

$$\mathcal{H}_{\mu\nu}^{ab} := (H_{\mu\nu} + W_{[\mu} W_{\nu]})^{ab} \quad (23)$$

and rewrite Eqs. (19–21) in the equivalent form

$$\hat{\nabla}_\mu \mathcal{F}_{ik}^{\mu\nu} = -\frac{1}{2} W_{\mu[ia} \hat{\nabla}^{[v} W^{\mu]a}{}_{|k]}, \quad (24)$$

$$\hat{\nabla}_\mu \mathcal{H}_{ab}^{\mu\nu} = -\frac{1}{2} W_{\mu[a|i} \hat{\nabla}^{[v} W^{\mu]i}{}_{|b]}, \quad (25)$$

$$\hat{\nabla}_\mu \hat{\nabla}^{[\mu} W^{\nu]ia} = -2[\hat{\nabla}^\mu, \hat{\nabla}^\nu] W_\mu^{ia} + \frac{\partial V}{\partial W_{via}}. \quad (26)$$

This representation permits to use the first generalized Bianchi identities,

$$[\hat{\nabla}_\mu, \hat{\nabla}_\nu] = \hat{R}_{\mu\nu} + \hat{F}_{\mu\nu} + \hat{H}_{\mu\nu}, \quad (27)$$

where the Riemann–Christoffel tensor for the background metric  $g_{\mu\nu}$  is

$$R_{\mu\nu}{}^\gamma{}_\lambda V^\lambda =: \left( \partial_{[\mu} \Gamma_{\nu]\lambda}^\gamma + \Gamma_{[\mu|\rho}^\gamma \Gamma_{|\nu]\lambda}^\rho \right) V^\lambda = [\nabla_\mu, \nabla_\nu] V^\gamma. \tag{28}$$

As a result, Eq. (26) for  $W_\mu^{ia}$  acquire the form

$$\begin{aligned} & \frac{1}{2} \hat{\nabla}_\mu \hat{\nabla}^{[\mu} W^{\nu]ia} - \mathcal{F}^{\mu\nu i}{}_k W_\mu^{ka} - \mathcal{H}^{\mu\nu a}{}_b W_\mu^{ib} \\ &= \frac{1}{2} \frac{\partial V}{\partial W_{\nu ia}} + ([W^\mu, W^\nu], W_\mu)^{ia} - R^{\mu\nu} W_\mu^{ia}, \end{aligned} \tag{29}$$

where  $R_{\nu\lambda} := R^\mu{}_{\nu\mu\lambda}$  is the Ricci tensor. Further we use the relation

$$\frac{1}{4} \frac{\partial}{\partial W_{\nu ia}} (W_\mu [[W^\mu, W^\rho], W_\rho])^i{}_i = ([W^\mu, W^\nu], W_\mu)^{ia} \tag{30}$$

including the commutators of  $\hat{W}_\mu$  and introduce the shifted potential  $\mathcal{V}$

$$\mathcal{V} := V + \frac{1}{2} Sp(W_\mu [[W^\mu, W^\rho], W_\rho]), \tag{31}$$

where the trace  $Sp(W_\mu [[W^\mu, W^\rho], W_\rho]) \equiv (W_\mu [[W^\mu, W^\rho], W_\rho])^i{}_i$ .

Then Eqs. (24), (25), and (29) of the model are represented in the form

$$\hat{\nabla}_\mu \mathcal{F}_{ik}^{\mu\nu} = -\frac{1}{2} W_{\mu[i|a} \hat{\nabla}^{[\nu} W^{\mu]a}{}_{|k]}, \tag{32}$$

$$\hat{\nabla}_\mu \mathcal{H}_{ab}^{\mu\nu} = -\frac{1}{2} W_{\mu[a|i} \hat{\nabla}^{[\nu} W^{\mu]i}{}_{|b]}, \tag{33}$$

$$\begin{aligned} & \frac{1}{2} \hat{\nabla}_\mu \hat{\nabla}^{[\mu} W^{\nu]ia} + \mathcal{F}^{\mu\nu i}{}_k W_\mu^{ka} + \mathcal{H}^{\mu\nu a}{}_b W_\mu^{ib} \\ &= \frac{1}{2} \frac{\partial \mathcal{V}}{\partial W_{\nu ia}} - R^{\mu\nu} W_\mu^{ia}, \end{aligned} \tag{34}$$

suitable for a comparison with the Gauss–Codazzi constraints (13–15).

