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Quantum quasi-Lie systems: properties and applications

J.F. Cariñena[1,a], **J. de Lucas**[2,b], **C. Sardón**[3,c]

[1] Department of Theoretical Physics and IUMA, University of Zaragoza, c. Pedro Cerbuna 12, 50009 Zaragoza, Spain
[2] Department of Mathematical Methods in Physics, University of Warsaw, ul. Pasteura 5, 02-093 Warsaw, Poland
[3] Department of Applied Mathematics, Polytechnic University of Madrid (UPM), c. José Gutiérrez Abascal 2, 28006 Madrid, Spain

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Abstract A Lie system is a non-autonomous system of ordinary differential equations describing the integral curves of a t-dependent vector field that is equivalent to a t-dependent family of vector fields within a finite-dimensional Lie algebra of vector fields. Lie systems have been generalised in the literature to deal with t-dependent Schrödinger equations determined by a particular class of t-dependent Hamiltonian operators, the quantum Lie systems, and other systems of differential equations through the so-called quasi-Lie schemes. This work extends quasi-Lie schemes and quantum Lie systems to cope with t-dependent Schrödinger equations associated with the here-called quantum quasi-Lie systems. To illustrate our methods, we propose and study a quantum analogue of the classical nonlinear oscillator searched by Perelomov, and we analyse a quantum one-dimensional fluid in a trapping potential along with quantum t-dependent Smorodinsky–Winternitz oscillators.

1 Introduction

A Lie system is a non-autonomous system of first-order ordinary differential equations whose general solution can be written via a (generally nonlinear) autonomous function, referred to as *superposition rule*, a finite set of particular solutions, and some constants related to the initial conditions [1–4]. For instance, non-autonomous inhomogeneous linear systems of first-order ordinary differential equations, Bernoulli equations, Riccati equations, and matrix Riccati equations are examples of Lie systems [3–5]. To make the terminology simpler, non-autonomous systems will be hereafter called t-dependent systems. In physical applications, the variable t will generally represent the time.

Lie systems are mathematically interesting due to their geometric and algebraic properties [5, 6]. In the theory of Lie systems and in its generalisations, it is frequently used that a t-dependent system of ordinary differential equations on an n-dimensional manifold M of the form

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = X^i(t,x), \qquad i = 1,\ldots,n, \qquad t \in \mathbb{R}, \qquad x \in M, \tag{1.1}$$

amounts to a t-dependent vector field $X : \mathbb{R} \times M \to TM$ given by $X = \sum_{i=1}^n X^i(t,x)\frac{\partial}{\partial x^i}$, namely a t-parametric family of vector fields on M of the form $X_t : x \in M \mapsto X(t,x) \in TM$ with $t \in \mathbb{R}$ [3]. In fact, X is frequently used to refer to (1.1) and X, indistinctly. The Lie–Scheffers theorem establishes that a Lie system is equivalent to a t-dependent vector field X whose associated vector fields $\{X_t\}_{t\in\mathbb{R}}$ are contained in a finite-dimensional Lie algebra W of vector fields, known as a Vessiot–Guldberg Lie algebra (VG Lie algebra henceforth) of the Lie system [2, 3]. As a consequence, it can be shown that the study of Lie systems leads to the investigation of superposition rules through projective foliations on an appropriate bundle [2], generalised distributions [2], Lie group actions [4, 6], etc. Furthermore, there has been a recent interest in Lie systems possessing VG Lie algebras of Hamiltonian vector fields regarding different geometric structures [5, 7, 8].

The Lie–Scheffers theorem implies that being a Lie system is a very restrictive condition [3, 5, 9, 10]. For instance, all Lie systems on the real line are locally diffeomorphic to subcases of the so-called Riccati equations [10]. Nonetheless, Lie systems play a relevant rôle in mathematics, physics, and control theory, e.g. [3] cites more than 200 works on Lie systems and their applications. In particular, the theory of Lie systems provides methods to determine the integrability of certain t-dependent systems of first-order

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[a] e-mail: jfc@unizar.es (corresponding author)
[b] e-mail: j.de-lucas-ara@uw.edu.pl
[c] e-mail: cristinasardon@usal.es

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differential equations [3, 11]. Furthermore, Lie systems have been proved to be very helpful in the study of geometric phases [12], the solution of relevant nonlinear oscillators [3], the analysis of Wei–Norman equations [3, 13–15], and the research on problems occurring in quantum mechanics [3, 7] and biology [5]. More particularly, Riccati equations and most of their generalisations are examples of Lie systems that are ubiquitous in physics, occurring, for instance, in cosmology and financial models [3, 11, 16–18]. Riccati equations appear also in the study of epidemiological models [19].

A generalisation of the concept of Lie system has recently been proposed in [3], where the theory of quasi-Lie systems was developed. Let us briefly explain the main elements of this theory. The theory of quasi-Lie systems is based on the so-called quasi-Lie scheme notion. A quasi-Lie scheme consists of a finite-dimensional vector space of vector fields, V, on a manifold M and a certain vector subspace $W \subset V$ such that W is a Lie algebra of vector fields, which is denoted by writing $[W, W] \subset W$, and the Lie bracket of any vector field of W with any vector field of V belongs to V, which is represented by writing $[W, V] \subset V$. The quasi-Lie scheme consisting of W and V is denoted by $S(W, V)$.

A quasi-Lie scheme $S(W, V)$ may be employed to investigate certain related t-dependent systems of first-order differential equations. More specifically, a quasi-Lie scheme $S(W, V)$ can be used to study any t-dependent vector field X whose associated vector fields, $\{X_t\}_{t\in\mathbb{R}}$, are contained in the finite-dimensional linear space of vector fields V. As V is not in general a Lie algebra, then X does not need to be a Lie system. The Lie algebra W of vector fields can be integrated to give rise to a local Lie group action of an associated connected Lie group, which can be employed to transform, in certain cases, X into a Lie system with a VG Lie algebra contained in V (cf. [3]). This idea allows one to analyse the integrability of non-harmonic classical oscillators, dissipative Milne–Pinney equations, and other systems that cannot be studied through Lie systems (see [3, 20] and references therein). In fact, quasi-Lie schemes are frequently used to transform a system X on manifold M that it is not a Lie system into a Lie system on M by means of a t-dependent transformation on M.

To make the introduction more accessible and to clarify the advantages of our methods, let us detail a simple example. The differential equations on $\mathbb{R}$ given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_0(t) + a_1(t)x + a_2(t)x^2 + a_3(t)x^3, \tag{1.2}$$

where $a_0(t), \ldots, a_3(t)$ are arbitrary t-dependent coefficients, appear in the study of Abel equations [20]. Geometrically, a differential equation (1.2) is described by a t-dependent vector field on $\mathbb{R}$ of the form

$$X(t, x) = \left(a_0(t) + a_1(t)x + a_2(t)x^2 + a_3(t)x^3\right)\frac{\partial}{\partial x}, \tag{1.3}$$

which gives rise to a t-parametric family of vector fields $\{X_t\}_{t\in\mathbb{R}}$ on $\mathbb{R}$. For every $t \in \mathbb{R}$, the vector field X_t is a linear combination $X_t = \sum_{\alpha=0}^{3} a_\alpha(t)X_\alpha$ of the vector fields

$$X_0 = \frac{\partial}{\partial x}, \quad X_1 = x\frac{\partial}{\partial x}, \quad X_2 = x^2\frac{\partial}{\partial x}, \quad X_3 = x^3\frac{\partial}{\partial x}.$$

In other words, X_t belongs, for each $t \in \mathbb{R}$, to the linear vector space $V = \langle X_0, \ldots, X_3\rangle$. If we fix $W = \langle X_0, X_1\rangle$, then W is a finite-dimensional Lie algebra of vector fields, and $[W, V] \subset V$, which turns W and V into a quasi-Lie scheme $S(W, V)$. Note that V is not a Lie algebra of vector fields. Moreover, it can be proved that V cannot be contained in any finite-dimensional Lie algebra of vector fields [3]. Then, the theory of quasi-Lie schemes uses W and V to integrate and study equations (1.2) in a geometric, elegant, and less *ad-hoc* manner than traditional methods (see [3, 20] and references therein).

Meanwhile, the theory of Lie systems has been extended to the quantum realm in [3, 21, 22]. A quantum Lie system is a t-dependent Hamiltonian operator $\widehat{H}(t)$ such that $\mathrm{i}\widehat{H}(t)$, for each fixed $t \in \mathbb{R}$, is contained in a finite-dimensional real Lie algebra of skew-Hermitian operators, a so-called *Vessiot–Guldberg (VG) Lie algebra* of $\widehat{H}(t)$ [3, 21]. For instance, the t-dependent Hamiltonian operator $\widehat{H}(t) = \widehat{p}^2/2+\omega^2(t)\widehat{x}^2/2$, where $\widehat{x}$ and $\widehat{p}$ are the position and momentum operators for a particle in $\mathbb{R}$, respectively, describes a one-dimensional quantum harmonic oscillator with a t-dependent frequency $\omega(t)$. Then, $\widehat{H}(t)$ is a quantum Lie system because $\mathrm{i}\widehat{H}(t) = \mathrm{i}\widehat{H}_2 + \omega^2(t)\mathrm{i}\widehat{H}_1$, with

$$\mathrm{i}\widehat{H}_1 = \mathrm{i}\widehat{x}^2/2, \qquad \mathrm{i}\widehat{H}_2 = \mathrm{i}\widehat{p}^2/2.$$

Then, one can understand that, for each $t \in \mathbb{R}$, the skew-Hermitian operator $\mathrm{i}\widehat{H}(t)$ belongs to $\mathfrak{W} = \langle \mathrm{i}\widehat{H}_1, \mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_3\rangle$, where $\mathrm{i}\widehat{H}_3 = \mathrm{i}(\widehat{x}\widehat{p} + \widehat{p}\widehat{x})/4$. In fact, $\mathfrak{W}$ is a three-dimensional real Lie algebra of skew-Hermitian operators relative to the commutator operator. The form of $\mathfrak{W}$ is chosen so as to ensure that $\mathrm{i}\widehat{H}(t)$ belongs to $\mathfrak{W}$ for every $t \in \mathbb{R}$, which explains why $\mathrm{i}\widehat{H}_1, \mathrm{i}\widehat{H}_2 \in \mathfrak{W}$, while $\mathrm{i}\widehat{H}_3$ was added to $\mathfrak{W}$ so that it becomes a Lie algebra. In fact, $-2\mathrm{i}\widehat{H}_3 = [\mathrm{i}\widehat{H}_1, \mathrm{i}\widehat{H}_2]$ and $[\mathrm{i}\widehat{H}_3, \mathrm{i}\widehat{H}_1] = \mathrm{i}\widehat{H}_1$, $[\mathrm{i}\widehat{H}_3, \mathrm{i}\widehat{H}_2] = -\mathrm{i}\widehat{H}_2$, which turns $\mathfrak{W}$ into a VG Lie algebra for a quantum Lie system.

The quantum Lie system notion allows for the development of powerful techniques to study their associated t-dependent Schrödinger equations [3, 7, 21]. For instance, t-dependent Schrödinger equations associated with quantum Lie systems can be investigated through Lie systems on Lie groups and the corresponding VG Lie algebras for quantum Lie systems are a clue to solve them [3]. For instance, quantum t-dependent dissipative Calogero–Moser harmonic oscillators can be analysed via such techniques [3]. Moreover, if a quantum Lie system is related to a solvable quantum VG Lie algebra, then its associated t-dependent Schrödinger equation can be integrated by quadratures [3].

To enlarge the field of applications of quantum Lie systems, we propose as a main result a quantum analogue of the theory of quasi-Lie schemes based on the hereupon denominated quantum quasi-Lie schemes. Very roughly speaking, the basic idea to develop the quantum quasi-Lie schemes theory is to mimic the quasi-Lie schemes theory via the substitution of t-dependent vector fields and Lie brackets of vector fields by t-dependent skew-Hermitian operators and quantum commutators, respectively. In this manner, a quantum quasi-Lie scheme consists of a Lie algebra $\mathfrak{W}$ of skew-Hermitian operators on a Hilbert space $\mathcal{H}$ and a finite-dimensional real linear space of skew-Hermitian operators on the same Hilbert space, $\mathfrak{V}$, so that $\mathfrak{W} \subset \mathfrak{V}$ and $[\mathfrak{W}, \mathfrak{V}] \subset \mathfrak{V}$. Such a quantum quasi-Lie scheme is denoted by $S(\mathfrak{W}, \mathfrak{V})$, and it can be then used to study a t-dependent Schrödinger equation on $\mathcal{H}$ when its t-dependent Hamiltonian operator $\widehat{H}(t)$ is such that each particular $\mathrm{i}\widehat{H}(t)$, for $t \in \mathbb{R}$, belongs to $\mathfrak{V}$.

