Eur. Phys. J. Spec. Top. (2023) 232:3637–3643
https://doi.org/10.1140/epjs/s11734-023-00850-4

THE EUROPEAN
PHYSICAL JOURNAL
SPECIAL TOPICS

Review

Kronecker coefficients from algebras of bi-partite ribbon graphs

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Received 1 November 2022 / Accepted 25 April 2023 / Published online 26 May 2023

Abstract Bi-partite ribbon graphs arise in organizing the large N expansion of correlators in random matrix models and in the enumeration of observables in random tensor models. There is an algebra $\mathcal{K}(n)$, with basis given by bi-partite ribbon graphs with n edges, which is useful in the applications to matrix and tensor models. The algebra $\mathcal{K}(n)$ is closely related to symmetric group algebras and has a matrix-block decomposition related to Clebsch–Gordan multiplicities, also known as Kronecker coefficients, for symmetric group representations. Quantum mechanical models which use $\mathcal{K}(n)$ as Hilbert spaces can be used to give combinatorial algorithms for computing the Kronecker coefficients.

1 Introduction

Kronecker coefficients are widely studied in mathematics from many points of view (e.g., symmetric polynomials, complexity theory, and combinatorics). In this contribution to the volume on non-commutativity and physics, motivated by work on tensor models in physics, we review how the structure of a class of non-commutative, associative algebras $\mathcal{K}(n)$ (one for each integer n) leads to new algorithms for computing Kronecker coefficients. These algorithms have a geometrical interpretation in terms of sub-lattices of a lattice whose basis vectors are labeled by bi-partite ribbon graphs. Bi-partite ribbon graphs themselves are related to holography in a fundamental way (which relates quantum field theory on one geometry to string theory on another geometry). The story we present has, as central figures, the algebra $\mathcal{K}(n)$ and stringy geometrical structures (lattices, holography). This can be viewed as part of a new direction in the study of non-commutative geometric structures in mathematical physics, which encompasses stringy holography as well as more direct approaches to the emergence of geometry from algebras in the traditional non-commutative geometry [1–3]. The standard mathematical approach to Kronecker coefficients is to think of them as the structure constants of the commutative fusion ring of representations of symmetric groups. Bringing the non-commutative associative algebra $\mathcal{K}(n)$ to bear on the Kronecker coefficients evidences the idea pioneered in the traditional non-commutative geometry, that underlying commutative mathematical structures, there are deeper non-commutative physical constructions which carry interesting hidden information.

Following the classic work of 't Hooft [4], it has been understood that the combinatorics of large gauge theories with gauge symmetries, such as $U(N)$, $SO(N)$, $andSp(N)$, simplifies in the large N limit. This simplification is related to the emergence of string world-sheet combinatorics in the $1/N$ expansion of physical observables, and underlies examples of gauge–string duality, such as the AdS/CFT correspondence [5]. These simplifications are based on the appearance of double-line diagrams or equivalently ribbon graphs in large N computations. Ribbon graphs are

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S.I.: Noncommutativity and Physics. Guest editors: George Zoupanos, Konstantinos Anagnostopoulos, and Peter Schupp.

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also related to the mathematics of holomorphic maps between two-dimensional surfaces, and in particular a special class of such maps called Belyi maps. This leads to simple mathematical models of gauge–string duality based on the correspondence between correlators of invariant matrix polynomials in matrix models (the gauge theory) and the counting of Belyi maps (which can be viewed as a combinatorial topological string theory) [6].

The solution of counting problems for tensor model observables can also be formulated in terms of ribbon graph counting [7, 8]. Specifically, we consider a complex tensor variable Φ_{ijk}, with $i, j, k \in \{1, 2, \cdots, N\}$ which transforms in the threefold tensor product $(V_N \otimes V_N \otimes V_N)$ of the fundamental representation V_N of $U(N)$. The space of polynomials of degree n in Φ and degree n in the complex conjugate $\bar{\Phi}$ contains a subspace of $U(N)$ invariants of dimension, which somewhat non-trivially turns out to be equal to the number of bi-partite ribbon graphs with n edges, for $N \geq n$. It was also understood [10] that there is an associative algebra $\mathcal{K}(n)$ with basis labeled by these bi-partite ribbon graphs. The dimension of the algebra, equivalently the number of bi-partite ribbon graphs with n edges, is equal to the sum of squares of Kronecker coefficients

$$\mathrm{Dim}(\mathcal{K}(n)) = \sum_{R_1, R_2, R_3 \vdash n} (C(R_1, R_2, R_3))^2 \quad (1)$$

for all triples R_1, R_2, R_3 of Young diagrams with n boxes. The algebra $\mathcal{K}(n)$ has a Fourier basis $Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$ labeled by triples of Young diagrams along with a pair of multiplicity indices τ_1, τ_2 which each range over $\{1, 2, \cdots, n\}$ [9, 10]. The Fourier basis is defined using Clebsch–Gordan coefficients of the symmetric group S_n and gives the Weddernurn–Artin decomposition of $\mathcal{K}(n)$ as a direct sum of matrix algebras, where the matrix algebras exist for triples of Young diagrams having non-vanishing Kronecker coefficient. The papers [11–13] have related results on tensor model observables with a similar algebraic perspective.

