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Review

Noncommutative gauge and gravity theories and geometric Seiberg–Witten map

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Abstract We give a pedagogical account of noncommutative gauge and gravity theories, where the exterior product between forms is deformed into a $\star$-product via an abelian twist (e.g. the Groenewold–Moyal twist). The Seiberg–Witten map between commutative and noncommutative gauge theories is introduced. It allows to express the action of noncommutative Einstein gravity coupled to spinor fields in terms of the usual commutative action with commutative fields plus extra interaction terms dependent on the noncommutativity parameter.

1 Introduction

Noncommutativity of phase space is a core feature of quantum mechanics. Noncommutativity of spacetime rather than phase space has been considered since the early days of quantum mechanics as a possible way to reconcile gravity with quantum theory. Indeed the dynamical variable in Einstein general relativity is spacetime itself with its metric structure, and noncommutativity of spacetime coordinates could lead to a regularization in the perturbative treatment of gravity as a quantum field theory. Noncommutativity of spacetime coordinates is further supported by Gedanken experiments that aim at probing spacetime structures at very small distances. They show that due to gravitational backreaction one cannot test spacetime under the Planck scale. For example, in relativistic quantum mechanics the position of a particle can be detected with a precision of at most the order of its Compton wavelength $\lambda_C = \hbar/mc$. Probing spacetime at very short distances implies extremely energetic particles, that in turn produce high spacetime curvature. When λ_C is of the order of the Planck length, the spacetime curvature radius has the same order of magnitude of the Compton wavelength of the probe particle, and the attempt to measure spacetime structure under the Planck scale fails. Gedanken experiments of this type show that the description of spacetime as a continuum of points (a smooth manifold) is an assumption no more justified at the Planck scale. It is hence natural to relax this assumption and conceive a more general noncommutative spacetime, where uncertainty relations and discretization naturally arise. Space and time are then described by a Noncommutative Geometry. In this way the impossibility of testing spacetime under the Planck length, a dynamical feature due to gravitational backreaction, is encoded at a deeper kinematical level.

In general spacetime discretization is expected in quantum gravity theories, see the review [1]. For example in string theory the study of string scatterings leads to generalized uncertainty principles where a minimal length emerges. In other approaches, e.g. loop quantum gravity, minimal area and volumes are predicted.

Spacetime noncommutativity also arises by considering an electron in a strong magnetic field B. In this regime, due to the minimal coupling with the background gauge field $(A_x, A_y) = (0, Bx)$ associated to the flux B, the dynamics takes place in the reduced phase space $q = y, p = Bx$. Thus the electron's coordinates become noncommutative: $[x, y] = -\frac{i\hbar}{B}$ (cf. [2] for an extended discussion). Hence quantum theory in the presence of a magnetic field B leads to a noncommutative spacetime. Similarly, low energy effective actions of open strings in the presence of a background Neveu–Schwarz B-field can be described by gauge theories on noncommutative spaces. The study

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of Yang–Mills (and Born–Infeld) theories on noncommutative spaces has proven very fruitful: it allows to realize string theory T-duality symmetry within the low energy physics of noncommutative (super) Yang–Mills theories [3]. It provides exact low energy D-brane effective actions, in a given $\alpha' \to 0$ sector of string theory where closed strings decouple, see Ref. [4]. In that paper Seiberg and Witten provided an explicit map (change of variables) between commutative and noncommutative gauge theories.

In this paper we study gravity on noncommutative spacetime as a noncommutative gauge theory of the above type. The noncommutative geometry is first formulated in a geometric (coordinate independent) language, useful for studying diffeomorphism invariant theories. Then the specific gauge theory describing noncommutative gravity in first order formalism (with independent vierbein and spin connection fields) is presented. As we discuss in Sect. 5.2, a generic feature of noncommutative gauge theories is that they are well defined for $U(N)$ or $GL(N)$ gauge groups in the fundamental or the adjoint representation but not for a generic representation, or for a generic gauge group G (like e.g. $SU(N)$). This general feature implies that the Lorentz gauge invariance of the first order gravity action becomes a $\star$-gauge invariance that enlarges the classical $SO(1,3)$ group to $GL(2,C)$.

The enlargement of the gauge group corresponds to an increase in the number of fundamental fields of the theory. This increase can be mitigated by imposing charge conjugation constraints on the noncommutative gauge action and can be fully avoided by the use of the Seiberg–Witten map [4], that relates the fields in the deformed action (the "quantum" fields) to the classical fields, in such a way that the ordinary gauge variations of the classical fields induce the $\star$-gauge variations on the quantum fields. We thus obtain a gravity theory on noncommutative spacetime with the same degrees of freedom as classical gravity.

Among other approaches to noncommutative gravity we mention gravity on fuzzy spaces [5, 6], emerging from matrix theory in the presence of fuzzy extra dimensions [7], a metric approach [8, 9] where the noncommutative Levi-Civita connection is constructed and the braided gauge symmetry approach of [10].

Finally, a noncommutative hamiltonian formalism for twisted geometric theories has been developed in [11], and applied to noncommutative vierbein gravity. It allows an algorithmic construction of the canonical $\star$-gauge generators.

The plan of the paper is as follows. Section 2 deals with the origin of twisted products, i.e. Weyl quantization and Groenewold–Moyal product. Section 3 transfers noncommutativity of coordinates to noncommutativity of functions and of exterior forms (via $\star$-wedge products), and summarizes the basic results of the corresponding noncommutative geometry. Section 4 illustrates the procedure in the case of Yang–Mills theory, by deforming its classical action. In Sect. 5 the $\star$-deformation of the gravity action is discussed in detail reviewing the results of [12]. Section 6 contains a discussion on the Seiberg–Witten map. In Sect. 7 it is shown how the Seiberg–Witten map allows to construct noncommutative gauge theories with any gauge group. The Seiberg–Witten map for the noncommutative gravity action is then described in detail, providing a noncommutative action with the same degrees of freedom as the commutative one [13–15]. Expanding this action in power series of the noncommutative deformation parameter we obtain an action on commutative spacetime with interaction terms dictated by noncommutativity of spacetime. We have thus constructed a modified gravity action that is expected to capture some quantum gravity aspects.

2 Noncommutative algebras, Weyl quantization and $\star$-products

The easiest way to describe a noncommutative spacetime is via the noncommutative algebra of its coordinates, i.e., giving a set of generators and relations. We list three typical examples of commutation relations:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \quad \text{canonical} \tag{2.1}$$

$$[x^\mu, x^\nu] = i f^{\mu\nu}{}_\sigma x^\sigma \quad \text{Lie algebra} \tag{2.2}$$

$$x^\mu x^\nu - q x^\nu x^\mu = 0 \quad \text{quantum (hyper)plane} \tag{2.3}$$

where $\theta^{\mu\nu}$ (a real antisymmetric matrix), $f^{\mu\nu}{}_\sigma$ (real structure constants), q (a complex number, e.g. a phase) are the respective noncommutativity parameters. When the noncommutativity parameters are turned off, the algebra becomes commutative and is the algebra of polynomial functions on d-dimensional space $\mathbb{R}^d$. We can also impose further constraints, for example periodicity of the coordinates describing the canonical noncommutative spacetime (2.1). This leads to a noncommutative torus rather than to a noncommutative (hyper)plane. Similarly, constraining the coordinates of the quantum (hyper)plane relations (2.3) we obtain a quantum (hyper)sphere.

This algebraic description should then be complemented by a topological approach, leading for example to the notion of continuous functions. This is achieved by completing the algebra generated by the noncommutative coordinates to a $C^\star$-algebra. Typically $C^\star$-algebras arise as algebras of operators on a Hilbert space. Connes noncommutative geometry [16] starts from these notions and enriches the $C^\star$-algebra structure and its representation on a Hilbert space, generalizing to the noncommutative case also the notions of smooth functions and metric structure.

A complementary approach to noncommutative space is given by the $\star$-product, retaining the usual space of functions but deforming the pointwise product into a noncommutative one. Historically the $\star$-product originated as a noncommutative product for functions on phase space. The quantization of phase space coordinates q, p with Poisson structure $\{q,p\}=1$ to operators $\Phi(q)=\hat{q}$, $\Phi(p)=\hat{p}$ on Hilbert space with $[\hat{q},\hat{p}]=i\hbar$ is extended à la Weyl to functions $f(p,q)\to\Phi(f)(\hat{p},\hat{q})$. The operator product then induces a $\star$-product, or Groenewold–Moyal product, on functions on phase space:

$$f\star g=\Phi^{-1}(\Phi(f)\Phi(g)).$$

On polynomial functions Weyl quantization amounts to replace p, q with the operators $\hat{p},\hat{q}$ and to symmetrize in $\hat{p}$ and $\hat{q}$: $p^mq^n\mapsto\Phi(p^mq^n)=\mathrm{Sym}(\hat{p}^m\hat{q}^n)$ where $\mathrm{Sym}(\hat{p}^m\hat{q}^n)$ is the symmetrized polynomial in $\hat{p},\hat{q}$ normalized so that $\mathrm{Sym}(p^mq^n)=p^mq^n$. It is defined by $(\hat{p}+\hat{q})^\ell=\sum_{m+n=\ell}\frac{(m+n)!}{m!n!}\mathrm{Sym}(\hat{p}^m\hat{q}^n)$. For example, $\mathrm{Sym}(\hat{p}^m)=\hat{p}^m$, $\mathrm{Sym}(\hat{q}^n)=\hat{q}^n$, $\mathrm{Sym}(\hat{p}\hat{q})=\frac{1}{2}(\hat{p}\hat{q}+\hat{q}\hat{p})$, $\mathrm{Sym}(p^2q)=\frac{1}{3}(\hat{p}^2\hat{q}+\hat{p}\hat{q}\hat{p}+\hat{q}\hat{p}^2)$. The corresponding $\star$-product explicitly reads

$$(f\star g)(p,q)=e^{\frac{i}{2}\hbar(\frac{\partial}{\partial q}\frac{\partial}{\partial p'}-\frac{\partial}{\partial p}\frac{\partial}{\partial q'})}f(p,q)g(p',q')|_{p=p',q=q'}.$$

Since the operator product is associative so is the $\star$-product.