We prove that every quantum quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ gives rise to a group of t-dependent gauge transformations determined by $\mathfrak{W}$, the group of the quantum quasi-Lie scheme, which enables one to transform certain t-dependent Schrödinger equations determined by the quantum quasi-Lie scheme into new ones that can be studied through quantum Lie systems. Such t-dependent Schrödinger equations, described by the here defined quantum quasi-Lie systems (QQLS), cover the t-dependent Schrödinger equations associated with quantum Lie systems as a particular case. In this manner, quantum quasi-Lie systems can be investigated via the theory of quantum Lie systems.

One of the main uses of quantum quasi-Lie schemes in this work is to use an element $U(t)$ of the group of a quantum quasi-Lie scheme to map a t-dependent Hamiltonian operator $-\mathrm{i}\widehat{H}(t)$ described through it into a quantum Lie system or a simpler t-dependent Hamiltonian operator $-\mathrm{i}\widehat{H}'(t)$. Instead of applying straightforwardly a generic $U(t)$ to transform $-\mathrm{i}\widehat{H}(t)$ into $-\mathrm{i}\widehat{H}'(t)$, which is operationally complicated and tedious but ubiquitous in the literature, we provide a series of results, e.g. Propositions 6.1 and 6.2, giving easy criteria to predict some interesting properties of $-\mathrm{i}\widehat{H}'(t)$. This will eventually give hints on the form of $U(t)$ to simplify $-\mathrm{i}\widehat{H}'(t)$ or to ensure that $-\mathrm{i}\widehat{H}(t)$ is a quantum quasi-Lie system. Mathematically, Propositions 6.1 and 6.2 concern the study of real non-semi-simple Lie algebra representations. This is the least studied case in the literature, which mainly focuses upon representations of complex semi-simple Lie algebras.

As applications, we show that quantum quasi-Lie schemes allow us to study the quantum evolution in problems in which the theory of quantum Lie systems does not apply. In particular, we develop a quantum analogue of the classical anharmonic oscillator formerly studied by Perelomov [23]. This solves the problem proposed by Perelomov of finding a quantum analogue of his oscillator. Subsequently, we focus on the application of quantum quasi-Lie systems to t-dependent Schrödinger equations describing a one-dimensional quantum fluid in a t-dependent trapping potential [24] and a quantum t-dependent frequency Winternitz–Smorodinsky oscillator.

The paper is organised as follows. Section 2 reviews fundamental notions of differential geometry on Hilbert spaces. Section 3 presents quantum Lie systems and their properties. Section 4 introduces the theory of quantum quasi-Lie schemes, while the theory of quantum quasi-Lie systems is presented in Sect. 5. Section 6 addresses the description of new methods to study t-dependent Schrödinger equations by means of quantum quasi-Lie schemes. Subsequently, we present an application of the theory of quantum quasi-Lie schemes and systems to quantum nonlinear oscillators with a homogeneous potential in Sect. 7. Section 8 is devoted to the application of the theory of quantum quasi-Lie schemes to the particular case of homogeneous potentials of degree minus two. In particular, this entails studying quantum t-dependent frequency Calogero–Moser systems, quantum fluids in a t-dependent homogeneous trapping potential, and t-dependent quantum Smorodinsky–Winternitz systems. To conclude, Sect. 9 contains a summary of the obtained results and a commentary on future research prospects. Table 1 summarizes many of the notions used and/or introduced in this work.

2 Differential geometry in Hilbert spaces

This section surveys the differential geometry of (possibly infinite-dimensional) Hilbert spaces. For further details on the geometry of infinite-dimensional manifolds and the theory of skew-Hermitian operators, we refer to [25–28].

Let $\mathcal{H}$ be a complex Hilbert space endowed with the topology inherited from the norm $\|\cdot\|$ associated with its scalar product $\langle\cdot|\cdot\rangle$. It is hereupon assumed that $\mathcal{H}$ is separable, as it usually happens in quantum mechanics. The separability of $\mathcal{H}$ means that there exists a countable orthonormal basis. The orthonormal basis, let us say $\{\psi_n\}_{n\in\mathbb{N}}$, gives rise to a global chart

$$\varphi : \psi \in \mathcal{H} \mapsto \{\mathfrak{Re}\langle\psi_n|\psi\rangle, \mathfrak{Im}\langle\psi_n|\psi\rangle\}_{n\in\mathbb{N}} \in \ell^2(\mathbb{R})$$

onto the real Hilbert space $\ell^2(\mathbb{R})$ of square summable sequences, which allows us to consider $\mathcal{H}$ as a manifold $\mathcal{H}_\mathbb{R}$ modelled over a Hilbert space [3, 7].

Let $C^\infty_\phi(U)$ be the space of germs of smooth real functions on an open $U \subset \mathcal{H}$ at $\phi \in U$, i.e. the space of equivalence classes of smooth functions on U that are equal on an open neighbourhood of ϕ. We call *operational tangent vector* at ϕ a linear mapping $D_\phi : C^\infty_\phi(U) \to \mathbb{R}$ satisfying that $D_\phi(fg) = (D_\phi f)g(\phi) + f(\phi)(D_\phi g)$ for every $f, g \in C^\infty_\phi(U)$. Every $\psi \in \mathcal{H}$ induces a derivative $\dot\psi_\phi : C^\infty_\phi(U) \to \mathbb{R}$ of the form

$$(\dot\psi_\phi f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\phi + t\psi), \quad \forall f \in C^\infty_\phi(U). \tag{2.1}$$

Table 1 Sketch of the main structures studied in the work. Note that $c_{\alpha\beta}{}^{\gamma}$ for $\alpha = 1, \ldots, r$ and $\beta, \gamma = 1, \ldots, s$ are real constants. The $b_1(t), \ldots, b_s(t)$ and $d_1(t), \ldots, d_r(t)$ are arbitrary real functions, while the $c_1(t), \ldots, c_s(t)$ are chosen to obtain a quasi-Lie system or a quantum quasi-Lie system. The $d_{\alpha,\beta}{}^{\gamma}$ for α, β, γ are real constants.

Lie system	Quantum Lie system
$X = \sum_{\alpha=1}^{r} b_\alpha(t) X_\alpha$	$\widehat{H}(t) = \sum_{\alpha=1}^{r} b_\alpha(t)\widehat{H}_\alpha$
$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^{\gamma} X_\gamma \quad \alpha, \beta = 1, \ldots, r$	$[\mathrm{i}\widehat{H}_\alpha, \mathrm{i}\widehat{H}_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^{\gamma}\mathrm{i}\widehat{H}_\gamma \quad \alpha, \beta = 1, \ldots, r$
$W = \langle X_1, \ldots, X_r\rangle$	$\mathfrak{W} = \langle X_1, \ldots, X_r\rangle$
Quasi-Lie scheme $S(W, V)$	**Quantum quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$**
$X = \sum_{\alpha=1}^{s} b_\alpha(t) X_\alpha$	$\widehat{H}(t) = \sum_{\alpha=1}^{s} b_\alpha(t)\widehat{H}_\alpha$
$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{s} c_{\alpha\beta}{}^{\gamma} X_\gamma \quad \alpha = 1, \ldots, r, \quad \beta = 1, \ldots, s$	$[\mathrm{i}\widehat{H}_\alpha, \mathrm{i}\widehat{H}_\beta] = \sum_{\gamma=1}^{s} c_{\alpha\beta}{}^{\gamma}\mathrm{i}\widehat{H}_\gamma, \quad \alpha = 1, \ldots, r, \ \beta = 1, \ldots, s$
$W = \langle X_1, \ldots, X_r\rangle, \ V = \langle X_1, \ldots, X_s\rangle$	$\mathfrak{W} = \langle \mathrm{i}\widehat{H}_1, \ldots, \mathrm{i}\widehat{H}_r\rangle, \ \mathfrak{V} = \langle \mathrm{i}\widehat{H}_1, \ldots, \mathrm{i}\widehat{H}_s\rangle$
G_W Group of $S(W, V)$	$G_{\mathfrak{W}}$ Group of $S(\mathfrak{W}, \mathfrak{V})$
Quasi-Lie system of $S(W, V)$	**Quantum quasi-Lie system of $S(\mathfrak{W}, \mathfrak{V})$**
$X = \sum_{\alpha=1}^{s} c_\alpha(t) X_\alpha$	$\widehat{H}(t) = \sum_{\alpha=1}^{s} c_\alpha(t)\widehat{H}_\alpha$
$f(t) \in \mathcal{G}_W \ \Rightarrow f(t)_\bigstar X(t) = \sum_{\alpha=1}^{p} d_\gamma(t) Y_\gamma$	$U(t) \in \mathcal{G}_{\mathfrak{W}} \ \Rightarrow U(t)_\bigstar \mathrm{i}\widehat{H}(t) = \sum_{\alpha=1}^{p} d_\gamma(t)\mathrm{i}\widehat{H}_\gamma$
$V \supset \langle Y_1, \ldots, Y_p\rangle, \ \ [Y_\alpha, Y_\beta] = \sum_{\gamma=1}^{p} d_{\alpha\beta}{}^{\gamma} Y_\gamma$	$\mathfrak{V} \supset \langle \mathrm{i}\widehat{H}_1, \ldots, \mathrm{i}\widehat{H}_p\rangle, \ \ [\mathrm{i}\widehat{H}_\alpha, \mathrm{i}\widehat{H}_\beta] = \sum_{\gamma=1}^{p} d_{\alpha\beta}{}^{\gamma}\mathrm{i}\widehat{H}_\gamma$

The derivative $\dot{\psi}_\phi$ is called a *kinematic tangent vector* at the foot point ϕ. There is, in general, no one-to-one relation between both types of tangent vectors on infinite-dimensional manifolds (see e.g. [29, 30]). The term 'kinematic' will be hereafter skipped to simplify the presentation of our results. The *tangent space* $T_\phi\mathcal{H}_\mathbb{R}$ at $\phi \in \mathcal{H}_\mathbb{R}$ is the space of all tangent vectors with foot point ϕ [26–28]. It is immediate to prove, e.g. by inspection of (2.1) on smooth functions $f_{\widetilde{\psi}} : \widetilde{\phi} \in \mathcal{H} \mapsto \mathfrak{Re}\langle\widetilde{\psi}|\widetilde{\phi}\rangle$ for $\widetilde{\psi} \in \mathcal{H}$, that $\dot{\psi}_\phi \neq \dot{\psi}'_\phi$, for $\psi, \psi', \phi \in \mathcal{H}$, if and only if $\psi \neq \psi'$. Hence, each $T_\phi\mathcal{H}_\mathbb{R}$ is isomorphic to $\mathcal{H}_\mathbb{R}$ for every $\phi \in \mathcal{H}_\mathbb{R}$. The isomorphism associates $\psi \in \mathcal{H}_\mathbb{R}$ with the tangent vector $\dot{\psi}_\phi \in T_\phi\mathcal{H}_\mathbb{R}$. To simplify the notation, $\dot{\psi}_\phi$ will be denoted by $\dot{\psi}$ if the foot point is known from the context. The space $T_\phi\mathcal{H}_\mathbb{R}$ admits a unique norm turning the isomorphism with $\mathcal{H}_\mathbb{R}$ into an isometry. Similarly, its *tangent bundle*, $T\mathcal{H}_\mathbb{R} := \bigsqcup_{\phi\in\mathcal{H}} T_\phi\mathcal{H}_\mathbb{R}$, where $\bigsqcup$ stands for a *disjoint union* [28, 31], is naturally diffeomorphic to $\mathcal{H}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R}$. The space $T\mathcal{H}_\mathbb{R}$ is indeed a vector bundle relative to $\pi : \dot{\psi}_\phi \in T\mathcal{H}_\mathbb{R} \mapsto \phi \in \mathcal{H}_\mathbb{R}$.

A *vector field* is a mapping $X : D(X) \subset \mathcal{H}_\mathbb{R} \to T\mathcal{H}_\mathbb{R}$ from a dense subspace $D(X) \subset \mathcal{H}_\mathbb{R}$ satisfying that $\pi \circ X = \mathrm{id}_{\mathcal{H}_\mathbb{R}}|_{D(X)}$. For any vector field X on $\mathcal{H}_\mathbb{R}$ and $\phi \in D(X)$, we denote $X(\phi) := (\phi, X_\phi) \in T_\phi\mathcal{H}_\mathbb{R} \subset T\mathcal{H}_\mathbb{R} \simeq \mathcal{H}_\mathbb{R} \oplus \mathcal{H}_\mathbb{R}$.