In the paper [7], we observed that the matrix subspace of $\mathcal{K}(n)$ associated with a triple of Young diagrams can be obtained as the eigenspace of a specified set of central elements in $\mathcal{K}(n)$, acting on $\mathcal{K}(n)$ by multiplication. This led to the realization of the square of the Kronecker coefficient for any triple (R_1, R_2, R_3) as the number of linearly independent null vectors of a certain combinatorially constructed integer matrix. Integer matrix algorithms give a constructive combinatorical algorithm for arriving at a basis in the space of null vectors. The construction has a natural interpretation in terms of quantum mechanics in a Hilbert space spanned by the bi-partite ribbon graphs with n edges. Furthermore, considering an involution operator on $\mathcal{K}(n)$, we arrive at the construction of the Kronecker coefficient itself, and therefore propose an answer to the old question of Murnaghan on a combinatorial interpretation of Kronecker coefficient [14, 15]. The complexity of the combinatorial algorithm is a left as a interesting problem for the future.

In Sect. 2, we describe the combinatorial model of gauge-string duality arising from complex matrix model correlators. In Sect. 3, we explain the quantum mechanics on the algebra of bi-partite ribbon graphs. In Sect. 4, we explain the construction of the integer matrix for every triple, whose null space has a dimension given by the square of the Kronecker coefficient for that triple. Further considerations lead us to the construction of several other pertinent integer sub-lattices in the lattice of bi-partite ribbon graphs. An interesting corollary is the identity (31) giving the sum of Kronecker coefficients for triples of Young diagrams with n boxes, in terms of the number of ribbon graphs invariant under an involution S defined in Sect. 4.

2 Bi-partite graphs and matrix model correlators

Consider a Gaussian complex matrix model of a complex matrix of size N with partition function

$$\mathcal{Z} = \int [dZ] e^{-\frac{1}{2} trZZ^\dagger}; \quad (2)$$

$[dZ] = \prod_{i,j} dZ^i_j d\bar{Z}^i_j$ is the standard measure on N^2 complex variables. Using standard formulae for multi-variable Gaussian integration, we find that the quadratic expectation value is

$$\langle Z^i_j (Z^\dagger)^k_l \rangle = \frac{1}{\mathcal{Z}} \int [dZ] e^{-\frac{1}{2} trZZ^\dagger} Z^i_j (Z^\dagger)^k_l = \delta^i_l \delta^k_j. \quad (3)$$

The complex matrix model measure is invariant under transformations by matrices U which are in the unitary group $U(N)$. The complex matrix model finds application in describing the half-BPS sector of the $\mathcal{N} = 4$ super-Yang Mills theory [16, 17]. Holomorphic gauge invariant polynomial functions of Z of degree n correspond to half-BPS quantum states of scaling dimension n. These can be parametrised using permutations in S_n

$$\mathcal{O}_\sigma(Z) = \sum_{i_1, \cdots, i_n} Z^{i_1}_{i_{\sigma(1)}} Z^{i_2}_{i_{\sigma(2)}} \cdots Z^{i_n}_{i_{\sigma(n)}} = \prod_{a=1}^{n} (\mathrm{tr} Z^a)^{C_a(\sigma)}, \quad (4)$$

where $C_a(\sigma)$ is the number of cycles of length a in σ. It is easy to verify that $\mathcal{O}_\sigma(Z) = \mathcal{O}_{\gamma\sigma\gamma^{-1}}(Z)$ for any $\gamma \in S_n$. The correlator of a holomorphic and an anti-holomorphic operator is defined by the integral

$$\langle \mathcal{O}_{\sigma_1}(Z) \mathcal{O}_{\sigma_2}(Z^\dagger) \rangle = \frac{1}{\mathcal{Z}} \int [dZ] e^{-\frac{1}{2} trZZ^\dagger} \mathcal{O}_{\sigma_1}(Z) \mathcal{O}_{\sigma_2}(Z^\dagger). \quad (5)$$