More in general (in formal deformation quantization) a $\star$-product on a manifold M with Poisson structure $\{\ ,\ \}$ is a noncommutative deformation of the usual pointwise product. It sends two smooth functions f, g to a third one $f\star g$ and is a differential operator on both its arguments. It satisfies the associative property

$$f\star(g\star h)=(f\star g)\star h,$$

the normalization property $f\star 1=1\star f=f$ and

$$f\star g=fg+\frac{i}{2}\hbar\{f,g\}+\mathcal{O}(\hbar^2)$$

so that in the semiclassical limit $\lim_{\hbar\to 0}\frac{-i}{\hbar}(f\star g-g\star f)=\{f,g\}$, realizing the correspondence principle between quantum and classical mechanics. For further reading on the topics of this section we refer to [17, Ch. 2, §3], [18, Ch. 6], [19].

3 $\star$-products from twists and noncommutative differential geometry

The $\star$-product on phase space of the previous section generalizes to $\mathbb{R}^d$ with coordinates x^μ as

$$(f\star g)(x)=\mathrm{e}^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}}f(x)h(y)|_{x=y}. \quad (3.1)$$

Here the antisymmetric matrix $\hbar\theta^{\mu\nu}$ has been for short denoted $\theta^{\mu\nu}$. Correspondingly, the classical limit $\hbar\to 0$ becomes $\theta\to 0$.

Notice that if we set

$$\mathcal{F}^{-1}=\mathrm{e}^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}}$$

then

$$f\star g=\mu(\mathcal{F}^{-1}(f\otimes g)) \quad (3.2)$$

where μ is the usual product of functions $\mu(f\otimes g)=fg$. The element $\mathcal{F}=\mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}}$ is an example of a Drinfeld twist. It is defined by the exponential series in powers of the noncommutativity parameters $\theta^{\mu\nu}$,

$$\begin{aligned}\mathcal{F}&=\mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}}\\&=1\otimes 1-\frac{i}{2}\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu\\&\quad-\frac{1}{8}\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\partial_{\mu_1}\partial_{\mu_2}\otimes\partial_{\nu_1}\partial_{\nu_2}+\cdots\end{aligned}$$

It is easy to see that $x^\mu\star x^\nu-x^\nu\star x^\mu=i\theta^{\mu\nu}$ thus recovering the noncommutative algebra abstractly defined in (2.1).

The method of constructing $\star$-products using Drinfeld twists [20] (see e.g. [21] for a quick introduction) is not the most general method (it does not apply to an arbitrary Poisson manifold [22]), however it is quite powerful, and the class of $\star$-products obtained is quite wide. For example choosing the appropriate twist we can obtain the noncommutative relations (2.1), (2.3) and also (depending on the explicit expression of the structure constants) some of the Lie algebra type (2.2). It is also well adapted to a coordinate free description of the $\star$-algebra of functions on a manifold M and to its differential geometry.

Let M be a smooth manifold. A twist is an invertible element $\mathcal{F}\in U\Xi\otimes U\Xi$ where $U\Xi$ is the universal enveloping algebra of vector fields, (i.e. the algebra generated by vector fields on M, where the element $XY-YX$ is identified with the vector field $[X,\ Y]$). The element $\mathcal{F}$ must satisfy some further conditions that we do not write here, but that hold true if we consider abelian twists, i.e., twists of the form

$$\begin{aligned}\mathcal{F}&=\mathrm{e}^{-\frac{i}{2}\theta^{IJ}X_I\otimes X_J}\\&=1\otimes 1-\frac{i}{2}\theta^{IJ}X_I\otimes X_J\\&\quad-\frac{1}{8}\theta^{I_1J_1}\theta^{I_2J_2}X_{I_1}X_{I_2}\otimes X_{J_1}X_{J_2}+\cdots \quad (3.3)\end{aligned}$$

where the vector fields X_I ($I=1,\ldots s$ with s not necessarily equal to $d=\dim M$) are mutually commuting $[X_I,X_J]=0$ (hence the name abelian twist).

It is convenient to introduce the following notation

$$\begin{aligned}\mathcal{F}^{-1}&=1\otimes 1+\frac{i}{2}\theta^{IJ}X_I\otimes X_J\\&\quad-\frac{1}{8}\theta^{I_1J_1}\theta^{I_2J_2}X_{I_1}X_{I_2}\otimes X_{J_1}X_{J_2}+\cdots\end{aligned}$$

$$= \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha$$

where a sum over the multi-index α is understood.

With a twist $\mathcal{F}$ we deform the whole differential geometry of M. Let $\mathcal{A}$ be the algebra of smooth functions on the manifold M. We deform $\mathcal{A}$ to a noncommutative algebra $\mathcal{A}_\star$ by defining the new product of functions

$$f \star g = \bar{\mathrm{f}}^\alpha(f)\,\bar{\mathrm{f}}_\alpha(g).$$

We see that this formula is a generalization of the Groenewold–Moyal star product on $\mathbb{R}^d$ defined in (3.1) or (3.2). Since the vector fields X_I are mutually commuting, this $\star$-product is associative. Note that only the algebra structure of $\mathcal{A}$ is changed to $\mathcal{A}_\star$ while, as vector spaces, $\mathcal{A}$ and $\mathcal{A}_\star$ are the same. We similarly consider the algebra of exterior forms $\Omega^\bullet$ with the wedge product $\wedge$, and deform it in the noncommutative exterior algebra $\Omega^\bullet_\star$ that is characterized by the graded noncommutative exterior product $\wedge_\star$ given by

$$\tau \wedge_\star \tau' = \bar{\mathrm{f}}^\alpha(\tau) \wedge \bar{\mathrm{f}}_\alpha(\tau'),$$

where τ and τ' are arbitrary exterior forms, and each vector field $X_{I_1}, X_{I_2}, X_{J_1}, X_{J_2} \ldots$ in (3.4) acts on forms via the Lie derivative. Only the product is deformed and hence $\Omega^\bullet_\star = \Omega^\bullet$ as (graded) vector spaces, in particular $\Omega^n_\star = \Omega^n$ for any degree n.

It is easy to show that the usual exterior derivative is compatible with the new $\wedge_\star$-product,

$$\mathrm{d}(\tau \wedge_\star \tau') = \mathrm{d}(\tau) \wedge_\star \tau' + (-1)^{\deg(\tau)} \tau \wedge_\star \mathrm{d}\tau' \tag{3.4}$$

since the exterior derivative commutes with the Lie derivative.

We also have compatibility with the usual undeformed integral (graded cyclicity property):

$$\int \tau \wedge_\star \tau' = (-1)^{\deg(\tau)\deg(\tau')} \int \tau' \wedge_\star \tau \tag{3.5}$$

with $\deg(\tau) + \deg(\tau') = d = \dim M$. In fact we have, up to boundary terms,

$$\int \tau \wedge_\star \tau' = \int \tau \wedge \tau' = (-1)^{\deg(\tau)\deg(\tau')} \int \tau' \wedge \tau$$
$$= (-1)^{\deg(\tau)\deg(\tau')} \int \tau' \wedge_\star \tau.$$

For example at first order in θ,

$$\int \tau \wedge_\star \tau' = \int \tau \wedge \tau' - \frac{i}{2}\theta^{IJ} \int \mathcal{L}_{X_I}(\tau \wedge \mathcal{L}_{X_J}\tau')$$
$$= \int \tau \wedge \tau' - \frac{i}{2}\theta^{IJ} \int d i_{X_I}(\tau \wedge \mathcal{L}_{X_J}\tau')$$

where we used the Cartan formula $\mathcal{L}_{X_I} = d i_{X_I} + i_{X_I} d$ and $\tau \wedge \mathcal{L}_{X_J}\tau'$ being a d-form so that its exterior derivative vanishes.

Finally, provided that the commuting vector fields $\{X_I\}$ defining an abelian twist are all (anti)hermitian, we have compatibility with the undeformed complex conjugation

$$(\tau \wedge_\star \tau')^* = (-1)^{\deg(\tau)\deg(\tau')} \tau'^* \wedge_\star \tau^*. \tag{3.6}$$

Indeed, sending i into $-i$ in the twist (3.3) amounts to send θ^{IJ} into $-\theta^{IJ} = \theta^{JI}$, i.e. to exchange the order of the factors in the $\star$-product.

4 Noncommutative Yang–Mills actions

It is straightforward to write a $U(N)$ Yang–Mills theory on noncommutative space given by Groenewold–Moyal star product,

$$S_{\mathrm{NCYM}} = \frac{-1}{2g^2} \int \mathrm{d}^4x \,\mathrm{Tr}(\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}) \tag{4.1}$$

where the noncommutative field strength $\hat{F}$ is defined by

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i(\widehat{A}_\mu \star \widehat{A}_\nu - \widehat{A}_\nu \star \widehat{A}_\mu).$$

This action is invariant under the noncommutative gauge transformations

$$\hat{\delta}\hat{A}_\mu = \partial_\mu \hat{\epsilon} + i(\hat{\epsilon} \star \hat{A}_\mu - \hat{A}_\mu \star \hat{\epsilon}),$$

which imply $\hat{\delta}\widehat{F}_{\mu\nu} = i(\hat{\epsilon} \star \widehat{F}_{\mu\nu} - \widehat{F}_{\mu\nu} \star \hat{\epsilon})$. Using the differential geometry developed in the previous section this action can also be rewritten as

$$S_{\mathrm{NCYM}} = \frac{-1}{2g^2} \int \mathrm{Tr}(\hat{F} \wedge_\star *_H \hat{F})$$

where $A = A_\mu \star \mathrm{d}x^\mu$, $\hat{F} = \mathrm{d}\hat{A} - i\hat{A} \wedge_\star \hat{A}$ and the Hodge star operator $*_H$ is the usual commutative one in flat Minkowski metric (recall that as vector spaces $\Omega^2_\star = \Omega^2$). The noncommutative gauge transformations now read

$$\hat{\delta}\widehat{A} = \mathrm{d}\hat{\epsilon} + i(\hat{\epsilon} \star \hat{A} - \hat{A} \star \hat{\epsilon}).$$

and imply $\hat{\delta}\widehat{F} = i(\hat{\epsilon} \star \widehat{F} - \widehat{F} \star \hat{\epsilon})$, $\hat{\delta}(*_H \widehat{F}) = i(\hat{\epsilon} \star (*_H \widehat{F}) - (*_H \widehat{F}) \star \hat{\epsilon})$.