A vector field X admits a *flow* if there exists a continuous map $Fl^X : (t, \phi) \in \mathcal{U} \subset \mathbb{R} \times \mathcal{H}_\mathbb{R} \mapsto Fl^X_t(\phi) \in \mathcal{H}_\mathbb{R}$ such that $\{0\} \times \mathcal{H}_\mathbb{R} \subset \mathcal{U}$, $Fl^X_0(\phi) = \phi$ for every $\phi \in \mathcal{H}_\mathbb{R}$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} Fl^X(t, \phi) = \lim_{s\to 0} \frac{Fl^X(t+s, \phi) - Fl^X(t, \phi)}{s} = X\Big(Fl^X(t, \phi)\Big),$$

where the limit is relative to the norm topology of $\mathcal{H}_\mathbb{R}$ [26, 32].

The main objects to be considered hereafter are the vector fields of the form

$$X_{\widehat{A}} : \psi \in D(\widehat{A}) \subset \mathcal{H}_\mathbb{R} \mapsto (\dot{\widehat{A}\psi})_\psi \in T_\psi\mathcal{H}_\mathbb{R} \subset T\mathcal{H}_\mathbb{R}$$

for a skew-Hermitian operator $\widehat{A} : D(\widehat{A}) \subset \mathcal{H} \to \mathcal{A}$, where $D(\widehat{A})$ is the domain of the operator $\widehat{A}$. It is worth noting that an operator on an infinite-dimensional Hilbert space only needs to be defined on a dense subspace of the Hilbert space [33]. Even if $\widehat{A}$ is not bounded, which implies in virtue of the *Hellinger–Toeplitz Theorem* [26] that $X_{\widehat{A}}$ can only be defined on a dense subset of $\mathcal{H}_\mathbb{R}$, the *Stone's Theorem* ensures that $\widehat{A}$ gives rise to a strongly continuous one-parameter group $\{e^{t\widehat{A}}\}_{t\in\mathbb{R}}$ of operators [34, 35].

Previous results ensure that the integral curves of $X_{\widehat{A}}$, namely $t \mapsto e^{t\widehat{A}}\phi$ for every $\phi \in \mathcal{H}$, are defined on the whole $\mathcal{H}_\mathbb{R}$ and $X_{\widehat{A}}$ is complete. This fact can be used to consider skew-Hermitian operators as fundamental vector fields relative to the standard action of the unitary group $U(\mathcal{H})$ on the Hilbert space $\mathcal{H}$, relative to the strong topology. If $\widehat{A} := -\mathrm{i}\widehat{H}$ for an autonomous self-Hermitian Hamiltonian $\widehat{H}$, then the solutions to the associated t-dependent Schrödinger equation describe the integral curves of $X_{-\mathrm{i}\widehat{H}}$ [3, 36, 37].

Assume that $\widehat{B}_1$ and $\widehat{B}_2$ are two, possibly not bounded, operators acting on $\mathcal{H}$ admitting a common dense domain D that is also invariant under $\widehat{B}_1$ and $\widehat{B}_2$, namely $\widehat{B}_1$, $\widehat{B}_2$ are defined on D and $\widehat{B}_1(D), \widehat{B}_2(D) \subset D$. The commutator of $\widehat{B}_1$ and $\widehat{B}_2$ is defined to be $[\widehat{B}_1, \widehat{B}_2]\phi = (\widehat{B}_1\widehat{B}_2 - \widehat{B}_2\widehat{B}_1)\phi$ for every $\phi \in D$.

For two vector fields $X_{\widehat{B}_1}, X_{\widehat{B}_2} \in \mathfrak{X}(\mathcal{H}_\mathbb{R})$ satisfying that $\widehat{B}_1$, $\widehat{B}_2$ are skew-Hermitian operators having a common invariant dense domain, we define

$$[X_{\widehat{B}_1}, X_{\widehat{B}_2}] := -X_{[\widehat{B}_1,\widehat{B}_2]}.$$

The above expression, when the integral curves of $\widehat{B}_1, \widehat{B}_2$ are smooth enough, arises as a consequence of the definition of the geometric Lie bracket of vector fields on a Banach manifold. Let us analyse this fact in detail. Assume that $X_{\widehat{B}_1}, X_{\widehat{B}_2}$ are such that $\widehat{B}_1, \widehat{B}_2$ admit a common dense invariant domain. If the curve $\alpha(t) := \exp(-t\widehat{B}_1)\exp(-t\widehat{B}_2)\exp(t\widehat{B}_1)\exp(t\widehat{B}_2)\phi$ is smooth at $\phi \in \mathcal{H}$, it is possible to retrieve the Lie bracket of the vector fields $X_{\widehat{B}_1}, X_{\widehat{B}_2}$ geometrically [38, Theorem 1.33]:

$$[[X_{\widehat{B}_1}, X_{\widehat{B}_2}]](\phi) = \frac{1}{2}\frac{\partial^2}{\partial t^2}\bigg|_{t=0}\left(Fl_{-t}^{X_{\widehat{B}_2}} \circ Fl_{-t}^{X_{\widehat{B}_1}} \circ Fl_{t}^{X_{\widehat{B}_2}} \circ Fl_{t}^{X_{\widehat{B}_1}}(\phi)\right) = \frac{1}{2}\frac{\partial^2}{\partial t^2}\bigg|_{t=0} e^{-t\widehat{B}_2}e^{-t\widehat{B}_1}e^{t\widehat{B}_2}e^{t\widehat{B}_1}\phi.$$

In particular, the above always holds when $\mathcal{H}$ is finite-dimensional. A *t-dependent vector field* on $\mathcal{H}_{\mathbb{R}}$ is a mapping $X : \mathcal{U} \subset \mathbb{R}\times\mathcal{H}_{\mathbb{R}} \to T\mathcal{H}_{\mathbb{R}}$, for a certain open subset $\mathcal{U}$ in $\mathbb{R}\times\mathcal{H}_{\mathbb{R}}$, such that $\pi \circ X(t,\psi) = \psi$ on $\mathcal{U}$ and the domain of each $X_t : \phi \in \mathcal{U}\cap(\{t\}\times\mathcal{H}_{\mathbb{R}}) \mapsto X(t,\phi) \in T\mathcal{H}_{\mathbb{R}}$, with $t \in \mathbb{R}$, is dense in $\mathcal{H}_{\mathbb{R}}$. The *integral curves* of X are sections $\gamma : \mathbb{R} \to \mathbb{R}\times\mathcal{H}_{\mathbb{R}}$ of the bundle $\pi_{\mathbb{R}} : \mathbb{R}\times\mathcal{H}_{\mathbb{R}} \to \mathbb{R}$ that are simultaneously integral curves of the vector field $\partial_t + X$ on $\mathbb{R}\times\mathcal{H}_{\mathbb{R}}$. If $\pi_2 : \mathbb{R}\times\mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ is the projection onto the second manifold, the curves γ are given by the solutions to the differential equation

$$\frac{\mathrm{d}(\pi_2\circ\gamma)}{\mathrm{d}t}(t) = X\circ\gamma(t).$$

Every t-dependent Hermitian operator $\widehat{H}(t)$ on $\mathcal{H}$ gives rise to a t-dependent vector field $X_{-\mathrm{i}\widehat{H}(t)}$ on $\mathcal{H}_{\mathbb{R}}$ and an associated t-dependent Schrödinger equation

$$\frac{\partial\psi}{\partial t} = -\mathrm{i}\widehat{H}(t)\psi, \qquad \psi\in\mathcal{H}, \qquad t\in\mathbb{R}.$$

3 Quantum Lie systems

Let us survey the extension to quantum mechanics of the theory of Lie systems [3]. On a finite-dimensional manifold N, every finite-dimensional real Lie algebra V of vector fields on N can be integrated to give rise to a local Lie group action on N. If N is an infinite-dimensional manifold, additional technical conditions on V may be necessary to integrate V so as to give rise to a local Lie group action on N. In particular, we are interested in the integration to a Lie group action of finite-dimensional real Lie algebras of vector fields induced by skew-Hermitian operators on (possibly infinite-dimensional) manifolds (see [39–41] for details). In all cases studied in this work, V can be found to be integrable to a local Lie group action. This Lie group action allows us to study quantum Lie systems via the standard theory of Lie systems [22, 42]. To focus on the applications of our ideas, quantum quasi-Lie schemes, and quantum Lie systems to be studied in next sections, further comments on the integrability of V to a Lie group action will be omitted.

Definition 3.1 A *quantum Lie system* is a t-dependent operator $\widehat{H}(t)$ on $\mathcal{H}$ of the form

$$\widehat{H}(t) := \sum_{\alpha=1}^{r} b_\alpha(t)\widehat{H}_\alpha, \tag{3.1}$$

where $b_1(t),\ldots,b_r(t)$ are t-dependent real functions and $\mathfrak{W} := \langle \mathrm{i}\widehat{H}_1,\ldots,\mathrm{i}\widehat{H}_r\rangle$ is a r-dimensional real Lie algebra of skew-Hermitian operators. We call $\mathfrak{W}$ a *Vessiot–Guldberg (VG) Lie algebra* for the quantum Lie system (3.1) [3, 22, 42].

Each quantum Lie system (3.1) determines a t-dependent Schrödinger equation

$$\frac{\partial\psi}{\partial t} = -\mathrm{i}\widehat{H}(t)\psi = -\sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{i}\widehat{H}_\alpha\psi, \qquad \psi\in\mathcal{H}, \qquad t\in\mathbb{R}. \tag{3.2}$$

Its particular solutions are then integral curves of the t-dependent vector field $X_{-\mathrm{i}\widehat{H}(t)}$ (see [3] for details).

Let us illustrate quantum Lie systems through an example. Consider the quantum one-dimensional *Caldirola–Kanai oscillator* [43–46], which is determined by the t-dependent operator

$$\widehat{H}_{\mathrm{CK}}(t) := \exp(-2\gamma t)\frac{\widehat{p}^2}{2} + \omega^2\exp(2\gamma t)\frac{\widehat{x}^2}{2}, \qquad \gamma\in\mathbb{R}, \quad \omega\in\mathbb{R}, \tag{3.3}$$

where $\widehat{x}$ and $\widehat{p}$ refer to the usual position and momentum operators on $\mathbb{R}$. This model is a quantum analogue of the classical harmonic oscillator with a t-dependent mass $m(t) := \exp(2\gamma t)$ and a constant frequency ω [3, 47].

The t-dependent operator $\widehat{H}_{\mathrm{CK}}(t)$ can be written as a linear combination with t-dependent real functions of the Hermitian operators

$$\widehat{H}_1 := \widehat{x}^2/2, \qquad \widehat{H}_2 := \widehat{p}^2/2, \qquad \widehat{H}_3 := (\widehat{x}\widehat{p}+\widehat{p}\widehat{x})/4.$$

The real linear space $\mathfrak{W}_{\mathrm{CK}} := \langle \mathrm{i}H_1, \mathrm{i}H_2, \mathrm{i}H_3\rangle$ is a vector space of skew-Hermitian operators. Additionally,

$$[\mathrm{i}\widehat{H}_1, \mathrm{i}\widehat{H}_2] = -2\mathrm{i}\widehat{H}_3, \quad [\mathrm{i}\widehat{H}_1, \mathrm{i}\widehat{H}_3] = -\mathrm{i}\widehat{H}_1, \quad [\mathrm{i}\widehat{H}_3, \mathrm{i}\widehat{H}_2] = -\mathrm{i}\widehat{H}_2.$$

Hence, $\mathfrak{W}_{\mathrm{CK}}$ is a finite-dimensional Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Since $\mathrm{i}\widehat{H}_{\mathrm{CK}}(t)$ takes values in $\mathfrak{W}_{\mathrm{CK}}$, the t-dependent operator $\widehat{H}_{\mathrm{CK}}(t)$ is a quantum Lie system.

The solutions of the t-dependent Schrödinger equation associated with (3.3), namely

$$\frac{\partial\psi}{\partial t} = -\mathrm{i}\widehat{H}_{\mathrm{CK}}(t)\psi = -\mathrm{i}\left(\exp(-2\gamma t)\frac{\widehat{p}^2}{2} + \omega^2\exp(2\gamma t)\frac{\widehat{x}^2}{2}\right)\psi,$$

are the integral curves of $X_{\mathrm{CK}}(t) := \omega^2\exp(2\gamma t)X_{-\mathrm{i}\widehat{H}_1} + \exp(-2\gamma t)X_{-\mathrm{i}\widehat{H}_2}$.