This is calculated to be [17, 18]

$$\langle \mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}(Z^\dagger)\rangle = \sum_{\sigma_3\in S_n}\sum_{\gamma\in S_n}\delta(\sigma_1\gamma\sigma_2\gamma^{-1}\sigma_3)N^{C_{\sigma_3}}$$
$$= \frac{n!}{|\mathcal{C}_1||\mathcal{C}_2|}\sum_{\sigma_1'\in\mathcal{C}_1}\sum_{\sigma_2'\in\mathcal{C}_2}\sum_{\sigma_3\in S_n}\delta(\sigma_1'\sigma_2'\sigma_3)N^{C_{\sigma_3}}. \quad (6)$$

Here, $\mathcal{C}_1$ is the conjugacy class of σ_1, $\mathcal{C}_2$ is the conjugacy class of σ_2, and C_{σ_3} is the number of cycles in the permutation σ_3. We have used $|\mathcal{C}_1|, |\mathcal{C}_2|$ to denote the sizes of the conjugacy classes $\mathcal{C}_1, \mathcal{C}_2$. We can also separate out the sum over σ_3 into all the conjugacy classes, labeled by $\mathcal{C}_3$ in S_n

$$\langle \mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}(Z^\dagger)\rangle$$
$$= \frac{n!}{|\mathcal{C}_1||\mathcal{C}_2|}\sum_{\sigma_1'\in\mathcal{C}_1}\sum_{\sigma_2'\in\mathcal{C}_2}\sum_{\mathcal{C}_3}\sum_{\sigma_3\in\mathcal{C}_3}\delta(\sigma_1'\sigma_2'\sigma_3)N^{C(\mathcal{C}_3)}. \quad (7)$$

In the last expression, we have used $C(\mathcal{C}_3)$ for the number of cycles in any permutation belonging to the conjugacy class $\mathcal{C}_3$.

It is convenient to normalize the observables as follows:

$$N^{-n}\frac{N^{C(\mathcal{C}_1)+C(\mathcal{C}_2)}|\mathcal{C}_1||\mathcal{C}_2|}{n!\,n!}\langle \mathcal{O}_{\sigma_1}(Z)\mathcal{O}_{\sigma_2}(Z^\dagger)\rangle$$
$$= \sum_{\mathcal{C}_3}\left(\frac{1}{n!}\sum_{\sigma_1'\in\mathcal{C}_1}\sum_{\sigma_2'\in\mathcal{C}_2}\sum_{\sigma_3\in\mathcal{C}_3}\mathcal{C}\delta(\sigma_1'\sigma_2'\sigma_3)N^{C(\mathcal{C}_1)+C(\mathcal{C}_2)+C(\mathcal{C}_3)-n}\right). \quad (8)$$

The sum for fixed conjugacy classes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ has a nice geometrical interpretation in terms of holomorphic maps from a two-dimensional surface to a sphere, with three branch points on the sphere. In the inverse image of the branch points, the branching structure is described by the conjugacy classes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. The genus h of the surface is given by

$$(2-2h) = C(\mathcal{C}_1) + C(\mathcal{C}_2) + C(\mathcal{C}_3) - n. \quad (9)$$

The formula (8) shows that the naturally normalized matrix model correlators can be interpreted as observables in a topological string with sphere (complex projective plane) target, where the string path integral is localized on holomorphic maps with three branch points, which can be chosen to be $0, 1, \infty$. Such maps are distinguished maps of interest in number theory, called Belyi maps. There are also nice combinatorial objects, namely bi-partite ribbon graphs associated with these Belyi maps. These can be thought of as the inverse image of the interval [0, 1]. For a review of the link between bi-partite ribbon graphs, and references to the original literature, we refer the reader to [6]. A good textbook discussion of bi-partite ribbon graphs and Belyi maps is in [19].

3 Counting, algebra, and quantum mechanics of tensor model observables

Counting of rank d tensor model observables. The counting of rank d complex tensor observables or tensor invariants, containing n copies of a complex tensor variable Φ and its conjugate $\bar{\Phi}$, has been performed in [8]. The enumeration method of tensor invariants used therein is based on the counting of equivalence classes in Cartesian products of the symmetric group S_n of n elements, generated by certain subgroup actions. We describe it here at rank $d = 3$, using the "gauge-fixed formulation" from [8, 10], while the generalization to any d is straightforward.

All the contractions between the indices of n tensors and n conjugate tensors producing $U(N)\times U(N)\times U(N)$ invariants can be described by triples of permutations $\sigma_1, \sigma_2, \sigma_3 \in S_n$ depicted in Fig. 1.