In this action the gauge potential and the field strength are valued in $n \times n$ hermitian matrices, that define the Lie algebra of $U(N)$. Other representations of $U(N)$ and gauge groups are in general problematic. Indeed consider an infinitesimal gauge transformation $\epsilon = \varepsilon^A T^A$, where the generators T^A belong to some representation of a Lie group G. The commutator of two infinitesimal gauge transformations is

$$[\epsilon,\ \epsilon']_\star \equiv \epsilon \star \epsilon' - \epsilon' \star \epsilon = \frac{1}{2}\{\varepsilon^A,\ \varepsilon'^B\}_\star [T^A, T^B] + \frac{1}{2}[\varepsilon^A,\ \varepsilon'^B]_\star \{T^A, T^B\}, \tag{4.2}$$

where $\{U,V\}_\star := U\star V - V\star U$, $[U,V]_\star := U\star V - V\star U$. We see that also the anticommutator $\{T^A,T^B\}$ appears. We thus have two options:

(i) Consider gauge groups like $U(N)$ or $GL(N)$ in the (anti)fundamental or in the adjoint, since in this case $\{T^A,T^B\}$ is again in the Lie algebra.

(ii) Allow for more general representations of $U(N)$ or $GL(N)$, or more general Lie algebras (including all simple Lie algebras) with representations that do not close under the anticommutator. In this case we have to enlarge the Lie algebra to include also anticommutators besides commutators, i.e., we have to consider all possible (symmetrized) products $T^AT^B \ldots T^C$ of generators. The gauge potential will correspondingly have components

$$\hat{A} = \hat{A}^A T^A + \hat{A}^{AB} T^{AB} + \hat{A}^{ABC} T^{ABC} + \cdots$$

and therefore infinite degrees of freedom. The Seiberg–Witten map discussed in Sect. 6 allows to reduce them to the classical degrees of freedom, and therefore to construct noncommutative Yang–Mills theories with any gauge group.

5 Noncommutative vierbein gravity coupled to fermions

5.1 Classical action and symmetries

Here we apply the twist procedure to first order gravity in $d = 4$ coupled to fermions and obtain gravity on noncommutative spacetime. The usual action of first-order gravity coupled to a spin $\frac{1}{2}$ field ψ reads:

$$S = \epsilon_{abcd}\int R^{ab}\wedge V^c \wedge V^d - i\bar\psi\gamma^a V^b \wedge V^c \wedge V^d \wedge D\psi - i(D\bar\psi)\gamma^a \wedge V^b \wedge V^c \wedge V^d \psi \tag{5.1}$$

where the vierbein V^a and the spin connection ω^{ab} are independent one-forms:

$$V^a = V^a_\mu \mathrm{d}x^\mu, \quad \omega^{ab} = \omega^{ab}_\mu \mathrm{d}x^\mu \tag{5.2}$$

and the two-form (Lorentz) curvature R^{ab} is defined as

$$R^{ab} = \mathrm{d}\omega^{ab} - \omega^a{}_c \wedge \omega^{cb}. \tag{5.3}$$

The Dirac conjugate is defined as usual: $\bar\psi = \psi^\dagger\gamma_0$.

This action can be recast in an index-free notation [12, 24], convenient for generalization to the noncommutative case:

$$S = \int \mathrm{Tr}(iR\wedge V\wedge V\gamma_5) + \bar\psi V\wedge V\wedge V\gamma_5 D\psi + D\bar\psi \wedge V\wedge V\wedge V\gamma_5\psi \tag{5.4}$$

where

$$R = \mathrm{d}\Omega - \Omega\wedge\Omega, \quad D\psi = \mathrm{d}\psi - \Omega\psi, \quad D\bar\psi = \overline{D\psi} = \mathrm{d}\bar\psi + \bar\psi\Omega \tag{5.5}$$

with

$$V \equiv V^a\gamma_a, \quad \Omega \equiv \frac{1}{4}\omega^{ab}\gamma_{ab}, \quad R \equiv \frac{1}{4}R^{ab}\gamma_{ab} \tag{5.6}$$

taking value in Dirac gamma matrices (recalled in Appendix B). Use of the gamma matrix identities $\gamma_{abc} = i\epsilon_{abcd}\gamma^d\gamma_5$, $\mathrm{Tr}(\gamma_{ab}\gamma_c\gamma_d\gamma_5) = -4i\epsilon_{abcd}$ in computing the trace leads back to the usual action (5.1). Reality of the component fields V^a, ω^{ab}, is equivalent to the hermiticity conditions

$$\gamma_0 V\gamma_0 = V^\dagger, \quad -\gamma_0\Omega\gamma_0 = \Omega^\dagger. \tag{5.7}$$

The action (5.4) is real; for the proof compare it to its complex conjugate, obtained by taking the Hermitian conjugate of the 4-form inside the trace in the integral.

The action is invariant under local diffeomorphisms (it is the integral of a 4-form on a 4-manifold, hence the infinitesimal diffeomorphism $\mathcal{L}_X = \mathrm{d}i_X + i_X\mathrm{d}$ reduces to $\mathrm{d}i_X$, a total derivative). It is also invariant under local Lorentz rotations. These latter read

$$\delta_\epsilon V = -[V,\epsilon], \quad \delta_\epsilon\Omega = d\epsilon - [\Omega,\epsilon], \quad \delta_\epsilon\psi = \epsilon\psi, \quad \delta_\epsilon\bar\psi = -\bar\psi\epsilon \tag{5.8}$$

with $\epsilon = \frac{1}{4}\epsilon^{ab}\gamma_{ab}$. The local Lorentz invariance of the index free action follows from $\delta_\epsilon R = -[R,\epsilon]$ and $\delta_\epsilon D\psi = \epsilon D\psi$, the cyclicity of the trace Tr and the fact that the gauge parameter ϵ commutes with γ_5.

After substituting (5.6) and $\epsilon = \frac{1}{4}\epsilon^{ab}\gamma_{ab}$ into (5.8), simple gamma algebra yields the gauge variations of the component fields:

$$\delta_\epsilon V^a = \epsilon^a{}_b V^b, \quad \delta_\epsilon\omega^{ab} = d\epsilon^{ab} - \omega^a{}_c\epsilon^{cb} + \omega^b{}_c\epsilon^{ca} \equiv D\epsilon^{ab}. \tag{5.9}$$

Similarly, the variation of the curvature components is found to be

$$\delta_\epsilon R^{ab} = \epsilon^a{}_c R^{cb} - \epsilon^b{}_c R^{ca}, \tag{5.10}$$

while $\delta_\epsilon(\bar\psi\gamma^a D\psi) = \epsilon^a{}_b\bar\psi\gamma^b D\psi$. Thus all quantities in the action (5.4) transform homogeneously under

Lorentz local rotations, and since ϵ_{abcd} is an invariant tensor of $SO(1,3)$, the action is likewise invariant. Here the proof of invariance looks simple both in the index-free and in the component formulation. Note however that in general the index-free proof is much simpler.

5.2 Noncommutative gauge theory and Lorentz group

Before presenting the noncommutative version of the action (5.4), we discuss the $\star$-deformation of the Lorentz symmetry variations (5.8). These are generated by the gamma matrices $\frac{1}{4}\gamma_{ab}$, and the gauge parameter is $\epsilon = \frac{1}{4}\epsilon^{ab}\gamma_{ab}$. The commutator of two Lorentz transformations is again a Lorentz transformation, corresponding to the fact that the commutator of two γ_{ab} matrices contains only γ_{ab} matrices. The situation changes when considering the $\star$-deformation of this symmetry: as discussed in Sect. 4, the commutator of two $\star$-gauge transformations contains also anticommutators of the generators. The anticommutator of two $\frac{1}{4}\gamma_{ab}$ matrices yields the identity and the γ_5 matrices, so that the gauge parameter must now include them in its expansion:

$$\epsilon = \frac{1}{4}\epsilon^{ab}\gamma_{ab} + i\epsilon\mathbb{1} + \tilde{\epsilon}\gamma_5.$$

The extra gauge parameters $\epsilon, \tilde{\epsilon}$ can be chosen to be real (like ϵ_{ab}). Indeed the reality of ϵ_{ab}, $\epsilon, \tilde{\epsilon}$ is equivalent to the hermiticity condition

$$-\gamma_0\epsilon\gamma_0 = \epsilon^\dagger \tag{5.11}$$

and if the gauge parameters ϵ, ϵ' satisfy this condition then also $[\epsilon \overset{\star}{,} \epsilon']$ is easily seen to satisfy this hermiticity condition.

Thus we have centrally extended the Lorentz group to

$$SO(3,1) \rightarrow SO(3,1) \times U(1) \times R^+,$$

or more precisely, (since our manifold M has a spin structure and we have a gauge theory of the spin group $SL(2,\ C)$)

$$SL(2,C) \rightarrow GL(2,C).$$

The Lie algebra generator $i\mathbb{1}$ is the anti-hermitian generator corresponding to the $U(1)$ extension, while γ_5 is the hermitian generator corresponding to the noncompact R^+ extension.

Since under noncommutative gauge transformations we have

$$\delta_\epsilon\Omega = d\epsilon - \Omega\star\epsilon + \epsilon\star\Omega \tag{5.12}$$

also the spin connection and the curvature will be valued in the $GL(2,\ C)$ Lie algebra representation given by all the even gamma matrices,

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} + i\omega\mathbb{1} + \tilde{\omega}\gamma_5,$$
$$R = \frac{1}{4}R^{ab}\gamma_{ab} + ir\mathbb{1} + \tilde{r}\gamma_5. \tag{5.13}$$

Similarly the gauge transformation of the vierbein,

$$\delta_\epsilon V = -V\star\epsilon + \epsilon\star V, \tag{5.14}$$

closes in the vector space of odd gamma matrices (i.e. the vector space linearly generated by $\gamma^a, \gamma^a\gamma_5$) and not in the subspace of just the γ^a matrices. Hence the noncommutative vierbein are valued in the odd gamma matrices

$$V = V^a\gamma_a + \tilde{V}^a\gamma_a\gamma_5. \tag{5.15}$$

Reality of the component fields V^a, $\tilde{V}^a$, ω^{ab}, ω, and $\tilde{\omega}$ is equivalent to the hermiticity conditions

$$\gamma_0 V\gamma_0 = V^\dagger, \quad -\gamma_0\Omega\gamma_0 = \Omega^\dagger. \tag{5.16}$$

These hermiticity conditions are consistent with the gauge variations.