Indeed, in terms of the shifted field strengths  $\hat{\mathcal{F}}$  (22) and  $\hat{\mathcal{H}}$  (23), the G-C constraints (13–15) acquire a simple form:

$$\mathcal{F}_{\mu\nu}^{ik} = 0, \quad \mathcal{H}_{\mu\nu}^{ab} = 0, \quad \hat{\nabla}^{[\mu} W_{ia}^{\nu]} = 0, \tag{35}$$

compatible with Eqs. (32–34) provided that  $\mathcal{V}$  satisfies the condition

$$\frac{1}{2} \frac{\partial \mathcal{V}}{\partial W_{\nu ia}} - R^{\mu\nu} W_\mu^{ia} = 0. \tag{36}$$

Due to the independence of the background Ricci tensor  $R^{\mu\nu}$  of  $W_{\nu ia}$  we find the general solution of (36),

$$\mathcal{V} = R^{\mu\nu} W_\mu^{ia} W_{\nu ia} - c, \tag{37}$$

where the integration constant  $c$  is proportional to the cosmological constant connected with the background metric

$g_{\mu\nu}(\xi^\rho)$ . The substitution of  $\mathcal{V}$  (37) into (36) transforms it into the identity for any  $R^{\mu\nu}$  independent of  $\hat{W}_\mu$ . In view of this, various restrictions on  $R^{\mu\nu}$ , including the Bianchi identity, will not lead to a new relation on  $\hat{W}_\mu$ . Thus, Eq. (36) comes to us as the necessary condition for the potential  $\mathcal{V}$  needed to consider (35) as some initial data for Eqs. (32–34).

Therefore, we obtain the sought for Lagrangian of the  $SO(1, p) \times SO(D - p - 1)$ -invariant gauge model (16) compatible with the G-C constraints (35)

$$\begin{aligned} \mathcal{L} = & \frac{1}{4} Sp(F_{\mu\nu} F^{\mu\nu}) - \frac{1}{4} Sp(H_{\mu\nu} H^{\mu\nu}) \\ & + \frac{1}{2} \hat{\nabla}_\mu W_\nu^{ia} \hat{\nabla}^{[\mu} W_{ia}^{\nu]} - \hat{\nabla}_\mu W^{\mu ia} \hat{\nabla}_\nu W_{ia}^\nu \\ & + R^{\mu\nu} W_\mu^{ia} W_{\nu ia} - \frac{1}{2} Sp(W_\mu [[W^\mu, W^\rho], W_\rho]) + c, \end{aligned} \tag{38}$$

which produces the following Euler–Lagrange equations:

$$\hat{\nabla}_\mu \mathcal{F}_{ik}^{\mu\nu} = -\frac{1}{2} W_{\mu[i|a} \hat{\nabla}^{[\nu} W^{\mu]a}{}_{|k]}, \tag{39}$$

$$\hat{\nabla}_\mu \mathcal{H}_{ab}^{\mu\nu} = -\frac{1}{2} W_{\mu[a|i} \hat{\nabla}^{[\nu} W^{\mu]i}{}_{|b]}, \tag{40}$$

$$\frac{1}{2} \hat{\nabla}_\mu \hat{\nabla}^{[\mu} W^{\nu]ia} + \mathcal{F}^{\mu\nu i}{}_k W_\mu^{ka} + \mathcal{H}^{\mu\nu a}{}_b W_\mu^{ib} = 0 \tag{41}$$

for the gauge  $A_{\mu i}{}^k$ ,  $B_{\mu a}{}^b$  and the vector  $W_{\mu ia}$  fields in a curved background.