Use Fig. 1, and fixing a gauge $\sigma_3 = \mathrm{id}$, we obtain tensor invariants from permutation pairs, where pairs related by conjugating with a diagonal adjoint action on $S_n \times S_n$ are in the same equivalence class

$$(\tilde\sigma_1, \tilde\sigma_2) \sim (\tau\tilde\sigma_1\tau^{-1},\ \tau\tilde\sigma_2\tau^{-1}), \qquad \tilde\sigma_i, \tau \in S_n. \quad (10)$$

These equivalence classes are also known to enumerate bi-partite ribbon graphs. There is thus a graphical interpretation of the rank 3 tensor invariants in terms of bi-partite ribbon graphs.

Burnside's lemma allows us to write the number of equivalence classes under a group action in terms of the fixed points of the same action. This leads us to

$$\mathrm{Rib}(n) = \sum_{p\vdash n}\mathrm{Sym}(p), \quad (11)$$

where the sum is performed over all partitions p of n, denoted $p \vdash n$, and where $\mathrm{Sym}(p) := \prod_{i=1}^n (i^{p_i})(p_i!)$. There is another expression of the same counting as a sum over triples (R_1, R_2, R_3) of irreducible representations (irreps) [10]

$$\mathrm{Rib}(n) = \sum_{R_1,R_2,R_3\vdash n} C(R_1, R_2, R_3)^2. \quad (12)$$

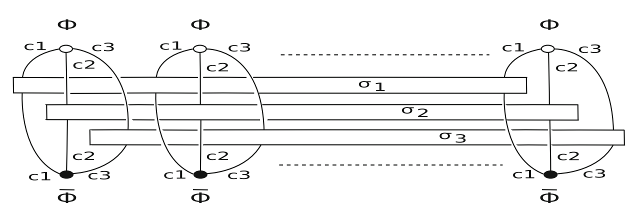

Fig. 1 The contraction of n tensors $\Phi_{c_1c_2c_3}$ and n tensors $\bar\Phi_{c_1c_2c_3}$ identified as permutation triple $(\sigma_1, \sigma_2, \sigma_3)$

The Kronecker coefficient $C(R_1, R_2, R_3)$ is a non-negative-integer that yields the number of times R_3 appears in the tensor-product decomposition $R_1 \otimes R_2$.

The algebra $\mathcal{K}(n)$ of bi-partite ribbon graphs. For the group action given in (10), we consider the set of orbits, each of which is associated with a ribbon graph. We introduce a label $r \in \{1, \ldots, \mathrm{Rib}(n)\}$.

Consider $\mathbb{C}(S_n) \otimes_{\mathbb{C}} \mathbb{C}(S_n)$, simply denoted $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$. For each ribbon graph labeled by r, consider its orbit representative given by the pair $(\tau_1^{(r)}, \tau_2^{(r)})$. The element E_r of $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$ is defined as

$$E_r = \frac{1}{n!} \sum_{\mu \in S_n} \mu \tau_1^{(r)} \mu^{-1} \otimes \mu \tau_2^{(r)} \mu^{-1}. \tag{13}$$

Now, we define the $\mathbb{C}$-vector subspace $\mathcal{K}(n) \subset \mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$ spanned by these elements

$$\mathcal{K}(n) = \mathrm{Span}_{\mathbb{C}} \Big\{ E_r \,, r = 1, \ldots \mathrm{Rib}(n) \Big\}. \tag{14}$$

The dimension of $\mathcal{K}(n)$ is the number of bi-partite ribbon graphs

$$\mathrm{Dim}(\mathcal{K}(n)) = \mathrm{Rib}(n); \tag{15}$$

$\mathcal{K}(n)$ is a permutation centralizer algebra (PCA) [9]—a subspace of a permutation algebra, here $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$, which commutes with a sub-algebra with basis labeled by permutations, here the diagonally embedded S_n permutations. $\mathcal{K}(n)$ is also semi-simple: it has a non-degenerate symmetric bilinear pairing given by

$$\boldsymbol{\delta}_2 : \mathbb{C}(S_n)^{\otimes 2} \times \mathbb{C}(S_n)^{\otimes 2} \to \mathbb{C}, \tag{16}$$

which is defined in terms of the usual delta function on the group. $\boldsymbol{\delta}_2(\otimes_{i=1}^{2} \sigma_i; \otimes_{i=1}^{2} \sigma'_i) = \prod_{i=1}^{2} \delta(\sigma_i \sigma'_i)$, with $\delta(\sigma) = 1$, if $\sigma = \mathrm{id}$, and 0 otherwise. This extends to linear combinations with complex coefficients. Semi-simplicity implies that, by the Wedderburn–Artin theorem, $\mathcal{K}(n)$ admits a decomposition in simple matrix algebras. This decomposition is made manifest using what we denote as the Fourier basis