Finally, the infinitesimal gauge transformations of the fields considered close the Lie algebra of $GL(2,\ C)$,

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]_\star = -\delta_{[\epsilon_1,\epsilon_2]_\star}. \tag{5.17}$$

5.3 Noncommutative gravity action and its symmetries

The abelian twist, defining the star products and compatible with usual integration on M, leads to the extension of the Lorentz gauge group to $GL(2,\ C)$. It allows to generalize to the noncommutative case the gravity action (5.4). The noncommutative action reads

$$S = \int \mathrm{Tr}(iR\wedge_\star V\wedge_\star V\gamma_5) + \bar{\psi}\star V\wedge_\star V\wedge_\star V\wedge_\star\gamma_5 D\psi$$
$$+ D\bar{\psi}\wedge_\star V\wedge_\star V\wedge_\star V\star\gamma_5\psi \tag{5.18}$$

with

$$R = d\Omega - \Omega\wedge_\star\Omega, \quad D\psi = d\psi - \Omega\star\psi,$$
$$D\bar{\psi} = d\bar{\psi} + \bar{\psi}\star\Omega. \tag{5.19}$$

Reality of this noncommutative action follows by comparing it to its complex conjugate (obtained by taking the Hermitian conjugate of the 4-form inside the trace in the integral).

Gauge invariance of the noncommutative action (5.18) under the $\star$-variations is proved in the same way as for the commutative case, noting that all the fields in the action transform homogeneously, cf. (5.12), (5.14) and

$$\delta_\epsilon \psi = \epsilon \star \psi, \quad \delta_\epsilon \bar{\psi} = -\bar{\psi} \star \epsilon, \quad \delta_\epsilon D\psi = \epsilon \star D\psi,$$
$$\delta_\epsilon D\bar{\psi} = -D\bar{\psi} \star \epsilon, \quad \delta_\epsilon R = -R \star \epsilon + \epsilon \star R. \tag{5.20}$$

Using that ϵ commutes with γ_5, and the cyclicity of the trace together with the graded cyclicity of the integral, the invariance of (5.18) follows.

Diffeomorphisms invariance. It is straightforward to prove that under an infinitesimal diffeomorphism $\mathcal{L}_X$ generated by a vector field X we have $\mathcal{L}_X S = 0$. For this we can use the Cartan identity $\mathcal{L}_X = i_X \mathrm{d} + \mathrm{d} i_X$ and recall that the action S in (5.18) is the integral of a 4-form. Under these diffeomorphisms the vector fields X_I defining the $\star$-product transform covariantly: $\delta_X X_I = \mathcal{L}_X X_I = [X, X_I]$, hence also the $\star$-product transforms.

For a fixed $*$-product (i.e., for fixed background fields X_I that do not transform under diffeomorphisms) the action is no more invariant under diffeomorphisms. The case of a $*$-product defined by vector fields X_I that are not fixed, but are dynamical thanks to an associated kinetic term, is discussed e.g. in [25].

Considering noncommutativity as a fixed background we are led to introduce $\star$-infinitesimal diffeomorphisms $\mathcal{L}^*_X$, satisfying a deformed Leibniz rule ([21], Sect. 4, eq. (4.13)). They also satisfy the Cartan identity $\mathcal{L}^*_X = i^*_X \mathrm{d} + \mathrm{d} i^*_X$ where i^*_X is the $\star$-contraction operator [21, 23]. Then as before $\mathcal{L}^*_X S = 0$. These deformed diffeomorphisms leave invariant the $\star$-product and the vector fields X_I (i.e., $\mathcal{L}^*_X X_I = 0$), therefore the action S with fixed noncommutativity background is invariant under $\star$-diffeomorphisms, cf. [18], Sect. 8.2.4.

Charge conjugation invariance. Noncommutative charge conjugation reads:

$$\psi \to \psi^C \equiv C(\bar{\psi})^{\mathrm{T}} = -\gamma_0 C \psi^*, \quad V \to V^C \equiv C V^{\mathrm{T}} C,$$
$$\Omega \to \Omega^C \equiv C \Omega^{\mathrm{T}} C \tag{5.21}$$

with $\star_\theta \to \star^C_\theta = \star_{-\theta}$ and consequently $\wedge_{\star_\theta} \to \wedge^C_{\star_\theta} = \wedge_{\star_{-\theta}}$, (see Appendix B for the properties of the charge conjugation matrix C). The action (5.18) is invariant under charge conjugation.

$$\begin{aligned}
S^C_{\text{bosonic}} &= i \int \mathrm{Tr}(R^C \wedge_{-\theta} V^C \wedge_{-\theta} V^C \gamma_5)^{\mathrm{T}} \\
&= -i \int \mathrm{Tr}(R^{\mathrm{T}} \wedge_{-\theta} V^{\mathrm{T}} \wedge_{-\theta} V^{\mathrm{T}} C \gamma_5 C^{-1})^{\mathrm{T}} \\
&= -i \int \mathrm{Tr}\big((V^{\mathrm{T}} \wedge_{-\theta} V^{\mathrm{T}} \gamma_5^{\mathrm{T}})^{\mathrm{T}} \wedge_\star R\big) \\
&= -i \int \mathrm{Tr}\big(-(V^{\mathrm{T}} \gamma_5^{\mathrm{T}})^{\mathrm{T}} \wedge_\star V \wedge_\star R\big) \\
&= i \int \mathrm{Tr}(\gamma_5 V \wedge_\star V \wedge_\star R) \\
&= i \int \mathrm{Tr}(R \wedge_\star \gamma_5 V \wedge_\star V) = i \int \mathrm{Tr}(R \wedge_\star V \wedge_\star V \gamma_5) \\
&= S_{\text{bosonic}}.
\end{aligned} \tag{5.22}$$

A similar proof holds for the fermionic part of the action: $S^C_{\text{fermionic}} = S_{\text{fermionic}}$.

Noncommutative action and gauge variations for the component fields. Finally, we give the bosonic noncommutative action in terms of the component fields V^a, ω^{ab}, $\tilde{V}^a$, ω, and $\tilde{\omega}$, and write the gauge variations of these fields.

$$\begin{aligned}
S_{\text{bosonic}} = \int & R^{ab} \wedge_\star (V^c \wedge_\star V^d - \tilde{V}^c \wedge_\star \tilde{V}^d) \epsilon_{abcd} \\
& - 2i\, R^{ab} \wedge_\star (V_a \wedge_\star \tilde{V}_b - \tilde{V}_a \wedge_\star V_b) \\
& - 4\, r \wedge_\star (V^a \wedge_\star \tilde{V}_a - \tilde{V}^a \wedge_\star V_a) \\
& + 4i\, \tilde{r} \wedge_\star (V^a \wedge_\star V_a - \tilde{V}^a \wedge_\star \tilde{V}_a)
\end{aligned} \tag{5.23}$$

with

$$\begin{aligned}
R^{ab} &= \mathrm{d}\omega^{ab} - \frac{1}{2} \omega^a{}_c \wedge_\star \omega^{cb} + \frac{1}{2} \omega^b{}_c \wedge_\star \omega^{ca} \\
&\quad - i(\omega^{ab} \wedge_\star \omega + \omega \wedge_\star \omega^{ab}) - \\
&\quad - \frac{i}{2} \epsilon^{ab}{}_{cd} (\omega^{cd} \wedge_\star \tilde{\omega} + \tilde{\omega} \wedge_\star \omega^{cd}) \\
r &= \mathrm{d}\omega - \frac{i}{8} \omega^{ab} \wedge_\star \omega_{ab} - i(\omega \wedge_\star \omega - \tilde{\omega} \wedge_\star \tilde{\omega}) \\
\tilde{r} &= \mathrm{d}\tilde{\omega} + \frac{i}{16} \epsilon_{abcd} \omega^{ab} \wedge_\star \omega^{cd} - i(\omega \wedge_\star \tilde{\omega} + \tilde{\omega} \wedge_\star \omega).
\end{aligned}$$

The noncommutative gauge variations read

$$\begin{aligned}
\delta_\epsilon V^a &= \frac{1}{2}(\epsilon^a{}_b \star V^b + V^b \star \epsilon^a{}_b) \\
&\quad + \frac{i}{4} \epsilon^a{}_{bcd} (\tilde{V}^b \star \epsilon^{cd} - \epsilon^{cd} \star \tilde{V}^b) \\
&\quad + i(\epsilon \star V^a - V^a \star \epsilon) - \tilde{\epsilon} \star \tilde{V}^a - \tilde{V}^a \star \tilde{\epsilon} \\
\delta_\epsilon \tilde{V}^a &= \frac{1}{2}(\epsilon^a{}_b \star \tilde{V}^b + \tilde{V}^b \star \epsilon^a{}_b) \\
&\quad + \frac{i}{4} \epsilon^a{}_{bcd} (V^b \star \epsilon^{cd} - \epsilon^{cd} \star V^b) \\
&\quad + i(\epsilon \star \tilde{V}^a - \tilde{V}^a \star \epsilon) - \tilde{\epsilon} \star V^a - V^a \star \tilde{\epsilon} \\
\delta_\epsilon \omega^{ab} &= \mathrm{d}\epsilon^{ab} + \frac{1}{2}(\epsilon^a{}_c \star \omega^{cb} - \epsilon^b{}_c \star \omega^{ca} \\
&\quad + \omega^{cb} \star \epsilon^a{}_c - \omega^{ca} \star \epsilon^b{}_c) \\
&\quad + i(\epsilon^{ab} \star \omega - \omega \star \epsilon^{ab}) \\
&\quad + \frac{i}{2} \epsilon^{ab}{}_{cd} (\epsilon^{cd} \star \tilde{\omega} - \tilde{\omega} \star \epsilon^{cd}) \\
&\quad + i(\epsilon \star \omega^{ab} - \omega^{ab} \star \epsilon) \\
&\quad + \frac{i}{2} \epsilon^{ab}{}_{cd} (\tilde{\epsilon} \star \omega^{cd} - \omega^{cd} \star \tilde{\epsilon}) \\
\delta_\epsilon \omega &= \mathrm{d}\epsilon - \frac{i}{8}(\omega^{ab} \star \epsilon_{ab} - \epsilon_{ab} \star \omega^{ab}) \\
&\quad + i(\epsilon \star \omega - \omega \star \epsilon - \tilde{\epsilon} \star \tilde{\omega} + \tilde{\omega} \star \tilde{\epsilon})
\end{aligned}$$

$$\delta_\epsilon \tilde{\omega} = \mathrm{d}\tilde{\epsilon} + \frac{i}{16}\epsilon_{abcd}(\omega^{ab} \star \epsilon^{cd} - \epsilon^{cd} \star \omega^{ab}) + i(\epsilon \star \tilde{\omega} - \tilde{\omega} \star \epsilon + \tilde{\epsilon} \star \omega - \omega \star \tilde{\epsilon}).$$