The Lagrangian (38) resembles the MacDowell and Mansouri one [61], which describes the Hilbert–Einstein Lagrangian  $L$  in terms of gauge fields of the group  $Sp(4)$  including its subgroup  $SO(1, 3)$  as an exact symmetry of  $L$ . In their approach the components of the vierbein appear as the gauge fields  $h_\mu^i$  associated with the spontaneously broken local translations of  $Sp(4)$ .

In our approach the components of  $W_\mu^{ia}$  realizing the Nambu–Goldstone modes of the  $SO(1, D - 1)$  symmetry spontaneously broken to its subgroup  $SO(1, p) \times SO(1, D - p - 1)$  have an analogous physical sense. In view of this analog of the H-E Lagrangian [61] in (38) it is represented by the sum of  $R^{\mu\nu} W_\mu^{ia} W_{\nu ia}$  and the quartic monomials forming the potential  $V$ . However, in Eq. (38) there are additional kinetic terms, one of which,  $Sp(F_{\mu\nu} F^{\mu\nu})$ , has an opposite sign. This points to the presence of ghosts produced by the spin connection gauge field  $A_\mu^{ik}$ . To exclude the ghosts we recall that the Lagrangian (38) is only the first-step Lagrangian, constructed without taking into account the M-C equations (11). As mentioned above, Eq. (11) just link the spin connection with the vielbein components  $\omega_\mu^i$ . Thus, to transform the Lagrangian (38) to the second-step Lagrangian really describing the hypersurfaces of the  $p$ -branes it is necessary to connect the ad hoc introduced metric  $g_{\mu\nu}(\xi^\rho)$  in (38) with  $A_\mu^{ik}$ . As a result, the ghost term transforms into an additional

term in the potential  $V$ , as shown in the next section, solving the ghost problem. Therefore, we will not discuss the Lagrangian (38) anymore, but we only note that it looks like a natural generalization of the 4-dimensional Dirac scale-invariant gravity theory [62] (see also [63]).

To develop this observation one can weaken the requirement for the gravitational field  $g_{\mu\nu}(\xi^\rho)$  to be treated as an external one. Then Eqs. (39–41) have to be completed by the variational equations with respect to  $g_{\mu\nu}$ . These equations connect  $g_{\mu\nu}$  with the gauge field strengths,  $W_{\mu ia}$ , and their covariant derivatives, and they may provide an alternative solution of the ghost problem. This requires an additional investigation.

Below we build the *second-step* action following from the modification of  $S$  (16) caused by the M-C equations (11). This modification yields a gauge-invariant action associated with the *minimal*  $p$ -branes in the Minkowski space.

#### 4 Branes as solutions of the gauge model

Since the Gauss–Codazzi equations (35) define some extremals of  $S$  (16) with the Lagrangian density (38) in a  $(p + 1)$ -dimensional curved space-time, it may be treated as the  $p$ -brane hypersurface  $\Sigma_{(p+1)}$ . For this purpose, we have to *identify* the metric  $g_{\mu\nu}$  of the  $(p + 1)$ -dimensional *space* introduced in (16) with the metric of the  $(p + 1)$ -dimensional *world hypersurface* swept by the  $p$ -brane in a  $D$ -dimensional Minkowski space. The identification requires one to take into account the remaining Maurer–Cartan equations (11), which connect the  $(p + 1)$ -bein  $\omega^i_\mu$  of the hypersurface  $\Sigma_{p+1}$  with the gauge and vector fields of the model (16).

In terms of these fields Eqs. (11) transform into the constraints

$$D_{[\mu}^{\parallel} \omega^i_{\nu]} \equiv \partial_{[\mu} \omega^i_{\nu]} + A_{[\mu}{}^i{}_k \omega_{\nu]}^k = 0, \tag{42}$$

$$\omega^i_{[\mu} W_{\nu]ia} = 0, \tag{43}$$

additionally to the G-C constraints (13–15). The vielbein  $\omega^i_\mu$  connects the orthonormal moving frame  $\mathbf{n}_i$  with the natural frame  $\mathbf{e}_\mu$ . Therefore, the metric tensor  $G_{\mu\nu}(\xi^\rho)$  of the hypersurface is represented as

$$\mathbf{e}_\mu = \omega^i_\mu \mathbf{n}_i, \quad G_{\mu\nu} = \omega^i_\mu \eta_{ik} \omega^k_\nu, \quad \omega^i_\mu \omega^{\mu k} = \delta^i_k \tag{44}$$

and is identified with the background metric  $g_{\mu\nu}(\xi^\rho) \equiv G_{\mu\nu}(\xi^\rho)$  of the model (16).