$$Q^{R_1,R_2,R_3}_{\tau_1,\tau_2} = \kappa_{R_1,R_2} \sum_{\sigma_1,\sigma_2 \in S_n} \sum_{i_1,i_2,i_3,j_1,j_2} C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau_2}_{j_1,j_2;i_3} \times D^{R_1}_{i_1 j_1}(\sigma_1) D^{R_2}_{i_2 j_2}(\sigma_2)\, \sigma_1 \otimes \sigma_2; \tag{17}$$

$D^R_{ij}(\sigma)$ are the matrix elements of the linear operator $D^R(\sigma)$ in an orthonormal basis for the irrep R. The indices τ_1, τ_2 run over an orthonormal basis for the multiplicity space of R_3 appearing in the tensor decomposition of $R_1 \otimes R_2$. This multiplicity is equal to the Kronecker coefficient $C(R_1, R_2, R_3)$. $\kappa_{R_1,R_2} = \frac{d(R_1)d(R_2)}{(n!)^2}$ is a normalization factor, where $d(R_i)$ is the dimension of the irrep R_i. $C^{R_1,R_2;R_3,\tau_1}_{i_1,i_2;i_3}$ are Clebsch–Gordan coefficients of the representations of S_n.

Quantum mechanics of bi-partite ribbon graphs. We define a sesquilinear form on $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$ as

$$g\left(\sum_i a_i \alpha_{1i} \otimes \alpha_{2i}, \sum_j b_j \beta_{1j} \otimes \beta_{2j} \right) = \sum_{i,j} \bar{a}_i b_j\; \delta\big(\alpha_{1i}^{-1} \beta_{1j}\big) \delta\big(\alpha_{2i}^{-1} \beta_{2j}\big), \tag{18}$$

where $a_i, b_i \in \mathbb{C}$, $\alpha_{1i}, \alpha_{2i}, \beta_{1j}, \beta_{2j} \in S_n$, and where the bar means complex conjugation. One checks that g is non-degenerate and, therefore, induces an inner product on $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$. We restrict g to an give an inner product on $\mathcal{K}(n) \subset \mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$, and consequently, $\mathcal{K}(n)$ is an algebra which is also an Hilbert space.

There is another operator that will be of prominent use in the following. Consider the linear conjugation operator $S : \mathbb{C}(S_n) \to \mathbb{C}(S_n)$ which maps a linear combination $A = \sum_i c_i \sigma_i \in \mathbb{C}(S_n)$ to $S(A) := \sum_i c_i \sigma_i^{-1}$. Extend this operation to $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$ by inverting the permutation in each tensor factor, we easily check that $S^2 = \mathrm{id}$ and call S an involution.

Let us discuss the Hermitian operators in our setting that could play the role of Hamiltonian operators. For a conjugacy class $\mathcal{C}_\mu \subset S_n$ labeled by $\mu \vdash n$, a partition of n, we have a central element $T_\mu = \sum_{\sigma \in \mathcal{C}_\mu} \sigma$ that obviously obeys $\gamma T_\mu \gamma^{-1} = T_\mu$, for any $\gamma \in S_n$. We are interested in particular $\mu = [k, 1^{n-k}]$ defined by a single cycle of length k and all remaining cycles of length 1. The corresponding operator denotes T_k.

There are operators that multiplicatively generate the center of $\mathcal{K}(n)$ [20]. At any $n \geq 2$, we define elements in $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$

$$T_k^{(1)} = T_k \otimes 1, \qquad T_k^{(2)} = 1 \otimes T_k, \qquad T_k^{(3)} = \sum_{\sigma \in \mathcal{C}_k} \sigma \otimes \sigma. \tag{19}$$

The sum of products of the $T_k^{(i)}$'s, $k = 1, \ldots, n$, generates the center $\mathcal{Z}(\mathcal{K}(n))$ of $\mathcal{K}(n)$. In fact, one does not need the entire set $k = 1, \ldots, n$ to generate the centre, only a smaller number of them is enough $k = 1, \ldots, k_*(n) \leq n$.