5.4 Classical limit and charge conjugation constraints

In the classical limit $\theta \to 0$ the $\star$-product becomes the usual pointwise product. The noncommutative gauge symmetry becomes a usual gauge symmetry with gauge group $GL(2, C)$ and the noncommutative vierbein in the classical limit leads to two independent vierbeins: V^a and $\tilde{V}^a$ transforming both only under the $SL(2, C)$ subgroup of $GL(2, C)$. As observed in [24] this is problematic because we obtain two massless gravitons and only one local Lorentz symmetry. That is not enough to kill the unphysical degrees of freedom. Either we concoct a mechanism such that the second graviton becomes massive or we further constrain the noncommutative theory so that in the classical limit the extra vierbein vanishes.

We present two methods of constraining the noncommutative fields. The first one is based on charge conjugation conditions. The second one will be presented in Sect. 7; it has a wider application and is based on the Seiberg–Witten map.

The fields in the noncommutative gravity action are in general θ-dependent as is clear by observing that the $\star$-gauge transformation of a field is θ dependent (because of the θ dependence of the $\star$-product). The vanishing of the $\tilde{V}^a$ components in the classical limit is achieved by imposing charge conjugation constraints on the fields [12]:

$$CV_\theta(x)C = V_{-\theta}(x)^{\mathrm{T}}, \quad C\Omega_\theta(x)C = \Omega_{-\theta}(x)^{\mathrm{T}},$$
$$C\epsilon_\theta(x)C = \epsilon_{-\theta}(x)^{\mathrm{T}} \tag{5.24}$$

where we have explicitly written the θ-dependence of the fields. Conditions (5.24) are consistent with $\star$-gauge transformations. For example, the field $CV_\theta(x)^{\mathrm{T}}C$ can be shown to transform in the same way as $V_{-\theta}(x)$.

These charge conjugation constraints imply that the fields and gauge parameter components $V^a, \omega^{ab}, \epsilon^{ab}$ are even in θ, while the components $\tilde{V}^a, \omega, \tilde{\omega}, \epsilon, \tilde{\epsilon}$ are odd.

We then conclude that the noncommutative gravity action in (5.18) with fields satisfying the charge conjugation constraints (5.24) is real, diffeomorphisms invariant, invariant under $GL(2, C)$ $\star$-gauge transformations and in the classical limit reduces to the usual gravity action with usual $SL(2, C)$ gauge invariance. Indeed, only the fields and gauge parameter components $V^a, \omega^{ab}, \epsilon^{ab}$ differ from zero in the classical limit.

As already observed the action is also charge conjugation invariant. In the presence of the charge conjugation constraints (5.24) the bosonic gravity action is furthermore even in θ. Indeed (5.24) implies $V^C = V_{-\theta}$, $\Omega^C = \Omega_{-\theta}$, $R^C = R_{-\theta}$. From the first equality in (5.22) we see that the bosonic action $S_{\text{bosonic}}(\theta)$ is mapped into $S_{\text{bosonic}}(-\theta)$ under charge conjugation, and since it is also invariant we conclude that it is even in θ.

6 Seiberg–Witten map

We first study the Seiberg–Witten map between commutative and noncommutative gauge theories with noncommutativity given by the Groenewold–Moyal product.

In a gauge theory physical quantities are gauge invariant: they do not depend on the gauge potential but on the gauge equivalence class of the potential given. The Seiberg–Witten map relates the noncommutative gauge fields to the commutative ones by requiring the noncommutative fields to have the same gauge equivalence classes as the commutative ones [4]. Explicitly, the noncommutative gauge potential $\hat{A} = A_\mu \mathrm{d}x^\mu$ and the noncommutative gauge parameters $\hat{\epsilon}$ depend on the ordinary A and ϵ so to satisfy:

$$\hat{A}(A + \delta_\epsilon A) = \hat{A}(A) + \hat{\delta}_{\hat{\epsilon}}\hat{A}(A) \tag{6.1}$$

with

$$\delta_\epsilon A_\mu = \partial_\mu \epsilon - iA_\mu \epsilon + i\epsilon A_\mu, \tag{6.2}$$

$$\hat{\delta}_{\hat{\epsilon}}\hat{A}_\mu = \partial_\mu \hat{\epsilon} - i\hat{A}_\mu \star \hat{\epsilon} + i\hat{\epsilon} \star \hat{A}_\mu. \tag{6.3}$$

This equation can be solved order by order in powers of the noncommutativity parameter θ yielding $\hat{A}$ and $\hat{\epsilon}$ as power series in θ:

$$\hat{A}(A) = A + A^1(A) + A^2(A) + \cdots + A^n(A) + \cdots \tag{6.4}$$

$$\hat{\epsilon}(\epsilon, A) = \epsilon + \epsilon^1(\epsilon, A) + \epsilon^2(\epsilon, A) + \cdots + \epsilon^n(\epsilon, A) + \cdots \tag{6.5}$$

where $A^n(A)$ and $\epsilon^n(\epsilon, A)$ are of order n in θ. Note that $\hat{\epsilon}$ depends on the ordinary ϵ and also on A. For example, up to first order it is readily checked that

$$\hat{A}_\kappa = A_\kappa + A^1_\kappa(A) + O(\theta^2) = A_\kappa - \frac{\theta^{\mu\nu}}{4}\{A_\mu, \partial_\nu A_\kappa + F_{\nu\kappa}\} + O(\theta^2) \tag{6.6}$$

$$\hat{\epsilon} = \epsilon + \epsilon^1 + O(\theta^2) = \epsilon - \frac{\theta^{\mu\nu}}{4}\{A_\mu, \partial_\nu \epsilon\} + O(\theta^2) \tag{6.7}$$

with $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - iA_\mu A_\nu + iA_\nu A_\mu$ and where $\{U, V\} = UV + VU$ is the anticommutator of two operators.

The Seiberg–Witten condition (6.1) holds for any value of the noncommutativity parameter θ. If we consider it at θ' and at θ we easily obtain that gauge equivalence classes of the θ'-noncommutative theory have to correspond to gauge equivalent classes of the θ-noncommutative theory, i.e., we generalize (6.1) to

$$\hat{A}'(\hat{A}+\hat{\delta}_{\hat{\epsilon}}\hat{A}) = \hat{A}'(\hat{A}) - \hat{\delta}'_{\hat{\epsilon}'}\hat{A}'(\hat{A}), \tag{6.8}$$

where we denoted by $\star'$, $\hat{A}'$, $\hat{\epsilon}'$, $\hat{\delta}'_{\hat{\epsilon}'}$ the star product, the gauge potential, the gauge parameter and the gauge variation: $\hat{\delta}'_{\hat{\epsilon}'}\hat{A}'_\kappa = \partial_\kappa\hat{\epsilon}' - i\hat{A}'_\kappa\star'\hat{\epsilon}' + i\hat{\epsilon}'\star'\hat{A}'_\kappa$ at noncommutativity parameter θ'. By considering θ and θ' infinitesimally close, so that $\theta' = \theta + \delta\theta$ and $\hat{A}' = \hat{A} + \delta\theta^{\mu\nu}\frac{\partial\hat{A}}{\partial\theta^{\mu\nu}}$ (we consider $\frac{\partial}{\partial\theta^{\mu\nu}}$ independent from $\frac{\partial}{\partial\theta^{\nu\mu}}$ and hence sum over all μ,ν indices) a rather straightforward computation, generalizing that for (6.6) and (6.7), shows that if $\hat{A}$ and $\hat{\epsilon}$ solve the differential equations

$$\frac{\partial}{\partial\theta^{\mu\nu}}\hat{A}_\kappa = -\frac{1}{8}\Big(\{\hat{A}_\mu, \partial_\nu\hat{A}_\kappa + \hat{F}_{\nu\kappa}\}_\star - \{\hat{A}_\nu, \partial_\mu\hat{A}_\kappa + \hat{F}_{\mu\kappa}\}_\star\Big) \tag{6.9}$$

$$\frac{\partial}{\partial\theta^{\mu\nu}}\hat{\epsilon} = -\frac{1}{8}\Big(\{\hat{A}_\mu, \partial_\nu\hat{\epsilon}\}_\star - \{\hat{A}_\nu, \partial_\mu\hat{\epsilon}\}_\star\Big) \tag{6.10}$$

where $\{U, V\}_\star = U\star V + V\star U$ and

$$\hat{F}_{\mu\nu} := \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i\hat{A}_\mu\star\hat{A}_\nu + i\hat{A}_\nu\star\hat{A}_\mu, \tag{6.11}$$

then $\hat{A}'(\hat{A})$ and $\hat{\epsilon}'(\hat{\epsilon}, \hat{A})$ satisfy also the Seiberg–Witten condition (6.8) for arbitrary values of θ' and θ. In particular, therefore, they solve the Seiberg–Witten condition (6.1).