The general solution of the algebraic constraints (43),

$$W_{\mu i}{}^a = -l_{\mu\nu}{}^a \omega^{\nu}_i, \quad l_{\mu\nu}{}^a := \mathbf{n}^a \frac{\partial^2 \mathbf{x}}{\partial \xi^\mu \partial \xi^\nu} \equiv \mathbf{n}^a \partial_{\mu\nu} \mathbf{x}, \tag{45}$$

includes the *second fundamental form*  $l_{\mu\nu}{}^a(\xi^\rho)$  of the hypersurface  $\Sigma_{p+1}$ .

Equation (42) have the solution fixed by the conditions

$$\nabla_{\mu}^{\parallel} \omega^i_{\nu} \equiv \partial_{\mu} \omega^i_{\nu} - \Gamma_{\mu\nu}^{\rho} \omega_{\rho}^i + A_{\mu}{}^i{}_k \omega_{\nu}^k = 0, \tag{46}$$

which express the *tetrad postulate*, well known from general relativity and linking the affine and metric connections on  $\Sigma_{p+1}$ .

Thus, the metric connection  $\Gamma_{\mu\nu}^{\rho}$  and  $\hat{A}_{\mu}$  turn out to be identified by means of the gauge transformation

$$\Gamma_{\nu\lambda}^{\rho} = \omega^{\rho}_i A_{\nu}{}^i{}_k \omega_{\lambda}^k + \partial_{\nu} \omega_{\lambda}^k \omega_k^{\rho} \equiv \omega^{\rho}_i D_{\nu}^{\parallel} \omega_{\lambda}^i, \tag{47}$$

where the transformation function  $\omega^i_\mu$  (44) coincides with the  $(p + 1)$ -bein (44) of the hypersurface. Equation (47) permit us to express the gauge field  $A_{\mu i}{}^k$  and its strength  $F_{\mu\nu i}{}^k$  by means of  $\Gamma_{\nu\lambda}^{\rho}$  and the Riemann tensor  $R_{\mu\nu}{}^{\gamma}{}_{\lambda}$  (28)

$$A_{\nu}{}^l{}_m = \omega^l_{\rho} \Gamma_{\nu\lambda}^{\rho} \omega^{\lambda m} + \omega^l_{\lambda} \partial_{\nu} \omega^{\lambda m}, \tag{48}$$

$$F_{\mu\nu}{}^i{}_k = \omega^i_{\gamma} R_{\mu\nu}{}^{\gamma}{}_{\lambda} \omega_k^{\lambda}, \tag{49}$$

respectively. Equation (49) shows the transformation of the *ghost kinetic term* in the Lagrangian (38) to the quadratic term into the Riemann tensor of curvature:  $\frac{1}{4} Sp(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4} R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda}$ , with the change of sign.

Then we have to substitute the metric connection  $\Gamma_{\nu\lambda}^{\rho}$  for the gauge field  $A_{\nu ik}$  into the G-C constraints (13–15). This substitution must be done together with the substitution of the *massless tensor* field  $l_{\mu\nu}{}^a = -\omega^i_{\nu} W_{\mu i}{}^a$  instead of the vector field  $W_{\mu i}{}^a$  using the relation (45).