We showed that $T_k^{(i)}$ are Hermitian operators on $\mathcal{K}(n)$ with respect to in the inner product defined by (18) $g(E_s, T_k^{(i)} E_r) = g(T_k^{(i)} E_s, E_r)$. (See proof of Proposition 3 in [7].) The operators $T_k^{(i)}$, for i ranging over $\{1, 2, 3\}$ and k ranging over some subset of $\{2, 3, \cdots, n\}$ form a set of commuting Hermitian operators on $\mathcal{K}(n)$. The commutativity follows from the fact that they are central elements of $\mathcal{K}(n)$. We consider such sets of operators as Hamiltonians and we define a quantum-mechanical time evolution of states in $\mathcal{K}(n)$ of the form

$$E_r(t) = e^{-itT_k^{(i)}} E_r, \tag{20}$$

where $E_r(t)$ becomes time-dependent ribbon graph states.

The action of the $T_k^{(i)}$'s on the ribbon graph bases yields a crucial fact. As linear operators, let $(\mathcal{M}_k^{(i)})_r^s$ be the matrix elements of $T_k^{(i)}$. We have

$$T_k^{(i)} E_s = \sum_s (\mathcal{M}_k^{(i)})_s^t E_t. \tag{21}$$

The matrix elements $(\mathcal{M}_k^{(i)})_r^s$ are non-negative integers (Proposition 2 in [7]).

$T_k^{(i)}$ operators act on the Fourier basis of $\mathcal{K}(n)$ as (Proposition 4 [7])

Proposition 1 For all $k \in \{2, 3, \cdots n\}$, $\{R_i \vdash n : i \in \{1, 2, 3\}\}$, $\tau_1, \tau_2 \in [\![1, C(R_1, R_2, R_3)]\!]$, the Fourier basis elements $Q_{\tau_1,\tau_2}^{R_1,R_2,R_3}$ are eigenvectors of $T_k^{(i)}$

$$T_k^{(i)} Q_{\tau_1,\tau_2}^{R_1,R_2,R_3} = \frac{\chi_{R_i}(T_k)}{d(R_i)} Q_{\tau_1,\tau_2}^{R_1,R_2,R_3}, \tag{22}$$

This means that the Fourier basis $Q_{\tau_1,\tau_2}^{R_1,R_2,R_3}$is an eigenbasis of the operators $T_k^{(i)}$. Furthermore, following Proposition 5 of [7], we have

Proposition 2 For any $\widetilde{k}_* \in \{k_*(n), k_*(n)+1, \cdots, n\}$, the list of eigenvalues of the reconnection operators $\left\{T_2^{(1)}, T_3^{(1)}, \cdots, T_{\widetilde{k}_*}^{(1)}; T_2^{(2)}, T_3^{(2)}, \cdots, T_{\widetilde{k}_*}^{(2)}; T_2^{(3)}, T_3^{(3)}, \cdots, T_{\widetilde{k}_*}^{(3)}\right\}$uniquely determines the Young diagram triples (R_1, R_2, R_3).

The sum of all permutations σ in the conjugacy class $\mathcal{C}_p$ in S_n for partition p is central elements in $\mathcal{Z}(\mathbb{C}(S_n))$. The irreducible normalized characters of these central elements are integers

$$\chi_R(T_p)/d(R) \in \mathbb{Z}. \tag{23}$$

The proof combines a known number theoretic fact about the normalized characters of a finite group being algebraic integers, along with the rationality of characters of irreducible representations of S_n which follows from the Murnaghan–Nakayama Lemma.

Stacking $T_k^{(i)}$ matrices and common eigenspace. Using Proposition 2, the Fourier subspace for a given triple (R_1, R_2, R_3) is uniquely specified as common eigenspace of the operators $T_k^{(i)}$, for $k \in \{2, \ldots, \widetilde{k}_*(n)\}$ and $i \in \{1, 2, 3\}$; with $k_*(n) \leq \widetilde{k}_* \leq n$, with specified eigenvalues for these reconnection operators. The numerator $\chi_R(T_k)$ is given by $\chi_R(T_k) = T_k \chi_R(\sigma)$ for $\sigma \in \mathcal{C}_k$. The character $\chi_R(\sigma)$ can be computed with the combinatorial Murnaghan–Nakayama rule [15]. The dimension $d(R)$ is obtained from the hook formula for dimensions.