Similarly to standard Lie systems, the associated t-dependent Schrödinger equation related to a quantum Lie system can be solved by means of a Lie system on a Lie group [1, 3]. Let us explain this fact (see [3] for applications). Reconsider again a general quantum Lie system (3.1), which satisfies that

$$[\mathrm{i}\widehat{H}_\alpha, \mathrm{i}\widehat{H}_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^{\gamma}\, \mathrm{i}\widehat{H}_\gamma, \qquad c_{\alpha\beta}{}^{\gamma} \in \mathbb{R}, \qquad \alpha, \beta, \gamma = 1, \ldots, r.$$

Let $\varphi : \mathfrak{g} \to \mathfrak{W}$ be an isomorphism of Lie algebras, where $\mathfrak{g}$ is an abstract Lie algebra, $G_{\mathfrak{g}}$ is the connected and simply connected Lie group related to $\mathfrak{g}$, and $\mathfrak{W}$ is the Lie algebra relative to the commutator of skew-Hermitian operators (more precisely defined on a common domain for all its elements given by a dense subspace $D_{\mathfrak{W}} \subset \mathcal{H}$). We define a continuous unitary action $\Phi : G_{\mathfrak{g}} \times \mathcal{H} \to \mathcal{H}$ (continuous relative to the norms induced by the one on $\mathcal{H}$ and the one in $G_{\mathfrak{g}}$), such that

$$\varphi(v)\psi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi(\exp(-tv), \psi), \qquad \forall\psi \in D_{\mathfrak{W}}, \quad \forall v \in \mathfrak{g}. \tag{3.4}$$

Note that Φ amounts to a Lie group morphism $\widehat{\Phi} : g \in G \mapsto \Phi^g \in U(\mathcal{H})$ where $\Phi^g(\psi) := \Phi(g, \psi)$ for every $g \in G_{\mathfrak{g}}$ and $\psi \in \mathcal{H}$, whilst $U(\mathcal{H})$ amounts to the space of unitary operators on $\mathcal{H}$. We define $\Phi_\psi : g \in G_{\mathfrak{g}} \mapsto \Phi(g, \psi) \in \mathcal{H}_{\mathbb{R}}$ for every $\psi \in \mathcal{H}_{\mathbb{R}}$. Note that φ is the infinitesimal Lie algebra morphism induced by Φ. Due to the relation $[X_{\widehat{B}_1}, X_{\widehat{B}_2}] = -X_{[\widehat{B}_1, \widehat{B}_2]}$, it follows from (3.4) that we can define $X_{-\varphi(v)}$ to be the fundamental vector field on $\mathcal{H}_{\mathbb{R}}$ corresponding to the element $v \in \mathfrak{g}$, and then, there exists a Lie algebra isomorphism $v \mapsto X_{-\varphi(v)}$ between the elements of the Lie algebra $\mathfrak{g}$ and the Lie algebra of vector fields W given by the vector fields on $\mathcal{H}_{\mathbb{R}}$ induced by the skew-Hermitian operators $\varphi(v)$ for $v \in \mathfrak{g}$.

Let $\{\mathrm{a}_1, \ldots, \mathrm{a}_r\}$ be the basis of $T_eG \simeq \mathfrak{g}$ given by $\mathrm{a}_\alpha := \varphi^{-1}(\mathrm{i}\widehat{H}_\alpha)$ for $\alpha = 1, \ldots, r$. Then, $\mathrm{a}_1, \ldots, \mathrm{a}_r$ have the same commutation relations, relative to the natural Lie bracket in T_eG (see [48]), as the operators $\mathrm{i}\widehat{H}_1, \ldots, \mathrm{i}\widehat{H}_r$, which in turn have the opposite structure constants of the vector fields $X_{\mathrm{i}\widehat{H}_\alpha}$. Hence, if $[\mathrm{i}\widehat{H}_\alpha, \mathrm{i}\widehat{H}_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^{\gamma}\, \mathrm{i}\widehat{H}_\gamma$, then $[X_{-\mathrm{i}\widehat{H}_\alpha}, X_{-\mathrm{i}\widehat{H}_\beta}] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^{\gamma} X_{-\mathrm{i}\widehat{H}_\gamma}$, where we assume $\alpha, \beta = 1, \ldots, r$. Hence,

$$[\mathrm{a}_\alpha, \mathrm{a}_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}{}^{\gamma}\, \mathrm{a}_\gamma, \qquad c_{\alpha\beta}{}^{\gamma} \in \mathbb{R}, \qquad \alpha, \beta, \gamma = 1, \ldots, r,$$

and

$$\Phi(\exp(-\lambda \mathrm{a}_\alpha), \psi) = \exp(-\mathrm{i}\lambda\widehat{H}_\alpha)\psi, \qquad \forall\psi \in D_{\mathfrak{W}}, \quad \forall\lambda \in \mathbb{R}, \qquad \alpha = 1, \ldots, r.$$

It is quite useful in applications to consider G to be a matrix Lie group. In such a case, the elements of $\mathfrak{g}$ are matrices and the commutator of the elements of T_eG is the commutator of the corresponding associated elements of the Lie algebra. It is also worth noting that $\varphi(\mathrm{Ad}_g(v)) = \mathrm{Ad}_{\Phi^g}\varphi(v)$, for all $g \in G$ and $v \in \mathfrak{g}$, where Ad_g stands for the adjoint action of g on $\mathfrak{g}$, and Ad_{Φ^g} is the adjoint action of $\Phi^g \in U(\mathcal{H})$ relative to $\varphi(v)$.

In particular, let us prove that solving the t-dependent Schrödinger equation associated with a quantum Lie system $\widehat{H}(t) = \sum_{\alpha=1}^{r} b_\alpha(t)H_\alpha$ reduces to solving the Lie system in G given by

$$\frac{\mathrm{d}g}{\mathrm{d}t} = -\sum_{\alpha=1}^{r} b_\alpha(t)X_\alpha^R(g),$$

where $X_\alpha^R(g) = R_{g*e}\mathrm{a}_\alpha$ for $\alpha = 1, \ldots, r$ and $R_g : g' \in G \mapsto g'g \in G$. Indeed, if we define $\psi(t) = \Phi(g(t), \psi(0))$, with $\psi(0) \in \mathcal{H}_{\mathbb{R}}$, then for every $t_0 \in \mathbb{R}$, one has that

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \Phi(g(t), \psi(t_0)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(g(t)g(t_0)^{-1}g(t_0), \psi(t_0))\\ &= \varphi\left(\frac{\mathrm{d}g}{\mathrm{d}t}(t_0)g^{-1}(t_0)\right)\Phi(g(t_0), \psi(0)) = -\mathrm{i}\widehat{H}(t_0)\psi(t_0),\end{aligned}$$

and $\psi(t)$ is the general solution to (3.2).

In the particular case of $\widehat{H}_{\mathrm{CK}}(t)$, which is related to a VG Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, the associated Lie system should be associated with the universal covering group, $\widetilde{SL}(2,\mathbb{R})$, of the Lie group $SL(2,\mathbb{R})$, which has a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. In consequence, it is defined on the Lie group $G := \widetilde{SL}(2,\mathbb{R})$ and takes the form

$$\frac{\mathrm{d}g}{\mathrm{d}t} = -e^{-2\gamma t} X_2^R(g) - \omega^2 e^{2\gamma t} X_1^R(g), \qquad g \in \widetilde{SL}(2,\mathbb{R}), \qquad t \in \mathbb{R}, \tag{3.5}$$

where $X_\alpha^R(g) := R_{g*e}\mathrm{a}_\alpha$ for $\alpha = 1, 2, 3$ and $\{\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3\}$ is a basis of $T_{\mathrm{Id}}SL_2 \simeq \mathfrak{sl}(2,\mathbb{R})$ satisfying the same commutation relations as $\mathrm{i}\widehat{H}_1, \mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_3$.

As $SL(2,\mathbb{R})$ has a fundamental group isomorphic to $\mathbb{Z}$, its universal covering, $\widetilde{SL}(2,\mathbb{R})$, is not a matrix Lie group, which is complicated to deal with in practical applications (cf. [33, p. 127]).

In applications, one is frequently interested in solutions of (3.5) for the variable t taking values close to zero. By the Ado theorem, every finite-dimensional Lie algebra can be written in a matrix form and, then, it admits a matrix Lie group [33]. Since all Lie groups with isomorphic Lie algebras are locally diffeomorphic close to their neutral elements, one can define an analogue of (3.5) on the matrix Lie group $SL(2,\mathbb{R})$ given by

$$\frac{\mathrm{d}g}{\mathrm{d}t} = -e^{-2\gamma t} X_2^R(g) - \omega^2 e^{2\gamma t} X_1^R(g), \qquad g \in SL(2,\mathbb{R}), \qquad t \in \mathbb{R}.$$

4 Quantum quasi-Lie schemes

This section develops the theory of quantum quasi-Lie schemes as an extension of the theory of quasi-Lie schemes to the quantum mechanical realm [3]. Its main aim is to investigate the integrability properties of a class of t-dependent Schrödinger equations describing the t-dependent Schrödinger equations studied via quantum Lie systems as a particular case. Additionally, we provide a criterion to ensure that a t-dependent Hermitian operator is not a quantum Lie system.

The following example illustrates the necessity of quantum quasi-Lie schemes. Consider an n-dimensional nonlinear quantum oscillator with a t-dependent anharmonic potential described by

$$\widehat{H}_{NH}(t) := \frac{1}{2}\sum_{i=1}^{n} \widehat{p}_i^2 + \omega(t)\frac{1}{2}\sum_{i=1}^{n}\widehat{x}_i^2 + \tilde{\omega}(t)\sum_{i=1}^{n}\widehat{x}_i^{\alpha}, \tag{4.1}$$

where α is a negative integer, the term *NH* stands for 'non-harmonic', and $\omega(t), \tilde{\omega}(t)$ are real t-dependent functions such that the points of the curve $(\omega(t), \tilde{\omega}(t))$ in $\mathbb{R}^2$ span the whole space $\mathbb{R}^2$. Let us prove that $\widehat{H}_{NH}(t)$ is not a quantum Lie system through the following no-go proposition.

Proposition 4.1 (Quantum Lie systems no − go proposition). *Let End($\mathcal{H}$) be the Lie algebra of operators from $\mathcal{H}$ to $\mathcal{H}$. Let $\widehat{H}(t)$ be a t-dependent Hermitian operator on $\mathcal{H}$ such that there exist $\widehat{H}_a, \widehat{H}_b \in \langle \widehat{H}(t)\rangle_{t\in\mathbb{R}}$ and a Hermitian operator $\widehat{H}_c$ on $\mathcal{H}$ satisfying that*

$$[i\widehat{H}_c, i\widehat{H}_a] = c_a i\widehat{H}_a, \qquad [i\widehat{H}_c, i\widehat{H}_b] = c_b i\widehat{H}_b,$$

for constants $c_a \in \mathbb{R}$ and $c_b \in \mathbb{R}\backslash\{0\}$. Let $pr : \mathfrak{U} \subset End(\mathcal{H}) \to End(\mathcal{H})$ be a Lie algebra morphism from a (possibly infinite-dimensional) Lie algebra $\mathfrak{U}$ of operators on $\mathcal{H}$ containing $\langle i\widehat{H}(t)\rangle_{t\in\mathbb{R}}$ and $i\widehat{H}_c$. If $ad^n_{pr(i\widehat{H}_b)}pr(i\widehat{H}_a) \neq 0$ for every natural number $n > 0$, then $\widehat{H}(t)$ is not a quantum Lie system.

Proof It follows by induction that $[\mathrm{i}\widehat{H}_c, \mathrm{ad}^n_{\mathrm{i}\widehat{H}_b}\mathrm{i}\widehat{H}_a] = (nc_b + c_a)\mathrm{ad}^n_{\mathrm{i}\widehat{H}_b}\mathrm{i}\widehat{H}_a$ for every $n \in \mathbb{N}$. Moreover, every $\mathrm{i}\widehat{H}_n := \mathrm{ad}^n_{\mathrm{i}\widehat{H}_b}\mathrm{i}\widehat{H}_a$, with $n \in \mathbb{N}\cup\{0\}$, belongs to the domain of pr because $\mathrm{i}\widehat{H}_a$ and $\mathrm{i}\widehat{H}_b$ are contained in $\mathfrak{U}$ and $\mathfrak{U}$ is a Lie algebra of operators. Since pr is a Lie algebra morphism, $\mathrm{ad}^n_{\mathrm{pr}(\mathrm{i}\widehat{H}_b)}\mathrm{pr}(\mathrm{i}\widehat{H}_a) = \mathrm{pr}(\mathrm{i}\widehat{H}_n)$ and, then, $\widehat{H}_n \neq 0$ for all $n \in \mathbb{N}$. Let $E := \langle \mathrm{i}\widehat{H}_n\rangle_{n\in\mathbb{N}}$ be an $\mathbb{R}$-linear space. Then, the operators $\{\mathrm{i}\widehat{H}_n\}_{n\in\mathbb{N}}$ are eigenvectors of the linear morphism $\mathrm{ad}_{\mathrm{i}\widehat{H}_c} : E \to E$ with different eigenvalues. Hence, the $\{\mathrm{i}\widehat{H}_n\}_{n\in\mathbb{N}}$ are linearly independent and $\dim E = +\infty$. In consequence, every Lie algebra of operators containing the $\mathrm{i}\widehat{H}(t)$ must be infinite-dimensional since it contains $\mathrm{i}\widehat{H}_a, \mathrm{i}\widehat{H}_b$ and all the $\{\mathrm{i}\widehat{H}_n\}_{n\in\mathbb{N}}$. □

Let us apply the results of the above proposition to (4.1). Let $\mathcal{L}^2(\mathbb{R}^n)$ be the space of equivalence classes of square-integrable complex functions on $\mathbb{R}^n$ that are equal almost everywhere in $\mathbb{R}^n$. Define the Hermitian operators on $\mathcal{L}^2(\mathbb{R}^n)$ of the form

$$\widehat{H}_a := \sum_{j=1}^{n}\widehat{x}_j^{\alpha}, \quad \widehat{H}_b := \sum_{j=1}^{n}\widehat{p}_j^2, \quad \widehat{H}_c := \sum_{j=1}^{n}\frac{(\widehat{x}_j\widehat{p}_j + \widehat{p}_j\widehat{x}_j)}{2}.$$

Note that $\widehat{H}_c$ is not included in (4.1). Its role in the quantum Lie systems no-go proposition is to ensure that an infinite family of its eigenvectors with different eigenvalues must be contained in any real Lie algebra containing $\langle \mathrm{i}\widehat{H}(t)\rangle_{r\in\mathbb{R}}$. Hence, $\mathrm{i}\widehat{H}_c$ does not need neither to belong to $\mathfrak{U}$ nor to appear in (4.1).