As a result, the G-C constraints (13–15) are transformed into

$$R_{\mu\nu}{}^{\gamma}{}_{\lambda} = l_{[\mu}{}^{\gamma a} l_{\nu]\lambda a}, \tag{50}$$

$$H_{\mu\nu}{}^{ab} = l_{[\mu}{}^{\gamma a} l_{\nu]\gamma}{}^b, \tag{51}$$

$$\nabla_{[\mu}^{\perp} l_{\nu]\rho a} = 0, \tag{52}$$

containing the general and  $SO(D - p - 1)$ -covariant derivative  $\nabla_{\mu}^{\perp}$

$$\nabla_{\mu}^{\perp} l_{\nu\rho}{}^a := \partial_{\mu} l_{\nu\rho}{}^a - \Gamma_{\mu\nu}^{\lambda} l_{\lambda\rho}{}^a - \Gamma_{\mu\rho}^{\lambda} l_{\nu\lambda}{}^a + B_{\mu}{}^{ab} l_{\nu\rho b}. \tag{53}$$

So, we observe the reduction of the M-C equations (11–12) to the modified field constraints (50–52). It should be emphasized that the constraint (50) expresses the generalization of the *Gauss Theorema Egregium* for *surfaces* in  $D = 3$  to  $(p + 1)$ -dimensional *hypersurfaces* embedded into a  $D$ -dimensional Minkowski space.

After the exclusion of  $\hat{A}_{\nu}$  and the substitution of  $l_{\nu\rho}{}^a$  for  $W_{\mu i}{}^a$  in (38) we obtain a  $(p + 1)$ -dimensional  $SO(D - p - 1)$ -invariant gauge action describing hypersurfaces equipped by the metric  $g_{\mu\nu}$ . The reduced action must have extremals compatible with the constraints (50–52). These constraints show that the ghost’s kinetic and  $R^{\mu\nu} W_{\mu}^{ia} W_{\nu ia}$  terms in (38) can be transformed into the terms quartic in  $l_{\mu\nu}^a$  and consequently

shifted to the potential  $V$ . Thus, the  $SO(D - p - 1)$ -invariant *second-step* action should have the form

$$S = \gamma \int d^{p+1} \xi \sqrt{|g|} \left\{ -\frac{1}{4} Sp(H_{\mu\nu} H^{\mu\nu}) + \frac{1}{2} \nabla_{\mu}^{\perp} l_{\nu\rho a} \nabla^{\perp[\mu} l^{\nu]\rho a} - \nabla_{\mu}^{\perp} l_{\rho a}^{\mu} \nabla^{\perp} l^{\nu\rho a} + V \right\}. \tag{54}$$

Variation of (54) in the dynamical fields  $l_{\mu\nu}^a$ ,  $B_{\mu}^{ab}$  gives their EOM

$$\nabla_{\nu}^{\perp} \mathcal{H}_{ab}^{\nu\mu} = \frac{1}{2} l_{\nu\rho[a} \nabla^{\perp[\mu} l^{\nu]\rho}{}_{b]}, \tag{55}$$

$$\frac{1}{2} \nabla_{\mu}^{\perp} \nabla^{\perp[\mu} l^{\nu]\rho a} = -[\nabla^{\perp\mu}, \nabla^{\perp\nu}] l_{\mu}^{\rho a} + \frac{1}{2} \frac{\partial V}{\partial l_{\nu\rho a}}, \tag{56}$$

where  $\mathcal{H}_{\mu\nu}^{ab} := H_{\mu\nu}^{ab} - l_{[\mu}^{\gamma a} l_{\nu]\gamma}^b$ . Equations (55–56) have the G-C constraints (51)–(52)

$$\mathcal{H}_{\mu\nu}^{ab} = 0, \quad \nabla_{[\mu}^{\perp} l_{\nu]\rho a} = 0, \tag{57}$$

as their particular solution, provided that

$$\frac{1}{2} \frac{\partial V}{\partial l_{\nu\rho a}} = [\nabla^{\perp\mu}, \nabla^{\perp\nu}] l_{\mu}^{\rho a}. \tag{58}$$

Using the Bianchi identities for  $\nabla^{\perp\mu}$  (53) we represent the r.h.s. of (58) as

$$[\nabla_{\gamma}^{\perp}, \nabla_{\nu}^{\perp}] l^{\mu\rho a} = R_{\gamma\nu}{}^{\mu}{}_{\lambda} l^{\lambda\rho a} + R_{\gamma\nu}{}^{\rho}{}_{\lambda} l^{\mu\lambda a} + H_{\gamma\nu}{}^a{}_b l^{\mu\rho b}. \tag{59}$$