The vectors in the Fourier subspace for a triple (R_1, R_2, R_3) solve the following matrix equation:

$$\begin{bmatrix} \mathcal{M}_2^{(1)} - \frac{\chi_{R_1}(T_2)}{d(R_1)} \\ \vdots \\ \mathcal{M}_{\widetilde{k}_*}^{(1)} - \frac{\chi_{R_1}(T_{\widetilde{k}})}{d(R_1)} \\ (1 \leftrightarrow 2) \\ (1 \leftrightarrow 3) \end{bmatrix} \cdot v = \mathbf{0}, \tag{24}$$

where the notation $(1 \leftrightarrow j)$, $j = 2, 3$, means that we replace the matrix block with elements labeled by 1 with matrix blocks labeled by $j = 2, 3$. This rectangular array gives the matrix elements of a linear operator mapping $\mathcal{K}(n)$ to $3(\widetilde{k}_* - 1)$ copies of $\mathcal{K}(n)$, using the geometric basis of ribbon graph vectors. From (23), the normalized characters are integers. Renaming as $\mathcal{L}_{R_1,R_2,R_3}$ the integer matrix in (24), we have

$$\mathcal{L}_{R_1,R_2,R_3} \cdot v = 0. \tag{25}$$

We then have, for each triple of Young diagrams, the problem of finding the null space of an integer matrix. Null spaces of integer matrices have integer null vector bases. These can be interpreted in terms of lattices and can be constructed via well-known algorithms.

4 The Hermite normal forms algorithm and lattice interpretation of kernels

We are interested in solving the linear system $X \cdot v = 0$, where $X = \mathcal{L}_{R_1,R_2,R_3}$ (25) has only integer matrix entries. A crucial fact about the null spaces of integer matrices is that they have bases given as integer vectors. This follows from the theory of Hermite normal forms (HNF) and has an interpretation in terms of sub-lattices [21, 22]. The null space of the integer matrix X is the span of a set of null vectors which can be chosen to be integer vectors, i.e., integral linear combinations of the E_r. A key result from the theory of integer matrices and lattices is that any integer matrix A (square or rectangular; we use $A = X^T$) has a unique HNF. This means that A has a decomposition $A = Uh$ with U a unimodular matrix, i.e., an integer matrix of determinant ± 1 and h is an integer matrix computed by an integrality perserving algorithm.

In the present application, we have a lattice $\mathbb{Z}^{\mathrm{Rib}(n)} \subset \mathbb{R}^{\mathrm{Rib}(n)}$ which is interpreted as the space of integer linear combinations of the geometric ribbon graph basis vectors E_r of the ribbon graph algebra $\mathcal{K}(n)$. Based on these facts, we provide the construction of $C(R_1, R_2, R_3)^2$ as the dimension of a sub-lattice of the lattice of ribbon graphs.

The HNF procedure achieves the proof of the following theorem (Theorem 1 in [7]):

Theorem 3 For every triple of Young diagrams (R_1, R_2, R_3) with n boxes, the lattice $\mathbb{Z}^{|\mathrm{Rib}(n)|}$ of integer linear combinations of the geometric basis vectors E_r of $\mathcal{K}(n)$ contains a sub-lattice of dimension $(C(R_1, R_2, R_3))^2$ spanned by a basis of integer null vectors of the operator X, which is $\mathcal{L}_{R_1,R_2,R_3}$ from (25).

The action of operator S (see section 3) on the geometrical ribbon graph basis E_r or on the Fourier basis Q of $\mathcal{K}(n)$ has key properties that will allow us to interpret the dimension of various lattice subspaces of $\mathbb{Z}^{|\mathrm{Rib}(n)|}$ in terms of sums of products of Kronecker coefficients.

Let us denote the vector space of ribbon graphs, which is the underlying vector space of the algebra $\mathcal{K}(n)$ by $V^{\mathrm{Rib}(n)}$. $V^{\mathrm{Rib}(n)}$ has a decomposition according to the eigenvalues of S

$$V^{\mathrm{Rib}(n)} = V^{\mathrm{Rib}(n)}_{S=1} \oplus V^{\mathrm{Rib}(n)}_{S=-1}. \tag{26}$$

The action of S on the Fourier basis Q leads to

$$V^{\mathrm{Rib}(n)} = \bigoplus_{R_1,R_2,R_3} V^{\mathrm{Rib}(n):\, R_1,R_2,R_3}, \tag{27}$$

where $V^{\mathrm{Rib}(n):\, R_1,R_2,R_3}$ has dimension $C(R_1, R_2, R_3)^2$ and is spanned by the $Q^{R_1,R_2,R_3}_{\tau_1,\tau_2}$ for all τ_1 and τ_2. Then, $V^{\mathrm{Rib}(n):\, R_1,R_2,R_3} = V^{\mathrm{Rib}(n):\, R_1,R_2,R_3}_{S=1} \oplus V^{\mathrm{Rib}(n):\, R_1,R_2,R_3}_{S=-1}$. Combining this with (27), we then have

$$V^{\mathrm{Rib}(n)} = \bigoplus_{R_1,R_2,R_3} \left(V^{\mathrm{Rib}(n):R_1,R_2,R_3}_{S=1} \oplus V^{\mathrm{Rib}(n):R_1,R_2,R_3}_{S=-1} \right). \tag{28}$$