The differential equations (6.9) and (6.10) admit solutions in terms of formal power series in θ. These are given recursively by

$$A^{n+1}_\mu = -\frac{1}{4(n+1)}\theta^{\rho\sigma}\{\hat{A}_\rho, \partial_\sigma\hat{A}_\mu + \hat{F}_{\sigma\mu}\}^n_\star, \tag{6.12}$$

$$\epsilon^{n+1} = -\frac{1}{4(n+1)}\theta^{\rho\sigma}\{\hat{A}_\rho, \partial_\sigma\hat{\epsilon}\}^n_\star, \tag{6.13}$$

where $\{\hat{f}, \hat{g}\}^n_\star$ is the n-th order term in $\{\hat{f}, \hat{g}\}_\star$, so that for example

$$\{\hat{A}_\rho, \partial_\sigma\hat{\epsilon}\}^n_\star \equiv \sum_{r+s+t=n}(A^r_\rho\star^s\partial_\sigma\epsilon^t + \partial_\sigma\epsilon^t\star^s A^r_\rho). \tag{6.14}$$

Here $\star^s$ indicates the s-th order term in the star product expansion [26]. There is a simple proof of (6.12), (6.13) [12]: multiplying the differential equations by $\theta^{\mu\nu}$ and analysing them order by order yields

$$\theta^{\mu\nu}\frac{\partial}{\partial\theta^{\mu\nu}}A^{n+1}_\rho = (n+1)A^{n+1}_\rho = -\frac{1}{4}\theta^{\mu\nu}\{\hat{A}_\mu, \partial_\nu\hat{A}_\rho + \hat{F}_{\nu\rho}\}^n_\star,$$

$$\theta^{\mu\nu}\frac{\partial}{\partial\theta^{\mu\nu}}\epsilon^{n+1} = (n+1)\epsilon^{n+1} = -\frac{1}{4}\theta^{\mu\nu}\{\hat{A}_\mu, \partial_\nu\hat{\epsilon}\}^n_\star$$

since A^{n+1}_ρ and ϵ^{n+1} are homogeneous functions of θ of order $n+1$.

Similar considerations hold for matter fields ϕ transforming in the fundamental or in the adjoint representation of the gauge group. The Seiberg–Witten condition reads, cf. [27],

$$\hat{\phi}(A + \delta_\epsilon A, \phi + \delta_\epsilon\phi) = \hat{\phi}(A, \phi) + \hat{\delta}_{\hat{\epsilon}}\hat{\phi}(A, \phi), \tag{6.15}$$

or more generally,

$$\hat{\phi}'(\hat{A} + \delta_{\hat{\epsilon}}\hat{A}, \hat{\phi} + \delta_{\hat{\epsilon}}\hat{\phi}) = \hat{\phi}'(\hat{A}, \hat{\phi}) + \hat{\delta}'_{\hat{\epsilon}'}\hat{\phi}'(\hat{A}, \hat{\phi}), \tag{6.16}$$

and it is satisfied if the matter fields solve the differential equation

$$\begin{aligned}\delta\theta^{\mu\nu}\frac{\partial\hat{\phi}}{\partial\theta^{\mu\nu}} &= -\frac{1}{4}\delta\theta^{\mu\nu}\hat{A}_\mu\star(\partial_\nu\hat{\phi} + D_\nu\hat{\phi})\\ &\text{fundamental rep., i.e., } \hat{\delta}_{\hat{\epsilon}}\hat{\phi} = i\hat{\epsilon}\star\hat{\phi},\\ \delta\theta^{\mu\nu}\frac{\partial\hat{\Psi}}{\partial\theta^{\mu\nu}} &= -\frac{1}{4}\delta\theta^{\mu\nu}\{\hat{A}_\mu, (\partial_\nu\hat{\Psi} + D_\nu\hat{\Psi})\}_\star\\ &\text{adjoint rep., i.e., } \hat{\delta}_{\hat{\epsilon}}\hat{\Psi} = i\hat{\epsilon}\star\hat{\Psi} - i\hat{\Psi}\star\hat{\epsilon}.\end{aligned} \tag{6.17}$$

The explicit solutions order by order in θ are

$$\phi^{n+1} = -\frac{1}{4(n+1)}\theta^{\mu\nu}(\hat{A}_\mu\star(\partial_\nu\hat{\phi} + D_\nu\hat{\phi}))^n \quad \text{(fundamental)} \tag{6.18}$$

$$\Psi^{n+1} = -\frac{1}{4(n+1)}\theta^{\mu\nu}\{\hat{A}_\mu, \partial_\nu\hat{\Psi} + D_\nu\hat{\Psi}\}^n_\star \quad \text{(adjoint)} \tag{6.19}$$

where

$$D_\nu\hat{\phi} = \partial_\nu\hat{\phi} - i\hat{A}_\nu\star\hat{\phi}, \quad D_\nu\hat{\Psi} = \partial_\nu\hat{\Psi} - i[\hat{A}_\nu, \hat{\Psi}]_\star$$

are the covariant derivative in the fundamental and in the adjoint, with $[S, T]_\star := S\star T - T\star S$.

The Seiberg–Witten differential equations (6.9), (6.10), (6.17) are not the most general solutions to the gauge equivalence condition (6.1). For example, from the differential equation for the gauge potential it is easy to see that the field strength $\hat{F}_{\mu\nu}$ satisfies the differential equation

$$\delta\theta^{\rho\sigma}\frac{\partial\hat{F}_{\mu\nu}}{\partial\theta^{\rho\sigma}} = -\frac{1}{4}\theta^{\rho\sigma}(\{\hat{A}_\rho, \partial_\sigma\hat{F}_{\mu\nu} + D_\sigma\hat{F}_{\mu\nu}\}_\star - 2\{\hat{F}_{\mu\rho}, \hat{F}_{\nu\sigma}\}_\star)$$

which has an extra addend with respect to the differential equation (6.17) for fields transforming in the adjoint. The most general Seiberg–Witten differential equations are presented in Appendix A. The freedom in the Seiberg–Witten differential equations may be useful for their integration, see e.g. Appendix A.

6.1 Geometric Seiberg–Witten map

The Seiberg–Witten map considered for Groenewold–Moyal noncommutativity can be generalized to the case of an abelian twist

$$\mathcal{F} = \mathrm{e}^{-\frac{i}{2}\theta^{IJ}X_I\otimes X_J} \tag{6.20}$$

where $\{X_I\}$ is a set of mutually commuting vector fields globally defined on a manifold M and θ^{IJ} is a constant antisymmetric matrix. The corresponding $\star$-product is obtained composing the usual pointwise multiplication $\mu(f\otimes g) = fg$ with the inverse twist $\mathcal{F}^{-1} = \mathrm{e}^{\frac{i}{2}\theta^{IJ}X_I\otimes X_J}$,

$$f\star g = \mu(\mathcal{F}^{-1}(f\otimes g)). \tag{6.21}$$

This $\star$-product is in general position dependent because the vector fields X_I are in general x-dependent. Associativity of the $\star$-product is guaranteed by mutual commutativity of the vector fields X_I. In the special case that $M = \mathbb{R}^d$ and $X_I = \frac{\partial}{\partial x^I}$, $I = 1,\ldots,d$ we recover the Groenewold–Moyal $\star$-product (3.1).

The use of vector fields $\{X_I\}$ on a manifold M suggests a coordinate independent approach to the Seiberg–Witten map. The resulting NC gauge potential $\hat{A}$ is then a 1-form that depends on A, on the mutually commuting vector fields X_I and on the deformation matrix $\theta = (\theta^{IJ})_{I,J=1,\ldots,d}$. The coordinate independent expression of the Seiberg–Witten differential eqs. (6.6),(6.10) reads

$$\frac{\partial}{\partial\theta^{IJ}}\hat{A} = -\frac{1}{4}\{i_{X_{[I}}\hat{A}, \mathcal{L}_{X_{J]}}\hat{A} + i_{X_{J]}}\hat{F}\}_\star \tag{6.22}$$

$$\frac{\partial}{\partial\theta^{IJ}}\hat{\epsilon} = -\frac{1}{4}\{i_{X_{[I}}\hat{A}, \mathcal{L}_{X_{J]}}\hat{\epsilon}\}_\star \tag{6.23}$$

where $\hat{F} \equiv \mathrm{d}\hat{A} - i\hat{A}\wedge_\star\hat{A}$ is a two-form, i_{X_I} and $\mathcal{L}_{X_I}$ are respectively the contraction and the Lie derivative along the mutually commuting vector fields X_I. When the abelian twist reduces to the Groenewold–Moyal case of the preceding section, the curvature becomes $\hat{F} = \frac{1}{2}\hat{F}_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$, where the $\hat{F}_{\mu\nu}$ components are given in (6.11). Note that in the Groenewold–Moyal case $\mathrm{d}x^\mu\wedge_\star\mathrm{d}x^\nu = \mathrm{d}x^\mu\wedge\mathrm{d}x^\nu$ since $\mathcal{L}_{X_I}\mathrm{d}x^\mu = \mathrm{d}\mathcal{L}_{X_I}x^\mu = 0$. Proceeding as in the Groenewold–Moyal case the recursive solutions are given by:

$$A^{n+1} = -\frac{1}{4(n+1)}\theta^{IJ}\{i_{X_I}\hat{A}, \mathcal{L}_{X_J}\hat{A} + i_{X_J}\hat{F}\}^n_\star \tag{6.24}$$

$$\epsilon^{n+1} = -\frac{1}{4(n+1)}\theta^{IJ}\{i_{X_I}\hat{A}, \mathcal{L}_{X_J}\hat{\epsilon}\}^n_\star. \tag{6.25}$$

Similarly one proves the generalization of Eqs. (6.18)–(6.19):

$$\begin{aligned}\phi^{n+1} &= -\frac{1}{4(n+1)}\theta^{IJ}\Big(i_{X_I}\hat{A}\star(2\mathcal{L}_{X_J}\hat{\phi} - i(i_{X_J}\hat{A})\star\hat{\phi})\Big)^n,\\ \hat{\delta}_{\hat{\epsilon}}\hat{\phi} &= i\hat{\epsilon}\star\hat{\phi}\end{aligned} \tag{6.26}$$

$$\begin{aligned}\Psi^{n+1} &= -\frac{1}{4(n+1)}\theta^{IJ}\{i_{X_I}\hat{A},\, 2\mathcal{L}_{X_J}\hat{\psi} - i(i_{X_J}\hat{A})\star\hat{\Psi}\\ &\quad + i\hat{\Psi}\star(i_{X_J}\hat{A})\}^n_\star,\\ \hat{\delta}_{\hat{\epsilon}}\hat{\Psi} &= i\hat{\epsilon}\star\hat{\Psi} - i\hat{\Psi}\star\hat{\epsilon}.\end{aligned} \tag{6.27}$$

In this subsection we have constructed the geometric Seiberg–Witten map for noncommutative gauge theories with gauge group $U(N)$ or $GL(N)$ (or products thereof) and with $\star$-product given by a general abelian twist (6.20). For abelian gauge groups the Seiberg–Witten map can be constructed for any $\star$-product associated with an arbitrary Poisson tensor. The map is obtained in [28] using Kontsevich formality theorem [29]. The study of its global geometric aspects shows that the Seiberg–Witten map quantizes line bundles with connections on a Poisson manifold to quantum (noncommutative) line bundles with noncommutative connections [30] The Seiberg–Witten map for nonabelian gauge groups and with arbitrary Poisson tensors is in general an open problem, we refer to [31] for interesting insights. The global geometric aspects of the Seiberg–Witten map are well understood for nonabelian $U(n)$-gauge fields on noncommutative tori: the Seiberg–Witten map defined in (A.9), (A.10), with $\gamma = -3$, $\rho = i$, quantizes vector bundles on tori with connections to vector bundles on noncommutative tori with noncommutative connections and the results are nonformal in the sense that they do not rely on power series expansion in the noncommutativity parameter θ.