Further, the use of the G-C representation of the Riemann tensor  $R_{\gamma\nu}{}^{\mu}{}_{\lambda}$  (50) and the field strength  $H_{\gamma\nu}{}^a{}_b$  (51) permits one to express the r.h.s. of (59) in the form of a cubic polynomial in  $l^{\mu\nu a}$ . The substitution of the polynomial into Eqs. (58) transforms them into the analytically solvable system of conditions

$$\frac{1}{2} \frac{\partial V}{\partial l_{\nu\rho a}} = (l^a l^b)^{\rho\nu} Sp(l_b) + \left( 2l_b l^a l^b - l^a l_b l^b - l_b l^b l^a \right)^{\rho\nu} - l^{\rho\nu b} Sp(l_b l^a), \tag{60}$$

where  $Sp(l^a) = g^{\mu\nu} l_{\mu\nu}^a$  and  $Sp(l^a l^b) = g^{\mu\nu} l_{\mu\rho}^a l^{b\rho\nu}$ .

Equation (60) have the following solution:

$$V = -\frac{1}{2} Sp(l_a l_b) Sp(l^a l^b) + Sp(l_a l_b l^a l^b) - Sp(l_a l^a l_b l^b) + c, \tag{61}$$

$$Sp(l^a) = 0, \tag{62}$$

fixing the potential term  $V$  and the trace of the second form  $l_{\mu\nu}^a$ .

Equation  $Sp(l^a) = 0$  are the well-known *minimality* conditions for the world volume of the  $(p + 1)$ -dimensional hypersurface. They are *equivalent* to the EOM [25,26] of

the fundamental  $p$ -branes in the  $D$ -dimensional Minkowski space,

$$\square^{(p+1)} \mathbf{x} = 0, \tag{63}$$

where  $\square^{(p+1)}$  is the invariant Laplace–Beltrami operator on  $\Sigma_{p+1}$ ,

$$\square^{(p+1)} := \frac{1}{\sqrt{|g|}} \partial_{\alpha} \sqrt{|g|} g^{\alpha\beta} \partial_{\beta}.$$

The equivalence of Eq. (62) for the  $l$ -traces (45)–(63) follows from Eq. (3), showing orthogonality between  $\mathbf{n}^a$  and the vectors  $\partial_{\beta} \mathbf{x}$  tangent to  $\Sigma_{p+1}$ :  $\mathbf{n}^a \frac{\partial \mathbf{x}}{\partial \xi^{\beta}} = 0$ . Thus the metric connection contribution to  $Sp(l^a)$  vanishes.

Equation (63) follows from the Dirac action for the  $p$ -branes,

$$S = T \int d^{p+1} \xi \sqrt{|g|}, \tag{64}$$

where  $g$  is the determinant of the induced metric  $g_{\alpha\beta} := \partial_{\alpha} \mathbf{x} \partial_{\beta} \mathbf{x}$ .

This proves that the  $SO(D - p - 1)$  gauge-invariant model

$$S = \gamma \int d^{p+1} \xi \sqrt{|g|} \mathcal{L},$$

$$\mathcal{L} = -\frac{1}{4} Sp(H_{\mu\nu} H^{\mu\nu}) + \frac{1}{2} \nabla_{\mu}^{\perp} l_{\nu\rho a} \nabla^{\perp[\mu} l^{\nu]\rho a} - \nabla_{\mu}^{\perp} l_{\rho a}^{\mu} \nabla^{\perp} l^{\nu\rho a} - \frac{1}{2} Sp(l_a l_b) Sp(l^a l^b) + Sp(l_a l_b l^a l^b) - Sp(l_a l^a l_b l^b) + c \tag{65}$$

for the gauge  $B_{\mu}^{ab}$  and the tensor  $l_{\mu\nu}^a$  fields in the background  $g_{\mu\nu}$  possesses the solutions (50–52) and (62). The latter describe electrically and magnetically neutral Dirac  $p$ -branes with minimal world volumes. The cosmological constant  $\gamma c$  in (65) is not essential in the considered case of the *background* gravitational field  $g_{\mu\nu}$ . However, it becomes a significant parameter in the process of quantization of the model.