Let us define

$$\mathrm{pr}(\mathrm{i}\widehat{H}_{NH}(t)) = \frac{\mathrm{i}}{2}\widehat{p}_1^2 + \omega(t)\frac{\mathrm{i}}{2}\widehat{x}_1^2 + \mathrm{i}\tilde{\omega}(t)\widehat{x}_1^\alpha,\ \mathrm{pr}(\mathrm{i}\widehat{H}_c) = \mathrm{i}\frac{(\widehat{x}_1\widehat{p}_1 + \widehat{p}_1\widehat{x}_1)}{2},\ \mathrm{pr}(\mathrm{i}\widehat{H}_a) = \mathrm{i}\widehat{x}_1^\alpha,\ \mathrm{pr}(\mathrm{i}\widehat{H}_b) = \mathrm{i}\widehat{p}_1^2.$$

Since $[\mathrm{i}\widehat{H}_c, \mathrm{i}\widehat{H}_a] = \alpha\mathrm{i}\widehat{H}_a$, $[\mathrm{i}\widehat{H}_c, \mathrm{i}\widehat{H}_b] = -2\mathrm{i}\widehat{H}_b$ and

$$\left[\mathrm{i}\widehat{p}_1^2, \mathrm{i}\widehat{x}_1^\alpha\widehat{p}_1^k\right] = 2\alpha\mathrm{i}\widehat{x}_1^{\alpha-1}\widehat{p}_1^{k+1} + \alpha(\alpha-1)\widehat{x}_1^{\alpha-2}\widehat{p}_1^k,$$

it follows that

$$[\mathrm{pr}(\mathrm{i}\widehat{H}_b), \mathrm{pr}(\mathrm{i}\widehat{H}_a)] = 2\alpha\mathrm{i}\widehat{x}_1^{\alpha-1}\widehat{p}_1 + \alpha(\alpha-1)\widehat{x}_1^{\alpha-2},$$

and, by induction and recalling that $\alpha < 0$, we obtain

$$\mathrm{ad}^n_{\mathrm{pr}(\mathrm{i}\widehat{H}_b)}(\mathrm{pr}(\mathrm{i}\widehat{H}_a)) = \sum_{k=0}^n a_k\widehat{x}_1^{\alpha-(n+k)}\widehat{p}_1^{n-k},$$

for certain complex constants $a_0, \ldots, a_n$. In particular, it can be proved that $a_0 = 2^n i\alpha(\alpha-1)\cdots(\alpha-n+1)$, while $a_n = (-i)^{n-1}\alpha(\alpha-1)\cdots(\alpha-2n+1)$, which are different from zero since $\alpha < 0$. Since $\mathrm{ad}^n_{\mathrm{pr}(\mathrm{i}\widehat{H}_b)}\mathrm{pr}(\mathrm{i}\widehat{H}_a) \neq 0$, the quantum Lie systems no-go proposition ensures that (4.1) is not a quantum Lie system.

To fully understand the relevance of Proposition 4.1, it is worth stressing that it may be difficult to prove that $\{\mathrm{i}\widehat{H}(t)\}_{t\in\mathbb{R}}$ and the successive Lie brackets of the elements of such a space span an infinite-dimensional Lie algebra and, hence, $\widehat{H}(t)$ is not a quantum Lie system. Instead, Proposition 4.1 gives conditions that ensure that $\widehat{H}(t)$ is not a quantum Lie system. In particular, a Lie algebra morphism $\mathrm{pr} : \mathfrak{U} \subset \mathrm{End}(\mathcal{H}) \to \mathrm{End}(\mathcal{H})$ can be used to simplify calculations by making $\mathrm{pr}(\mathrm{i}\widehat{H}_a)$ and $\mathrm{pr}(\mathrm{i}\widehat{H}_b)$ to be as simple as possible, which in turn will turn the condition $\mathrm{ad}^n_{\mathrm{pr}(\mathrm{i}\widehat{H}_b}\mathrm{pr}(\mathrm{i}\widehat{H}_a) \neq 0$ easier to be verified.

Note that the operators $\widehat{H}_a$, $\widehat{H}_b$, and $\widehat{H}_c$ have been chosen so as to ensure that the conditions of the quantum Lie systems no-go proposition hold. For instance, if we had chosen $\widehat{H}_a = \sum_{j=1}^n \widehat{x}_j^2$ instead of $\widehat{H}_a = \sum_{j=1}^n \widehat{p}_j^\alpha$, we would have obtained that $\mathrm{ad}^3_{\mathrm{pr}(\mathrm{i}\widehat{H}_b)}\mathrm{pr}\left(\mathrm{i}\sum_{j=1}^n \widehat{x}_j^2\right) = 0$, which would have shown that Proposition 4.1 cannot be applied.

Consider now a t-dependent Schrödinger equation

$$\frac{\partial\psi}{\partial t} = -\mathrm{i}\widehat{H}(t)\psi, \qquad \psi \in \mathcal{H}, \qquad t \in \mathbb{R}, \tag{4.2}$$

where $-\mathrm{i}\widehat{H}(t)$ is a t-dependent operator such that every particular skew-Hermitian operator $-\mathrm{i}\widehat{H}(t)$, for any $t \in \mathbb{R}$, belongs to a finite-dimensional real linear space $\mathfrak{V}$ of skew-Hermitian operators.

Hence, the t-dependent skew-Hermitian operator $-\mathrm{i}\widehat{H}(t)$ can be understood as a curve in $\mathfrak{V}$.

This time we do not impose the skew-Hermitian operators $\{-\mathrm{i}\widehat{H}(t)\}_{t\in\mathbb{R}}$ and their successive commutators to span a real finite-dimensional Lie algebra of operators (with respect to the operator commutator) as in the case of quantum Lie systems. Instead, suppose that the operators $\{-\mathrm{i}\widehat{H}(t)\}_{t\in\mathbb{R}}$ belong to a finite-dimensional real linear space of skew-Hermitian operators $\mathfrak{V}$ and we also assume that there exists a nonzero real Lie algebra $\mathfrak{W} \subset \mathfrak{V}$ such that $[\mathfrak{W}, \mathfrak{V}] \subset \mathfrak{V}$.

Definition 4.2 A *quantum quasi-Lie scheme*, $S(\mathfrak{W}, \mathfrak{V})$, is a pair of nonzero finite-dimensional $\mathbb{R}$-linear spaces $\mathfrak{W}, \mathfrak{V}$ of skew-Hermitian operators on a Hilbert space $\mathcal{H}$ satisfying that

$$\mathfrak{W} \subset \mathfrak{V}, \qquad [\mathfrak{W}, \mathfrak{W}] \subset \mathfrak{W}, \qquad [\mathfrak{W}, \mathfrak{V}] \subset \mathfrak{V}. \tag{4.3}$$

To illustrate quasi-Lie schemes, let us consider the $\mathbb{R}$-linear space of skew-Hermitian operators

$$\mathfrak{V}_{NH} := \left\langle \mathrm{i}\widehat{H}_0 := \mathrm{i}\sum_{j=1}^n \widehat{p}_j^2,\ \mathrm{i}\widehat{H}_1 := \mathrm{i}\sum_{j=1}^n \widehat{x}_j^\alpha,\ \mathrm{i}\widehat{H}_2 := \mathrm{i}\sum_{j=1}^n \widehat{x}_j^2,\ \mathrm{i}\widehat{H}_3 := \mathrm{i}\sum_{j=1}^n \frac{(\widehat{x}_j\widehat{p}_j + \widehat{p}_j\widehat{x}_j)}{2} \right\rangle,$$

for a fixed $\alpha \in \mathbb{N}\backslash\{2\}$ and $\mathfrak{W}_{NH} := \langle \mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_3 \rangle$.

Since $[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_3] = -2\mathrm{i}\widehat{H}_2$, the linear space $\mathfrak{W}_{NH}$ is a two-dimensional real Lie algebra. We can see that $[\mathfrak{W}_{NH}, \mathfrak{V}_{NH}] \subset \mathfrak{V}_{NH}$ since

$$[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_1] = 0, \qquad [\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_0] = -\mathrm{i}\widehat{H}_3, \qquad [\mathrm{i}\widehat{H}_3, \mathrm{i}\widehat{H}_0] = -2\mathrm{i}\widehat{H}_0, \qquad [\mathrm{i}\widehat{H}_3, \mathrm{i}\widehat{H}_1] = \alpha\mathrm{i}\widehat{H}_1.$$

It follows that $\mathfrak{W}_{NH}$ and $\mathfrak{V}_{NH}$ satisfy the conditions (4.3), and therefore, the pair $\mathfrak{W}_{NH}, \mathfrak{V}_{NH}$ gives rise to a quantum quasi-Lie scheme $S(\mathfrak{W}_{NH}, \mathfrak{V}_{NH})$. Moreover, $-\mathrm{i}\widehat{H}_{NH}(t)$ takes values in $\mathfrak{V}_{NH}$. Indeed,

$$\widehat{H}_{NH}(t) = \frac{1}{2}\widehat{H}_0 + \omega(t)\frac{1}{2}\widehat{H}_2 + \tilde{\omega}(t)\widehat{H}_1.$$

The last expression will be a clue to study the t-dependent Schrödinger equation related to $\widehat{H}_{NH}(t)$ through quasi-Lie schemes. Indeed, it suggests us, along with other examples to be studied in the forthcoming sections, to propose the following definition.

Definition 4.3 The t-dependent Schrödinger equation (4.2) admits a *compatible quantum quasi-Lie scheme* $S(\mathfrak{W}, \mathfrak{V})$ if $-\mathrm{i}\widehat{H}(t)$ is such that every operator $-\mathrm{i}\widehat{H}(t)$ for a fixed $t \in \mathbb{R}$ belongs to $\mathfrak{V}$, namely $-\mathrm{i}\widehat{H}(t)$ can be considered as a curve in $\mathfrak{V}$.

Let us stress that the idea behind Definition 4.3 is that, given a t-dependent Schrödinger equation (4.2), we can apply the theory of quantum quasi-Lie schemes to study it only via a quantum quasi-Lie scheme that is compatible with the t-dependent Schrödinger equation in the sense given above. Note that a t-dependent skew-Hermitian operator can be understood as a function from $\mathbb{R}$ to the space of skew-Hermitian operators from $\mathcal{H}$ to $\mathcal{H}$. In this manner, one can say that $-\mathrm{i}\widehat{H}(t)$ is compatible with the quantum quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ if and only if $-\mathrm{i}\widehat{H}(t)$ takes values in $\mathfrak{V}$.

If not otherwise stated, we hereafter assume that every quantum quasi-Lie scheme and t-dependent skew-Hermitian operator is defined on a generic (possibly infinite-dimensional) Hilbert space $\mathcal{H}$.

5 Quantum quasi-Lie systems

Many of the works in the literature try to map a certain t-dependent operator into a simpler one by means of a t-dependent gauge transformation (see, e.g. [24]). Similarly, we introduce quantum quasi-Lie systems in this section as a class of t-dependent Schrödinger equations that can be transformed into a t-dependent Schrödinger equation related to a quantum-Lie system via a quantum quasi-Lie scheme. To achieve this goal, we will extend some previous results on quantum Lie systems to a more general realm.