We also mention that the Seiberg–Witten map for Chern–Simons gauge theories can be studied to all orders in the noncommutativity parameter θ [32, 33].

7 Noncommutative gauge theories with any gauge group

Up to now we have considered noncommutative gauge theories with gauge group $U(N)$ or $GL(N)$ in the fundamental or adjoint, and more generally representations of gauge groups such that the generators T^A of the Lie algebra close also in the usual matrix product. This is needed for the closure of infinitesimal gauge transformations, cf. (4.2). If on the other hand we consider an arbitrary gauge group G, Eq. (4.2) shows that infinitesimal gauge transformations do not close in the Lie algebra Lie(G), but in the universal enveloping algebra $\mathcal{U}(\mathrm{Lie}(G))$. This latter is the product of all generators T^A modulo the relations $T^AT^B - T^BT^A = [T^A, T^B]$. When considering an arbitrary gauge group G the noncommutative gauge potential is therefore universal enveloping algebra valued

$$\hat{A} = \hat{A}^A T^A + \hat{A}^{AB} T^A T^B + \hat{A}^{ABC} T^A T^B T^C + \cdots$$

and hence with infinitely many (symmetric) components $\hat{A}^A, \hat{A}^{AB}, \hat{A}^{ABC}, \ldots$. The Seiberg–Witten map is well defined also in this case. It then constrains these infinite components to depend on the commutative ones $A^A = A^A_\mu \mathrm{d}x^\mu$ in $A = A^A T^A$. Similarly, the gauge parameters and the matter fields depend on the commutative gauge parameters and matter fields, besides the commutative gauge potential.

This Seiberg–Witten map approach to noncommutative gauge theories is called universal enveloping algebra valued approach [27]. It has been used to propose noncommutative standard and grand unified particle physics models [34, 35] having the same degrees of freedom as in the commutative models. Renormalizability and scattering amplitudes using the Seiberg–Witten map have been studied e.g. in [36–39], either considering a power series expansion in θ or a θ-exact approach (i.e. to all orders in θ) [37, 40], where the power series is instead in the gauge coupling constant. For recent literature on scattering amplitudes of noncommutative particle models using the Seiberg–Witten map see [41, 42] and references therein.

7.1 Expansion of gauge and matter fields to first order in θ

Up to first order in θ the solution to the Seiberg–Witten conditions for the gauge potential, the infinitesimal gauge transformation parameter, the matter fields and the field strength reads:

$$\hat{A} = A - \frac{1}{4}\theta^{IJ}\{A_I, \mathcal{L}_J A + F_J\} + \mathcal{O}(\theta^2) \tag{7.1}$$

$$\hat{\epsilon} = \epsilon - \frac{1}{4}\theta^{IJ}\{A_I, \mathcal{L}_J \epsilon\} + \mathcal{O}(\theta^2) \tag{7.2}$$

$$\hat{\psi} = \psi - \frac{1}{4}\theta^{IJ} A_I(\mathcal{L}_J + L_J)\psi + \mathcal{O}(\theta^2) \tag{7.3}$$

$$\hat{F} = F - \frac{1}{4}\theta^{IJ}(\{A_I, (\mathcal{L}_J + L_J)F\} - [F_I, F_J]) + \mathcal{O}(\theta^2) \tag{7.4}$$

where A_I, F_I are defined as the contraction along the tangent vector X_I of the exterior forms A, F, i.e., $A_I \equiv i_I A$, $F_I \equiv i_I F$, (i_I being the contraction along X_I). We have also introduced the Lie derivative $\mathcal{L}_I$ along the vector field X_I, and the covariant Lie derivative L_I along the vector field X_I. L_I acts on F and ψ as $L_I F = \mathcal{L}_I F - iA_I F + iF A_I$ and $L_I \psi = \mathcal{L}_I \psi - A_I \psi$. The covariant Lie derivative L_I has the Cartan form:

$$L_I = i_I D + D i_I$$

where D is the covariant derivative. We refer to [13] for higher order in θ expressions.

The Seiberg–Witten map (7.1)–(7.4), and its higher order in θ terms, allows to expand noncommutative gauge theory actions in terms of commutative fields. Noncommutative gauge theories are therefore seen as commutative ones with specific interaction terms due to noncommutativity of spacetime.

We exemplify this general procedure by studying the Seiberg–Witten map for the gravity action (5.18).

7.2 Expansion of noncommutative gravity action at first order in θ

As usual in this case we use the notation $\Omega = A$, $R = F$. The Seiberg–Witten solutions (7.1)–(7.4) are *not* $SO(1,\,3)$ -gauge covariant, due to the presence of the "naked" connection Ω and the non-covariant Lie derivative $\mathcal{L}_I = i_I \mathrm{d} + \mathrm{d} i_I$. However, when inserted in the NC action the resulting action is gauge invariant order by order in θ. Indeed usual gauge variations induce the $\star$-gauge variations under which the noncommutative action is invariant. Therefore the NC action, re-expressed in terms of ordinary fields via the SW map, is invariant under usual gauge transformations. Moreover the action, once re-expressed in terms of ordinary fields remains geometric, and hence invariant under diffeomorphisms. This is the case because the noncommutative action and the SW map are geometric: indeed only coordinate independent operations like the contraction i_I and the Lie derivatives $\mathcal{L}_I$ and L_I appear in the Seiberg–Witten map.

We replace the noncommutative fields appearing in the action with their expansions (7.1)–(7.4) in commutative fields, and integrating by parts we obtain the following gravity action coupled to spinors

$$\begin{aligned} S = \int & \mathrm{Tr}(iRVV\gamma_5) + \bar{\psi}V^3\gamma_5 D\psi + D\bar{\psi}V^3\gamma_5\psi \\ & + \frac{i}{4}\theta^{IJ}\Big(\bar{\psi}\{V^3, R_{IJ}\}\gamma_5 D\psi + D\bar{\psi}\{V^3, R_{IJ}\}\gamma_5\psi\Big) \\ & + \frac{i}{2}\theta^{IJ}\Big(2L_I\bar{\psi}R_J V^3\gamma_5\psi - 2\bar{\psi}V^3 R_I\gamma_5 L_J\psi \\ & - L_I\bar{\psi}V^3\gamma_5 L_J D\psi - L_I D\bar{\psi}\, V^3\gamma_5 L_J\psi \\ & + \bar{\psi}(\{L_I V L_J V, V\} + L_I V\, V L_J V)\gamma_5 D\psi \\ & + D\bar{\psi}(\{L_I V L_J V, V\} + L_I V\, V L_J V)\gamma_5\psi\Big) + O(\theta^2) \end{aligned} \tag{7.5}$$

where we have omitted writing the wedge product, and $V^3 = V \wedge V \wedge V$. The expression of the gravity action, up to second order in θ, in terms of the commutative fields has been given in [14], after a propaedeutical study of the Seiberg–Witten map for $\star$-products of fields.

In conclusion, we have constructed a gravity action in noncommutative spacetime—an expected feature of quantum spacetime—and shown its equivalence to the usual gravity action (in the first order formalism) on commutative spacetime with extra interaction terms.

These are obtained from spacetime noncommutativity using the Seiberg–Witten map between commutative and noncommutative gauge theories. This extended gravity action is invariant under local Lorentz transformations because it is expressed solely in terms of gauge covariant operators L_I, i_I, D, curvature R, vierbein V, spinor fields ψ, invariant vector fields $\{X_I\}$ and noncommutativity parameter θ. It is diffeomorphic invariant and charge conjugation invariant. This noncommutative gravity action can also be coupled to noncommutative scalar and gauge fields [25, 43]. Choosing an appropriate kinetic term, the vector fields $\{X_I\}$ can become dynamical, the idea being that both spacetime curvature and noncommutativity should depend on matter distribution. It would be interesting to study cosmological models as solutions of these extended gravity actions.