The zero-mode structure of the gauge model (65) is defined by rigid symmetries of the primary equations (3–4) for the collective coordinates  $\mathbf{x}(\xi^{\mu})$  of the model. It does not require explicit solution of the Maurer–Cartan equations (50–52). The latter are the integrability conditions of Eqs. (3–4) invariant under rigid translations and rotations of  $\mathbf{x}(\xi^{\mu})$ . As a result, the world vector  $\mathbf{x}(\xi^{\mu})$  is restored up to global translations and rotations. Thus, we obtain an infinitely degenerate family of branes with the same energy. The Dirac action (64) invariant under the mentioned symmetries just realizes the classical world volume theory for such world volume-minimizing configurations and does not include other terms.

One might ask whether the extremals defined by the first-order PDEs (50–52) with  $Sp l_a = 0$  can be interpreted as solitons; it is necessary to know their explicit time dependence. But, in view of the brane  $\leftrightarrow$  GFT map, this question reduces to the problem of the existence of solitonic solutions for the nonlinear wave equations (63). The exact solutions of Eq. (63) found in [25,26] include, in particular, the static ones, which may be understood as solitons. However, this gives rise to certain objections, since these static solutions have either an infinite total energy for infinitely extended branes, or a singular world volume metric for the closed ones. The absence of regular static solutions with a finite energy is explained by the observation that the branes undergo the action of anharmonic elastic forces tending to contract them. This instability does not occur if the branes rotate, and the centrifugal force is sufficient to compensate for the elastic force. Another way for compensating the brane instability is the introduction of additional forces (fluxes) that, however, will modify the original Dirac action (64). These arguments also explain why the constraints (50–52) and (62) cannot be treated as BPS-like conditions. In its turn, the existence of such conditions strongly depends on the boundary conditions for the gauge-invariant action (65).

## 5 Summary

We unified the Gauss map with the brane dynamics and introduced new dynamical variables alternative to the world vectors  $\mathbf{x}$ . These variables have a clear interpretation as the Yang–Mills fields interacting with massless multiplets in curved backgrounds. The  $(p+1)$ -dimensional gauge-invariant models including these fields were constructed, and it was proven that the  $(p+1)$ -dimensional hypersurfaces embedded into  $D$ -dimensional Minkowski space are exact extremals of their Euler–Lagrange equations.

At the first step we built the  $SO(1, p) \times SO(D - p - 1)$  gauge-invariant model, where the spin and metric connections were treated as independent ones in a bundle space. This model describes interactions of the Yang–Mills fields with the massless vector multiplet in a  $(p+1)$ -dimensional curved background. The model has the ghost degrees of freedom carried by the  $SO(1, p)$  gauge fields associated with the spin connection. To cancel the ghosts we used the tetrad postulate identifying the metric and the  $SO(1, p)$  connections. As a result, the  $SO(1, p) \times SO(D - p - 1)$  gauge model of the first step was reduced to the  $SO(D - p - 1)$ -invariant gauge model of interacting gauge and massless tensor fields in the gravitational background. Then we found the exact solution of this gauge model, which described the hypersurfaces characterized by minimal  $(p+1)$ -dimensional world volumes. We identified these hypersurfaces as the funda-

mental Dirac  $p$ -branes embedded into the  $D$ -dimensional Minkowski space.

The gauge approach presented reformulates the problem of the fundamental  $p$ -brane quantization to that of the  $SO(D - p - 1)$  gauge-invariant model (65) along its Euler–Lagrange extremals constrained by the Gauss–Codazzi equations represented as the field constraints. This permits us to apply the well-known BFV–BRST and other methods of quantization of gauge theories to the quantization of the fundamental branes. This investigation is in progress.

**Acknowledgments** We would like to thank J. A. Brandenburg, D. Bykov, A. Golovnev, E. Ivanov, V. Kotlyar, M. Larsson, V. Pervushin, A. Rosly, V. Tugai, D. Uvarov, H. von Zur-Muhlen, and Y. Zinoviev for interesting discussions and help. We are grateful to the Physics Department of Stockholm University and Nordic Institute for Theoretical Physics NORDITA for kind hospitality and financial support.

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