Definition 5.1 The *representation of a quasi-Lie scheme* $S(\mathfrak{W}, \mathfrak{V})$ is a Lie algebra morphism

$$\begin{aligned} \rho \ : \mathfrak{W} \ &\rightarrow \quad \mathrm{End}(\mathfrak{V}), \\ \widehat{A} \ &\mapsto \rho_{\widehat{A}} := [\widehat{A}, \cdot], \end{aligned}$$

where we recall that $\mathrm{End}(\mathfrak{V})$ is the Lie algebra of endomorphisms on $\mathfrak{V}$ relative to the operator commutator. A *subrepresentation* of $S(\mathfrak{W}, \mathfrak{V})$ is a subspace $\mathfrak{V}_1 \subset \mathfrak{V}$ such that $S(\mathfrak{W}, \mathfrak{V}_1)$ is a quasi-Lie scheme.

As the subspace $\mathfrak{W}$ of a quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ is a real Lie algebra of skew-Hermitian operators on a Hilbert space $\mathcal{H}$, there exists a Lie group action $\Phi : (g, \psi) \in G \times \mathcal{H} \mapsto \Phi(g, \psi) \in \mathcal{H}$ such that the mappings $\Phi^g : \psi \in \mathcal{H} \mapsto \Phi(g, \psi) \in \mathcal{H}$, with $g \in G$, are unitary operators on $\mathcal{H}$ and, moreover, one has that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \Phi(\exp(tv), \psi) = \widetilde{\rho}(v)\Phi(\exp(t_0 v), \psi), \qquad \forall \psi \in \mathcal{H},$$

where $\mathfrak{g}$ is the Lie algebra of G, while $\widetilde{\rho} : \mathfrak{g} \rightarrow \mathfrak{W}$ is a Lie algebra isomorphism relative to the commutator operator on elements of $\mathfrak{W}$.

Note that there may be other Lie group action $\Phi' : G' \times \mathcal{H}_{\mathbb{R}} \rightarrow \mathcal{H}_{\mathbb{R}}$, where the Lie algebra $\mathfrak{g}'$ of G' is isomorphic to $\mathfrak{g}$ and G' is connected, satisfying similar properties. Let us study the relations between such Lie group actions.

Let us consider $\widehat{\Phi} : g \in G \mapsto \Phi^g \in U(\mathcal{H})$ and $\widehat{\Phi}' : g \in G' \mapsto \Phi'^g \in U(\mathcal{H})$. Now, we aim to show that $\widehat{\Phi}(G) = \widehat{\Phi}'(G')$. Due to the fact that Φ and Φ' are Lie group actions and $\widetilde{\rho}, \widetilde{\rho}\,'$ are their tangent maps at 0, respectively, one can consider the following commutative diagram

$$\begin{array}{ccccc} \mathfrak{g} & \xrightarrow{\ \widetilde{\rho}\ } & \mathfrak{W} & \xleftarrow{\ \widetilde{\rho}\,'\ } & \mathfrak{g}' \\ \downarrow{\scriptstyle \exp_G} & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp_{G'}} \\ G & \xrightarrow{\ \widehat{\Phi}\ } & U(\mathcal{H}) & \xleftarrow{\ \widehat{\Phi}'\ } & G' \end{array} \tag{5.1}$$

where $\exp_G, \exp_{G'}$ are the exponential maps from $\mathfrak{g}, \mathfrak{g}'$ into the Lie groups G, G', respectively, and exp stands for the exponential of operators in $\mathfrak{W}$, which exists because the elements of $\mathfrak{W}$ are skew-Hermitian operators and as a consequence of the Stone–von Neumann theorem. Since G is connected, every element of G can be written as a product of elements in the image of the $\exp_G$ [49, p. 224]. Hence, the group generated by the elements of $\widehat{\Phi}(\exp_G(\mathfrak{g}))$ is $\widehat{\Phi}(G)$. Repeating the same argument concerning the right-hand side of diagram (5.1), using that it is commutative, and the fact that $\mathrm{Im}\,\widetilde{\rho} = \mathrm{Im}\,\widetilde{\rho}\,'$, we obtain that $\widehat{\Phi}(G) = \widehat{\Phi}'(G')$.

The space $\mathcal{G}_{\mathfrak{W}}$ of curves $U(t)$ in $\widehat{\Phi}(G)$ with $U(0) = \mathrm{Id}_{\mathcal{H}}$ is a group relative to the multiplication

$$(U_1 \star U_2)(t) = U_1(t)U_2(t), \qquad U_1(t), U_2(t) \in \mathcal{G}_{\mathfrak{W}}, \qquad \forall t \in \mathbb{R}.$$

Note that we will hereafter consider $\mathcal{G}_{\mathfrak{W}}$ as an algebraic group. This implies that no additional manifold structure is assumed to be defined on it. Moreover, the neutral element of $\mathcal{G}_{\mathfrak{W}}$ is the curve in G mapping each $t \in \mathbb{R}$ to the neutral element, e, of G. The previous ideas justify the following definition.

Definition 5.2 If $S(\mathfrak{W}, \mathfrak{V})$ is a quasi-Lie scheme and $\Phi : G_{\mathfrak{W}} \times \mathcal{H}_{\mathbb{R}} \rightarrow \mathcal{H}_{\mathbb{R}}$ is a Lie group action obtained by integrating the Lie algebra $\mathfrak{W}$, we call *group* of the quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ the set $\mathcal{G}_{\mathfrak{W}}$ of curves $U(t)$ in $\widehat{\Phi}(G_{\mathfrak{W}})$ with $U(0) = \mathrm{Id}_{\mathcal{H}}$.

Recall that, given a t-dependent Schrödinger equation (4.2), one can define a family of unitary operators $U(t_2, t_1)$, with $t_2, t_1 \in \mathbb{R}$, such that, given a certain $\psi_0 \in \mathcal{H}$, then $U(t_2, t_1)\psi_0$ is the value of the particular solution $\psi(t)$ to (4.2) for $t = t_2$ with initial condition $\psi(t_1) = \psi_0$. To simplify the notation, we will call *t-dependent evolution operator of a Schrödinger equation* (4.2) the t-dependent family of unitary operators, $U(t)$, with $t \in \mathbb{R}$, such that $U(t) = U(t, 0)$.

Proposition 5.3 *The elements of the group $\mathcal{G}_{\mathfrak{W}}$ of a quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ are the t-dependent evolution operators of the t-dependent Schrödinger equations of the form*

$$\frac{\partial \psi}{\partial t} = -i\widehat{H}_0(t)\psi, \qquad \psi \in \mathcal{H}, \tag{5.2}$$

for a certain t-dependent operator $-i\widehat{H}_0(t)$ taking values in $\mathfrak{W}$. Conversely, the t-dependent evolution operator to (5.2) can be described by a t-dependent evolution operator U(t) in $\mathcal{G}_{\mathfrak{W}}$.

Proof Let $\mathcal{U}$ be an open neighbourhood of the neutral element, e, of the Lie group $G_{\mathfrak{W}}$, where canonical coordinates of the second-kind can be defined. Consider a curve $U(t)$ in $\mathcal{G}_{\mathfrak{W}}$. Then, $U(0) = \mathrm{Id}_{\mathcal{H}}$ and, if t is so close enough to 0, let us say $t \in (-\epsilon, \epsilon)$ for a certain real $\epsilon > 0$, we can assume that $U(t)$ takes values in $\widehat{\Phi}(\mathcal{U})$ for $t \in (-\epsilon, \epsilon)$, where we recall that $\widehat{\Phi}$ is the morphism of groups $\widehat{\Phi} : G_{\mathfrak{W}} \to U(\mathcal{H})$ related to the Lie group action $\Phi : G_{\mathfrak{W}} \times \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ induced by the integration of $\mathfrak{W}$. Using canonical coordinates of second-kind in $\mathcal{U}$ in a particular basis $\{\mathrm{i}\widehat{H}_1, \ldots, \mathrm{i}\widehat{H}_r\}$ of $\mathfrak{W}$, we obtain that $U(t)$ can be written as

$$U(t) = \exp\big(\mathrm{i} f_1(t)\widehat{H}_1\big) \times \ldots \times \exp\big(\mathrm{i} f_r(t)\widehat{H}_r\big), \tag{5.3}$$

for certain t-dependent real functions $f_1(t), \ldots, f_r(t)$ vanishing at $t = 0$.

A short calculation shows that

$$\frac{\mathrm{d}U}{\mathrm{d}t}(t)\psi = \sum_{j=1}^{r} \frac{\mathrm{d}f_j}{\mathrm{d}t}(t)\mathrm{Ad}_{\prod_{k=1}^{j-1}\exp(\mathrm{i} f_k(t)\widehat{H}_k)}(\mathrm{i}\widehat{H}_j)U(t)\psi, \qquad \forall \psi \in \mathcal{H}, \tag{5.4}$$

where $\mathrm{Ad}_{\exp(\mathrm{i} f_k(t)\widehat{H}_k)}\widehat{H}_j := \exp(\mathrm{i} f_k(t)\widehat{H}_k)\widehat{H}_j \exp(-\mathrm{i} f_k(t)\widehat{H}_k)$ for every $j, k = 1, \ldots, r$ and $t \in \mathbb{R}$. Note that the exponentials are arranged in ascending order relative to k. We also assume that $\mathrm{Ad}_{\prod_{k=1}^{0}\exp(\mathrm{i} f_k(t)\widehat{H}_k)} := \mathrm{Id}_{\mathfrak{W}}$. As $\mathfrak{W}$ is a real Lie algebra, each morphism $\mathrm{Ad}_{\exp(\lambda \mathrm{i}\widehat{H}_j)}$, with $\lambda \in \mathbb{R}$ and $j = 1, \ldots, r$, leaves $\mathfrak{W}$ invariant. Moreover, the right-hand side of (5.4) shows that

$$\frac{\mathrm{d}U}{\mathrm{d}t}(t)U(t)^{-1} = \sum_{j=1}^{r} \frac{df_j}{dt}(t)\mathrm{Ad}_{\prod_{k=1}^{j-1}\exp(\mathrm{i} f_k(t)\widehat{H}_k)}(\mathrm{i}\widehat{H}_j)$$

becomes a curve $-\mathrm{i}\widehat{H}_0(t)$ taking values in $\mathfrak{W}$. Consequently, $U(t)$ determines a curve $-\mathrm{i}\widehat{H}_0(t)$ in $\mathfrak{W}$ and the general solution to its t-dependent Schrödinger equation has general solution $U(t)\psi_0$ for $\psi_0 \in \mathcal{H}$. Recall that since $f_k(0) = 0$, for $k = 1, \ldots, r$, the $\psi_0 \in \mathcal{H}$ is the initial condition for the particular solution $\psi(t) = U(t)\psi_0$ and $U(0) = \mathrm{Id}_{\mathcal{H}}$.

Conversely, assume that we are given a curve $-\mathrm{i}\widehat{H}_0(t)$ in $\mathfrak{W}$. Let us determine a curve $U(t)$, defined at least for t in an open neighbourhood around zero, describing the t-dependent evolution operator for (5.2). Around $t = 0$, the curve $U(t)$ can be written in the form (5.3) because $U(0) = \mathrm{Id}_{\mathcal{H}}$. We can define a t-dependent family of mappings of the form $T_t : \mathfrak{W} \to \mathfrak{W}$, for each $t \in \mathbb{R}$, such that

$$-\mathrm{i}\sum_{j=1}^{r} \lambda_j \widehat{H}_j \overset{T_t}{\mapsto} -\sum_{j=1}^{r} \lambda_j \mathrm{Ad}_{\prod_{k=1}^{j-1}\exp(\mathrm{i} f_k(t)\widehat{H}_k)}(\mathrm{i}\widehat{H}_j)$$

for every set $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$. The functions $f_1(t), \ldots, f_r(t)$ are undetermined now, but $f_1(0) = \ldots = f_r(0) = 0$ and then $T(0) = \mathrm{Id}_{\mathfrak{W}}$. Since T_t depends continuously on t, then each particular T_{t_0} will be invertible for t_0 in a neighbourhood of zero. As a consequence, the system of differential equations given by

$$-\sum_{j=1}^{r} \frac{\mathrm{d}f_j}{\mathrm{d}t}(t)\mathrm{Ad}_{\left(\prod_{k=1}^{j-1}\exp(\mathrm{i} f_k(t)\widehat{H}_k\right)}(\mathrm{i}\widehat{H}_j) = -\mathrm{i}\widehat{H}_0(t)$$

amounts, after applying T_t^{-1}, to

$$-\mathrm{i}\sum_{k=1}^{r} \frac{df_k}{dt}(t)\widehat{H}_k = T_t^{-1}(-\mathrm{i}\widehat{H}_0(t)),$$

which always has a local solution $f_1(t), \ldots, f_r(t)$ for t in a neighbourhood of zero. This allows us to determine the values of $f_1(t), \ldots, f_r(t)$ close to $t = 0$ ensuring that $U(t)$ is the t-dependent evolution operator of (5.2).