We can show that

$$\mathrm{Dim}\left(V^{\mathrm{Rib}(n):R_1,R_2,R_3}_{S=-1}\right) = \frac{C(R_1, R_2, R_3)(C(R_1, R_2, R_3) - 1)}{2} = \mathrm{Dim}\left(P^{R_1,R_2,R_3} V^{\mathrm{Rib}(n)}_{\mathrm{pairs}^-}\right) \tag{29}$$

with P^{R_1,R_2,R_3} the projector onto $V^{\mathrm{Rib}(n):R_1,R_2,R_3}$ and $V^{\mathrm{Rib}(n)}_{S=-1} = V^{\mathrm{Rib}(n)}_{\mathrm{pairs}^-}$. Similarly, we can show

$$\mathrm{Dim}\left(V^{\mathrm{Rib}(n):R_1,R_2,R_3}_{S=+1}\right) = \frac{C(R_1, R_2, R_3)(C(R_1, R_2, R_3) + 1)}{2} = \mathrm{Dim}\left(P^{R_1,R_2,R_3} V^{\mathrm{Rib}(n)}_{\mathrm{pairs}^+}\right) + \mathrm{Dim}\left(P^{R_1,R_2,R_3} V^{\mathrm{Rib}(n)}_{\mathrm{singlets}}\right) \tag{30}$$

with the decomposition $V^{\mathrm{Rib}(n)}_{S=1} = V^{\mathrm{Rib}(n)}_{\mathrm{pairs}^+} \oplus V_{\mathrm{singlets}}$, that defines $V^{\mathrm{Rib}(n)}_{\mathrm{pairs}^+}$, the subspace spanned by $\{(E^{(n)}_r + E^{(\bar{n})}_r)\}$, and V_{singlets} the subspace spanned by $\{E^{(s)}_r\}$ that are ribbons obeying $SE^{(s)}_r = E^{(s)}_r$. Note that we do not have separate expressions for the two terms in the sum above in terms of Kronecker coefficients, since we do not expect the P^{R_1,R_2,R_3} to commute with the projection of $V^{\mathrm{Rib}(n)}_{S=1}$ into the separate summands $V^{\mathrm{Rib}(n)}_{\mathrm{singlets}}$ and $V^{\mathrm{Rib}(n)}_{\mathrm{pairs}^+}$.

Using once again the HNF procedure and the outcome of the above discussion, we reach the statement (Theorem 2 in [7])

Theorem 4 For every triple of Young diagrams (R_1, R_2, R_3) with n boxes, there are three constructible sub-lattices of $\mathbb{Z}^{\mathrm{Rib}(n)}$ of respective dimensions $C(R_1, R_2, R_3)(C(R_1, R_2, R_3) + 1)/2$, $C(R_1, R_2, R_3)(C(R_1, R_2, R_3) - 1)/2$, and $C(R_1, R_2, R_3)$.

If we perform the sum over R_1, R_2, R_3 in (30), we have $\mathrm{Dim}\left(V^{\mathrm{Rib}(n)}_{S=+1}\right) = \mathrm{Dim}\left(V^{\mathrm{Rib}(n)}_{\mathrm{pairs}^+}\right) + \mathrm{Dim}(V_{\mathrm{singlets}})$ and $\mathrm{Dim}\left(V^{\mathrm{Rib}(n)}_{S=-1}\right) = \mathrm{Dim}\left(V^{\mathrm{Rib}(n)}_{\mathrm{pairs}^-}\right)$. Since $\mathrm{Dim}\left(V^{\mathrm{Rib}(n)}_{\mathrm{pairs}^+}\right) = \mathrm{Dim}\left(V^{\mathrm{Rib}(n)}_{\mathrm{pairs}^-}\right)$, we have

$$\text{Number of bi-partite ribbon graphs with } n \text{ edges invariant under } S = \mathrm{Dim}\left(V^{\mathrm{Rib}(n)}_{\mathrm{singlets}}\right) = \sum_{R_1,R_2,R_3} C(R_1, R_2, R_3). \tag{31}$$

While the sum over triples of Young diagrams with n boxes of the square of Kronecker coefficients gives the number of ribbon graphs with n edges, the sum of the Kronecker coefficients gives the number of singlet ribbon graphs.

Acknowledgements The research of S.R. was supported by the STFC consolidated grant ST/P000754/1 "String Theory, Gauge Theory and Duality". The authors would like to thank Peter Schupp for suggestions on the introduction.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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