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Appendix A: Ambiguities in the Seiberg–Witten map

The solution to the Seiberg–Witten conditions (6.1), (6.15) is not unique. For example if $\hat{A}_\mu$ is a solution, any noncommutative gauge transformation of $\hat{A}_\mu$ gives another solution. Another source of ambiguities is that of field redefinitions of the gauge potential (e.g., if $\hat{A}_\mu$ is a solution then so is $\hat{A}_\mu + \theta^{\rho\sigma}\theta^{\lambda\eta}\hat{F}_{\rho\lambda} \star D_\sigma \hat{F}_{\eta\mu}$). We generalize the Seiberg–Witten equations (6.6), (6.10) and (6.17) allowing for three extra terms $\hat{D}_{\mu\nu\rho}(\hat{A}), \hat{E}_{\mu\nu}(\hat{A}, \hat{\epsilon})$, $\hat{C}_{\mu\nu}(\hat{A}, \hat{\phi})$ and $\hat{C}_{\mu\nu}(\hat{A}, \hat{\Psi})$ that are a priori arbitrary functions of their arguments and derivatives thereof, that are (formal) power series in θ and that are antisymmetric in the μ, ν indices. We consider the equations

$$\delta^\theta \hat{A}_\kappa = \delta\theta^{\mu\nu} \frac{\partial \hat{A}_\kappa}{\partial\theta^{\mu\nu}} = -\frac{1}{4}\delta\theta^{\mu\nu}\Big(\{\hat{A}_\mu, \partial_\nu \hat{A}_\kappa + \hat{F}_{\nu\kappa}\}_\star + \hat{D}_{\mu\nu\kappa}(\hat{A})\Big), \tag{A.1}$$

$$\delta^\theta \hat{\epsilon} = \delta\theta^{\mu\nu} \frac{\partial \hat{\epsilon}}{\partial\theta^{\mu\nu}} = -\frac{1}{4}\delta\theta^{\mu\nu}\Big(\{\partial_\mu \hat{\epsilon}, \hat{A}_\nu\}_\star + \hat{E}_{\mu\nu}(\hat{A}, \hat{\epsilon})\Big), \tag{A.2}$$

$$\delta^\theta \hat{\phi} = \delta\theta^{\mu\nu} \frac{\partial \hat{\phi}}{\partial\theta^{\mu\nu}} = -\frac{1}{4}\delta\theta^{\mu\nu}\Big(\hat{A}_\mu \star \partial_\nu \hat{\phi} + \hat{A}_\mu \star D_\nu \hat{\phi} + \hat{C}_{\mu\nu}(\hat{A}, \hat{\phi})\Big), \tag{A.3}$$

$$\delta^\theta \hat{\Psi} = \delta\theta^{\mu\nu} \frac{\partial \hat{\Psi}}{\partial\theta^{\mu\nu}} = -\frac{1}{4}\delta\theta^{\mu\nu}\Big(\{\hat{A}_\mu, \partial_\nu \hat{\Psi} + D_\nu \hat{\Psi}\}_\star + \hat{C}_{\mu\nu}(\hat{A}, \hat{\Psi})\Big) \tag{A.4}$$

and observe that $\hat{E}_{\mu\nu}$ must be linear in $\hat{\epsilon}$ since all terms in (A.2) but $\hat{E}_{\mu\nu}$ are linear in $\hat{\epsilon}$, similarly $\hat{C}_{\mu\nu}$ must be linear in $\hat{\phi}$ because of the linearity in $\hat{\phi}$ of all other terms in (A.3), and similarly for $\hat{C}_{\mu\nu}(\hat{A}, \hat{\Psi})$ in (A.4). Imposing the Seiberg–Witten conditions (6.8), (6.16) we obtain the conditions

$$\begin{aligned}
&\hat{D}_{\mu\nu\kappa}(\hat{A} + \hat{\delta}_{\hat{\epsilon}}\hat{A}) - \hat{D}_{\mu\nu\kappa}(\hat{A}) - i[\hat{\epsilon}, \hat{D}_{\mu\nu\kappa}(\hat{A})]_\star \\
&\quad = -D_\kappa \hat{E}_{\mu\nu}(\hat{A}, \hat{\epsilon}), \\
&\hat{C}_{\mu\nu}(\hat{A} + \hat{\delta}_{\hat{\epsilon}}\hat{A}, \hat{\phi} + \hat{\delta}_{\hat{\epsilon}}\hat{\phi}) - \hat{C}_{\mu\nu}(\hat{A}, \hat{\phi}) - i\hat{\epsilon} \star \hat{C}_{\mu\nu}(\hat{A}, \hat{\phi}) \\
&\quad = -i\hat{E}_{\mu\nu}(\hat{A}, \hat{\epsilon}) \star \hat{\phi}, \\
&\hat{C}_{\mu\nu}(\hat{A} + \hat{\delta}_{\hat{\epsilon}}\hat{A}, \hat{\Psi} + \hat{\delta}_{\hat{\epsilon}}\hat{\Psi}) - \hat{C}_{\mu\nu}(\hat{A}, \hat{\Psi}) - i[\hat{\epsilon}, \hat{C}_{\mu\nu}(\hat{A}, \hat{\Psi})]_\star \\
&\quad = -i[\hat{E}_{\mu\nu}(\hat{A}, \hat{\epsilon}), \hat{\Psi}]_\star.
\end{aligned} \tag{A.5}$$

In particular we notice that any $\hat{D}_{\mu\nu\kappa}$ and $\hat{C}_{\mu\nu}$ covariant under gauge transformations solve (A.5) with $\hat{E}_{\mu\nu} = 0$.

In summary, as discussed in [44], the most general solution $\hat{A}(A)$, $\hat{\epsilon}(A, \varepsilon)$, $\hat{\phi}(A, \phi)$, $\hat{\Psi}(A, \Psi)$ of the Seiberg–Witten conditions (6.1), (6.15) is given by the differential equations (A.1)–(A.4) where $\hat{D}, \hat{E}, \hat{C}$ are constrained by (A.5). Further constraints on the $\hat{D}, \hat{E}, \hat{C}$ terms are obtained by requiring that the Seiberg–Witten map respects hermiticity and charge conjugation in the sense that the hermiticity and charge conjugation properties of the commutative fields imply those of the noncommutative fields [13, 35]. If we ask $\hat{D}, \hat{E}, \hat{C}$ to have no explicit dependence on θ and to be covariant under constant $GL(d,\mathbb{R})$ coordinate transformations (the star product $f \star g$ is itself invariant under constant $GL(d,\mathbb{R})$ coordinate transformations: $x^\mu \to M^\mu{}_\rho x^\rho$, $\theta^{\mu\nu} \to M^\mu{}_\rho M^\nu{}_\sigma \theta^{\rho\sigma}$) we recover the results in [45] and in [46]:

$$\hat{D}_{\mu\nu\kappa} = \alpha D_\kappa \hat{F}_{\mu\nu} + \beta D_\kappa [\hat{A}_\mu, \hat{A}_\nu]_\star, \quad \hat{E}_{\mu\nu} = 2\beta[\partial_\mu \hat{\epsilon}, \hat{A}_\nu]_\star, \tag{A.6}$$

$$\hat{C}_{\mu\nu} = -2i\beta[\hat{A}_\mu, \hat{A}_\nu]_\star \star \hat{\phi} + \gamma \hat{F}_{\mu\nu} \star \hat{\phi} \quad \text{(fundamental)} \tag{A.7}$$

$$\hat{C}_{\mu\nu} = -2i\beta[[\hat{A}_\mu, \hat{A}_\nu]_\star, \hat{\Psi}]_\star + \gamma' \hat{F}_{\mu\nu} \star \hat{\Psi} + \tilde{\gamma} \hat{\Psi} \star \hat{F}_{\mu\nu} \quad \text{(adjoint)} \tag{A.8}$$

with $\alpha, \beta, \gamma, \gamma'$ and $\tilde{\gamma}$ arbitrary constants.

An interesting Seiberg–Witten differential equation is obtained considering (A.1)–(A.4) with $\hat{D}_{\mu\nu\kappa} = 0$, $\hat{E}_{\mu\nu} = 0$ and

$$\hat{C}_{\mu\nu}(\hat{A}, \hat{\phi}) = \gamma \hat{F}_{\mu\nu} \star \hat{\phi} + \rho D_\mu D_\nu \hat{\phi} \quad \text{for } \mu < \nu, \text{ and } \hat{C}_{\nu\mu} := -\hat{C}_{\mu\nu} \tag{A.9}$$

$$\hat{C}_{\mu\nu}(\hat{A}, \hat{\Psi}) = \gamma [\hat{F}_{\mu\nu}, \hat{\Psi}]_\star + \rho D_\mu D_\nu \hat{\Psi} \quad \text{for } \mu < \nu, \text{ and } \hat{C}_{\nu\mu} := -\hat{C}_{\mu\nu}. \tag{A.10}$$

Here ρ is a constant and $GL(d,\mathbb{R})$ covariance is broken because $D_\mu D_\nu \hat{\Psi}$ is not antisymmetric in the μ, ν indices, i.e., $\hat{C}_{\nu\mu}$ does not contain also the term $\rho D_\nu D_\mu \phi$. This choice, with $\gamma = -3$ and $\rho = i$, allows to solve the Seiberg–Witten map on noncommutative tori (obtained from the Groenewold–Moyal noncommutative plane) to all orders in θ for topologically nontrivial $U(N)$-gauge potentials with constant field strengths [44].

Appendix B: Gamma matrices in $D = 4$

We summarize in this Appendix our gamma matrix conventions in $D = 4$.

$$\eta_{ab} = (1, -1, -1, -1), \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad [\gamma_a, \gamma_b] = 2\gamma_{ab}, \tag{B.1}$$

$$\gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \gamma_5\gamma_5 = 1, \quad \varepsilon_{0123} = -\varepsilon^{0123} = 1, \tag{B.2}$$

$$\gamma_a^\dagger = \gamma_0 \gamma_a \gamma_0, \quad \gamma_5^\dagger = \gamma_5 \tag{B.3}$$

$$\gamma_a^T = -C\gamma_a C^{-1}, \quad \gamma_5^T = C\gamma_5 C^{-1}, \quad C^2 = -1, \quad C^\dagger = C^T = -C. \tag{B.4}$$

Useful identities

$$\gamma_a \gamma_b = \gamma_{ab} + \eta_{ab} \tag{B.5}$$

$$\gamma_{ab}\gamma_5 = \frac{i}{2}\varepsilon_{abcd}\gamma^{cd} \tag{B.6}$$

$$\gamma_{ab}\gamma_c = \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \tag{B.7}$$

$$\gamma_c\gamma_{ab} = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a - i\varepsilon_{abcd}\gamma_5\gamma^d \tag{B.8}$$

$$\gamma_a\gamma_b\gamma_c = \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abcd}\gamma_5\gamma^d \tag{B.9}$$

$$\gamma^{ab}\gamma_{cd} = -i\varepsilon^{ab}{}_{cd}\gamma_5 - 4\delta^{[a}_{[c}\gamma^{b]}{}_{d]} - 2\delta^{ab}_{cd} \tag{B.10}$$

$$\mathrm{Tr}(\gamma_a\gamma^{bc}\gamma_d) = 8\ \delta^{bc}_{ad} \tag{B.11}$$

$$\mathrm{Tr}(\gamma_5\gamma_a\gamma_{bc}\gamma_d) = -4i\,\varepsilon_{abcd} \tag{B.12}$$

where $\delta^{ab}_{cd} \equiv \frac{1}{2}(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d)$ and indices antisymmetrization in square brackets has total weight 1.

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