□

The main property of a quantum quasi-Lie scheme is given by the following theorem, which will be employed to simplify t-dependent Schrödinger equations 'compatible' with quantum quasi-Lie schemes.

Definition 5.4 Let $S(\mathfrak{W}, \mathfrak{V})$ be a quasi-Lie scheme and let $U(t)$ be the t-dependent evolution operator related to a t-dependent skew-Hermitian operator $-\mathrm{i}\widehat{H}(t)$ defining a curve in $\mathfrak{V}$. If $U'(t)$ stands for a curve in $\mathcal{G}_{\mathfrak{W}}$, then we write $U'(t)_{\bigstar}(-\mathrm{i}\widehat{H}(t))$, for the t-dependent skew-Hermitian operator associated with the t-dependent evolution operator $(U' \star U)(t)$. We also write $\mathfrak{W}_{\mathbb{R}}$ and $\mathfrak{V}_{\mathbb{R}}$ for the spaces of curves in $\mathfrak{W}$ and $\mathfrak{V}$, respectively.

Theorem 5.5 *(The main theorem of quantum Lie systems). Let $S(\mathfrak{W}, \mathfrak{V})$ be a quantum quasi-Lie scheme and let $-i\widehat{H}(t)$ be a curve in $\mathfrak{V}$. If $U_{\mathfrak{W}}(t)$ is an element of $\mathcal{G}_{\mathfrak{W}}$, then*

$$U_{\mathfrak{W}}(t)_{\bigstar}(-i\widehat{H}(t)) := \widehat{A}(t) + Ad_{U_{\mathfrak{W}}(t)}(-i\widehat{H}(t)), \tag{5.5}$$

where $\widehat{A}(t)$ is the t-dependent skew-Hermitian operator in $\mathfrak{W}_{\mathbb{R}}$ whose evolution is determined by $U_{\mathfrak{W}}(t)$, is an element of $\mathfrak{V}_{\mathbb{R}}$. If $\psi(t)$ is a particular solution to the t-dependent Schrödinger equation related to $-i\widehat{H}(t)$, then the curve in $\mathcal{H}$ of the form $\psi'(t) = U_{\mathfrak{W}}(t)\psi(t)$ is a solution to the Schrödinger equation related to $U_{\mathfrak{W}}(t)_{\bigstar}(-i\widehat{H}(t))$.

Proof As $U_{\mathfrak{W}}(t)$ is the t-dependent evolution operator related to the t-dependent skew-Hermitian operator $\widehat{A}(t)$, one has that $\widehat{A}(t)$ is an element of $\mathfrak{W}_{\mathbb{R}} \subset \mathfrak{V}_{\mathbb{R}}$.

Since $-\mathrm{i}\widehat{H}'(t) := U_{\mathfrak{W}}(t)_{\bigstar}(-\mathrm{i}\widehat{H}(t))$ is by definition the t-dependent skew-Hermitian operator admitting a t-dependent evolution operator $U'(t) := U_{\mathfrak{W}}(t)U(t)$, where $U(t)$ is the t-dependent evolution operator for $\widehat{H}(t)$, then

$$\begin{aligned}\frac{\mathrm{d}U'}{\mathrm{d}t}(t)U'^{\dagger}(t)\psi_0 &= \left[\frac{dU_{\mathfrak{W}}}{dt}(t)U^{\dagger}_{\mathfrak{W}}(t) + \mathrm{Ad}_{U_{\mathfrak{W}}(t)}\left(\frac{dU}{dt}(t)U^{\dagger}(t)\right)\right]\psi_0 \\ &= \widehat{A}(t)\psi_0 - \mathrm{Ad}_{U_{\mathfrak{W}}(t)}\big(\mathrm{i}\widehat{H}(t)\big)\psi_0 = \widehat{A}(t)\psi_0 + \left(\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{ad}^n_{\widehat{A}(t)}(-\mathrm{i}\widehat{H}(t))\right)\psi_0,\end{aligned} \tag{5.6}$$

for every $\psi_0 \in \mathcal{H}$. As $S(\mathfrak{W}, \mathfrak{V})$ is a quantum Lie system, the curve $-[\widehat{B}, \mathrm{i}\widehat{H}(t)] \subset \mathfrak{V}$, for any $\widehat{B} \in \mathfrak{W}$, is a curve in $\mathfrak{V}$ and $-\mathrm{i}\widehat{A}(t)$ is a curve in $\mathfrak{W}$. By induction, $\mathrm{ad}^n_{\widehat{A}(t)}(\mathrm{i}\widehat{H}(t))$ is also a curve contained in $\mathfrak{V}$ for every $n \in \mathbb{N} \cup \{0\}$ and, in view of (5.6), one has that

$$\frac{\mathrm{d}U'}{\mathrm{d}t}(t)U'^{\dagger}(t) = -\mathrm{i}\widehat{H}'(t)$$

and, for every $t \in \mathbb{R}$, the skew-Hermitian operator $-\mathrm{i}\widehat{H}'(t)$ belongs to $\mathfrak{V}$. □

Summarising, a quantum quasi-Lie scheme on a Hilbert space $\mathcal{H}$ induces a group of t-dependent transformations on $\mathcal{H}$ allowing us to transform the t-dependent Schrödinger equation on $\mathcal{H}$ described by a t-dependent skew-Hermitian operator in $\mathfrak{V}_{\mathbb{R}}$ of a quantum quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ into another t-dependent Schrödinger equations related to t-dependent skew-Hermitian operators in $\mathfrak{V}_{\mathbb{R}}$. A particular relevant case occurs when the quantum quasi-Lie scheme enables us to transform, via an element of $\mathcal{G}_{\mathfrak{W}}$, the initial t-dependent Schrödinger equation into a final one which can be described by means of the usual theory of quantum Lie systems.

This motivates the next definition.

Definition 5.6 The t-dependent skew-Hermitian operator $-\mathrm{i}\widehat{H}(t)$ is a *quantum quasi-Lie system* with respect to the quantum quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ if $-\mathrm{i}\widehat{H}(t)$ is a curve in $\mathfrak{V}$ and there exists a curve $U(t)$ in $\mathcal{G}_{\mathfrak{W}}$ such that $U(t)_{\bigstar}(-\mathrm{i}\widehat{H}(t))$ is a quantum Lie system.

To simplify the notation, we call quantum quasi-Lie system and quantum Lie system the t-dependent Schrödinger equations related to a quantum quasi-Lie system or a quasi-Lie system, respectively.

6 Transformation properties of quantum quasi-Lie schemes

The group, $\mathcal{G}_{\mathfrak{W}}$, of a quantum quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ allows us to map every t-dependent skew-Hermitian operator $-\mathrm{i}\widehat{H}(t)$ in $\mathfrak{V}_{\mathbb{R}}$ into new ones in $\mathfrak{V}_{\mathbb{R}}$. This fact may be helpful in integrating or, at least, simplifying t-dependent Schrödinger equations related to t-dependent Hamiltonian operators [24]. This section depicts new easily implementable techniques to accomplish such a study.

Proposition 6.1 *If $S(\mathfrak{W}, \mathfrak{V})$ is a quantum quasi-Lie scheme, $-i\widehat{H}(t)$ belongs to $\mathfrak{V}_{\mathbb{R}}$, and $P : \mathfrak{V} \to \mathfrak{W}$ is any projection of $\mathfrak{V}$ onto $\mathfrak{W}$, then there exists $U(t)$ in $\mathcal{G}_{\mathfrak{W}}$ such that $P[U(t)_{\bigstar}(-i\widehat{H}(t))] = 0$ at least for t in an open neighbourhood of 0 in $\mathbb{R}$.*

Proof An element $U(t)$ in $\mathcal{G}_{\mathfrak{W}}$ acts on a $-\mathrm{i}\widehat{H}(t)$ belonging to $\mathfrak{V}_{\mathbb{R}}$ in the form (5.5). Applying the projection P in both sides of (5.5) and using that $dU(t)/dt(t)U^{\dagger}(t)$ takes values in $\mathfrak{W}$ for every $t \in \mathbb{R}$, one obtains

$$P[U(t)_{\bigstar}(-\mathrm{i}\widehat{H}(t))] = P\left(\frac{\mathrm{d}U}{\mathrm{d}t}(t)U^{\dagger}(t) - \mathrm{Ad}_{U(t)}(\mathrm{i}\widehat{H}(t))\right) = \frac{\mathrm{d}U}{\mathrm{d}t}(t)U^{\dagger}(t) - P(\mathrm{Ad}_{U(t)}(\mathrm{i}\widehat{H}(t))).$$

Consider the differential equation for $U(t)$ of the form

$$\frac{dU}{dt}(t)U^\dagger(t) = P(\mathrm{Ad}_{U(t)}\mathrm{i}\widehat{H}(t)). \tag{6.1}$$

Any particular solution $U(t)$ of this equation with $U(0) = \mathrm{Id}_{\mathcal{H}}$ will map $-\mathrm{i}\widehat{H}(t)$ into a new t-dependent skew-Hermitian operator $-\mathrm{i}\widehat{H}'(t) := U(t)_\bigstar(-\mathrm{i}\widehat{H}(t))$ such that $P[U(t)_\bigstar(-\mathrm{i}\widehat{H}(t)] = 0$.

Let us consider a linear space, $\mathcal{V}$, isomorphic to $\mathfrak{V}$ via $\Theta : \mathcal{V} \to \mathfrak{V}$. Consider also $\mathcal{W} \subset \mathcal{V}$ to be the linear subspace $\mathcal{V} := \Theta^{-1}(\mathfrak{W})$. Then, $\mathcal{W}$ inherits via Θ a Lie algebra structure that makes that $\mathcal{W}$ can be considered as the Lie algebra of a Lie group $G_{\mathfrak{W}}$. The isomorphism $\Theta : \mathcal{V} \to \mathfrak{V}$ allows us to define a projector $\mathcal{P} : \mathcal{V} \to \mathcal{V}$ onto $\mathcal{W}$ of the form $\mathcal{P} := \Theta^{-1} \circ P \circ \Theta$. Consider the system of differential equations on $G_{\mathfrak{W}}$ of the form

$$R_{g^{-1}*g}\frac{\mathrm{d}g}{\mathrm{d}t} = \mathcal{P}(\mathrm{Ad}_g a(t)), \tag{6.2}$$

where $a(t) := \Theta^{-1}(\mathrm{i}\widehat{H}(t))$. Let us prove that the solution to this system satisfying that $g(0) = e$ is such that the related $U(t)$ is a solution to (6.1). Let $\Phi : G_{\mathfrak{W}} \times \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ be the Lie group action induced by the Lie algebra isomorphism $\Theta|_{\mathcal{W}}\colon \mathcal{W} \to \mathfrak{W}$. If we define $U(t) = \Phi^{g(t)}$ and $\psi \in \mathcal{H}$, we have that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} U(t)\psi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \Phi(g(t)g^{-1}(t_0), \Phi(g(t_0), \psi)) = \Theta(\mathcal{P}(\mathrm{Ad}_{g(t_0)}a(t_0)))U(t_0)\psi.$$

Since $\Theta(\mathrm{Ad}_g a(t)) = \Phi^g \mathrm{i}\widehat{H}(t)\Phi^{g^{-1}}$, we get from the definition of $\mathcal{P}$ that

$$P(\Phi^g \mathrm{i}\widehat{H}(t)\Phi^{g^{-1}}) = P(\Theta(\mathrm{Ad}_g a(t))) = \Theta(\mathcal{P}(\mathrm{Ad}_g a(t))).$$

Hence,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} U(t)\psi = \Theta(\mathcal{P}(\mathrm{Ad}_g a(t_0)))U(t_0)\psi = P(U(t)\mathrm{i}\widehat{H}(t)U^{-1}(t))U(t)\psi.$$

Since the differential equation (6.2) admits a local solution for every initial condition, the same is true for the differential equation (6.1). □

If $\mathfrak{V}$ is a Lie algebra, namely $-\mathrm{i}\widehat{H}(t)$ is a quantum Lie system, then we can chose $\mathfrak{W} = \mathfrak{V}$ and the previous proposition ensures that there exists $U(t)$ in $\mathcal{G}_{\mathfrak{W}}$ such that $-U(t)_\bigstar \mathrm{i}\widehat{H}(t) = 0$. Hence, $U^{-1}(t)\psi$, for any $\psi \in \mathcal{H}$ becomes the general solution to the t-dependent Schrödinger equation related to $-\mathrm{i}\widehat{H}(t)$.

Let us use Proposition 6.1 to simplify or to integrate $-\mathrm{i}\widehat{H}(t)$ in $\mathfrak{V}_{\mathbb{R}}$ when $\mathfrak{V}$ is not a Lie algebra. In such a case, if $-\mathrm{i}\widehat{H}(t) = \sum_{\alpha=1}^{r+s} b_\alpha(t)\mathrm{i}H_\alpha$ for certain t-dependent functions $b_1(t), \ldots, b_{r+s}(t)$ and a basis $\{\mathrm{i}H_1, \ldots, \mathrm{i}H_{r+s}\}$ of $\mathfrak{V}$ such that the first r elements form a basis for $\mathfrak{W}$, then Proposition 6.1 ensures that there exists an $U(t)$ in $\mathcal{G}_{\mathfrak{W}}$ such that $U(t)_\bigstar \mathrm{i}\widehat{H}(t) = \sum_{\alpha=r+1}^{r+s} b_\alpha(t)\mathrm{i}H_\alpha$. This simplifies the expression of the initial t-dependent Schrödinger equation related to $-\mathrm{i}\widehat{H}(t)$.

To provide further techniques to integrate t-dependent Schrödinger equations determined by a t-dependent skew-Hermitian operator $-\mathrm{i}\widehat{H}(t)$ in $\mathfrak{V}_{\mathbb{R}}$ when $\mathfrak{V}$ is not a Lie algebra, let us extend to the quantum case some structures defined for standard quasi-Lie schemes in [20].

The result of the following theorem allows us to determine when a t-dependent skew-Hermitian operator, $-\mathrm{i}\widehat{H}(t)$, belonging to the space $\mathfrak{V}_{\mathbb{R}}$ related to a quantum quasi-Lie scheme $S(\mathfrak{W}, \mathfrak{V})$ cannot be mapped into zero via an element $U_{\mathfrak{W}}(t)$ of the group $\mathcal{G}_{\mathfrak{W}}$. In this case, we will use a quantum quasi-Lie scheme to study the form of $-U(t)_\bigstar \mathrm{i}\widehat{H}(t)$ for each $U(t)$ in $\mathcal{G}_{\mathfrak{W}}$.

Theorem 6.2 *Let $S(\mathfrak{W}, \mathfrak{V})$ be a quantum quasi-Lie scheme with a subrepresentation $\mathfrak{V}_1$ of codimension one, namely $\dim \mathfrak{V}/\mathfrak{V}_1 = 1$, and let $\tau_{\mathfrak{V}_1} : \mathfrak{V} \to \mathfrak{V}/\mathfrak{V}_1$ be the canonical projection onto $\mathfrak{V}/\mathfrak{V}_1$. If $-i\widehat{H}(t)$ belongs to $\mathfrak{V}_{\mathbb{R}}$ while $\tau_{\mathfrak{V}_1}(-i\widehat{H}(t)) \neq 0$ and $\tau_{\mathfrak{V}_1}(\mathfrak{W}) = 0$, then $\tau_{\mathfrak{V}_1}(-U(t)_\bigstar i\widehat{H}(t)) \neq 0$ for every $U(t)$ in $\mathcal{G}_{\mathfrak{W}}$.*

Proof To make the notation clearer, we will simply denote elements of $\mathfrak{W}$ and $\mathfrak{V}$ by lower-case letters, e.g. $w \in \mathfrak{W}$, $v, v_1 \in \mathfrak{V}$ and the elements of $\mathcal{G}_{\mathfrak{W}}$ by curves of the form $g(t)$. Since $[\mathfrak{W}, \mathfrak{V}_1] \subset \mathfrak{V}_1$, each Lie algebra morphism $\rho_w = [w, \cdot]$, with $w \in \mathfrak{W}$, induced by the representation of $S(\mathfrak{W}, \mathfrak{V})$ leads to a well-defined morphism $\widehat{\rho}_w : \tau_{\mathfrak{V}_1}(v) \in \mathfrak{V}/\mathfrak{V}_1 \mapsto \tau_{\mathfrak{V}_1}(\rho_w(v)) \in \mathfrak{V}/\mathfrak{V}_1$. In fact, if $v_1, v_2 \in \mathfrak{V}$ and $\tau_{\mathfrak{V}_1}(v_1) = \tau_{\mathfrak{V}_1}(v_2)$, then $[w, v_1 - v_2] \in \mathfrak{V}_1$ for every $w \in \mathfrak{W}$, $\tau_{\mathfrak{V}_1}([w, v_1 - v_2]) = 0$, and

$$\widehat{\rho}_w(\tau_{\mathfrak{V}_1}(v_1)) := \tau_{\mathfrak{V}_1}([w, v_1]) = \tau_{\mathfrak{V}_1}([w, v_2]) =: \widehat{\rho}_w(\tau_{\mathfrak{V}_1}(v_2)).$$

This allows us to define a Lie algebra morphism $\widehat{\rho} : w \in \mathfrak{W} \mapsto \widehat{\rho}_w \in \mathrm{End}(\mathfrak{V}/\mathfrak{V}_1)$. Indeed,

$$[\widehat{\rho}_{w_1}, \widehat{\rho}_{w_2}](\tau_{\mathfrak{V}_1}(v)) := \widehat{\rho}_{w_1}(\widehat{\rho}_{w_2}(\tau_{\mathfrak{V}_1}(v))) - \widehat{\rho}_{w_2}(\widehat{\rho}_{w_1}(\tau_{\mathfrak{V}_1}(v))) = \widehat{\rho}_{w_1}(\tau_{\mathfrak{V}_1}([w_2, v])) - \widehat{\rho}_{w_2}(\tau_{\mathfrak{V}_1}([w_1, v])),$$

for all $w_1, w_2 \in \mathfrak{W}$ and every $v \in \mathfrak{V}$. Using the Jacobi identity and the definition of $\tau_{\mathfrak{V}_1}$,

$$[\widehat{\rho}_{w_1}, \widehat{\rho}_{w_2}](\tau_{\mathfrak{V}_1}(v)) := \tau_{\mathfrak{V}_1}([w_1, [w_2, v]] - [w_2, [w_1, v]]) = \widehat{\rho}_{[w_1, w_2]}(\tau_{\mathfrak{V}_1}(v)), \qquad \forall w_1, w_2 \in \mathfrak{W}, \forall v \in \mathfrak{V}.$$

Hence, $[\widehat{\rho}_{w_1}, \widehat{\rho}_{w_2}] = \widehat{\rho}_{[w_1,w_2]}$ and $\widehat{\rho}$ is a Lie algebra morphism. Moreover, $\widehat{\rho}$ and $\rho : w \in \mathfrak{W} \mapsto [w, \cdot] \in \text{End}(\mathfrak{V})$ are equivariant relative to $\tau_{\mathfrak{V}_1}$, namely $\widehat{\rho}_w \circ \tau_{\mathfrak{V}_1} = \tau_{\mathfrak{V}_1} \circ \rho_w$, for all $w \in \mathfrak{W}$. In fact,

$$\widehat{\rho}_w \circ \tau_{\mathfrak{V}_1}(v) = \tau_{\mathfrak{V}_1}([w, v]) = \tau_{\mathfrak{V}_1} \circ \rho_w(v), \qquad \forall w \in \mathfrak{W}, \forall v \in \mathfrak{V} \Longrightarrow \widehat{\rho}_w \circ \tau_{\mathfrak{V}_1} = \tau_{\mathfrak{V}_1} \circ \rho_w, \quad \forall w \in \mathfrak{W}.$$

Using that ρ and $\widehat{\rho}$ are equivariant relative to $\tau_{\mathfrak{V}_1}$, we are going to finally prove that if $v(t)$ takes values in $\mathfrak{V}$ and $\tau_{\mathfrak{V}_1}(v(t)) \neq 0$, then $\tau_{\mathfrak{V}_1}(g(t)_\bigstar v(t)) \neq 0$ for every $g(t)$ in $\mathcal{G}_{\mathfrak{W}}$. Since $\dim \mathfrak{V}/\mathfrak{V}_1 = 1$, there exists a unique function $\theta \in \mathfrak{W}^*$ such that $\widehat{\rho}_w(\tau_{\mathfrak{V}_1}(v)) = \theta(w)\tau_{\mathfrak{V}_1}(v)$.

Consider the case of $g(t) = \exp(f(t)w)$ for some $w \in \mathfrak{W}$, a t-dependent function $f(t)$, and an element $v(t)$ in $\mathfrak{V}_{\mathbb{R}}$. It is clear that

$$g(t)_\bigstar v(t) = \frac{dg}{dt}(t)g(t)^{-1} + \text{Ad}_{g(t)}v(t) = \frac{df}{dt}(t)w + v(t) + f(t)[w, v(t)] + \frac{f^2(t)}{2}[w, [w, v(t)]] + \ldots$$

Applying $\tau_{\mathfrak{V}_1}$ on both sides, using the assumptions on $\mathfrak{W}$ and $\mathfrak{V}$, and recalling the $\widehat{\rho}$ is equivariant relative to $\tau_{\mathfrak{V}_1}$, we obtain

$$\begin{aligned}\tau_{\mathfrak{V}_1}(g(t)_\bigstar v(t)) &= \tau_{\mathfrak{V}_1}\left(\frac{dg}{dt}g(t)^{-1}\right) + \tau_{\mathfrak{V}_1}(v(t)) + \tau_{\mathfrak{V}_1}\left([f(t)w, v(t)] + \frac{f^2(t)}{2}[w, [w, v(t)]] + \ldots\right) \\ &= (1 + f(t)\theta(w) + \frac{f^2(t)}{2}\theta^2(w) + \ldots)\tau_{\mathfrak{V}_1}(v(t)) \\ &= e^{f(t)\theta(w)}\tau_{\mathfrak{V}_1}(v(t)).\end{aligned}$$

Therefore, if $\tau_{\mathfrak{V}_1}(v(t)) \neq 0$, then $\tau_{\mathfrak{V}_1}(g(t)_\bigstar v(t)) \neq 0$. Since every $g(t)$ in $\mathcal{G}_{\mathfrak{W}}$ can be written as a product of elements of the form $g(t) = \prod_{i=1}^{s} \exp(f_i(t)w_i)$ for some elements $w_1, \ldots, w_s$ of $\mathfrak{W}$ and functions $f_1(t), \ldots, f_s(t)$, it follows from the above result that if $\tau_{\mathfrak{V}_1}(v(t)) \neq 0$, then $\tau_{\mathfrak{V}_1}(g(t)_\bigstar(v(t))) \neq 0$ for every $g(t)$ in $\mathcal{G}_{\mathfrak{W}}$. □

As an immediate consequence of the proof of Proposition 6.2, we obtain the following corollary.

Corollary 6.3 *Let us assume the same conditions of Theorem 6.2 and let $i\widehat{H}_1, \ldots, i\widehat{H}_r$ be a basis of $\mathfrak{V}$ such that $iH_1, \ldots, iH_{r-1}$ is a basis of $\mathfrak{V}_1$. If θ is the element of $\mathfrak{W}^*$ associated with the induced representation $\widehat{\rho} : \mathfrak{W} \to End(\mathfrak{V}/\mathfrak{V}_1)$, namely if $\tau_{\mathfrak{V}_1} : \mathfrak{V} \to \mathfrak{V}/\mathfrak{V}_1 \simeq \langle i\widehat{H}_r\rangle$, the θ is the only element in $\mathfrak{W}^*$ such that $\tau_{\mathfrak{V}_1}([a, i\widehat{H}_r]) = \theta(a)\tau_{\mathfrak{V}_1}(i\widehat{H}_r)$ for every $a \in \mathfrak{W}$, then, for any functions $a_1(t), \ldots, a_r(t)$, we have*

$$U(t) = \prod_{i=1}^{r} \exp(a_i(t)i\widehat{H}_i) \implies \tau_{\mathfrak{V}_1}(U(t)_\bigstar i\widehat{H}(t)) = \left[\prod_{i=1}^{r} \exp(a_i(t)\theta(i\widehat{H}_i))\right] i\widehat{H}_r.$$

7 Applications of quantum quasi-Lie schemes

This section concerns the application of the theory of quantum quasi-Lie schemes to the quantum anharmonic oscillator model given by n interacting particles via a t-dependent Hermitian operator of the form

$$\widehat{H}_{nH}(t) := \frac{1}{2}\sum_{i=1}^{n}\left(\widehat{p}_i^2 + \omega^2(t)\widehat{x}_i^2\right) + c(t)U_{nH}(\widehat{x}_1, \ldots, \widehat{x}_n), \tag{7.1}$$

where $c(t)$ is a non-vanishing real function, $\omega(t)$ is any real t-dependent function describing a sort of t-dependent frequency, and $U_{nH}(\widehat{x}_1, \ldots, \widehat{x}_n)$ is a quantum potential determined by an homogeneous polynomial of order k depending on the position operators $\widehat{x}_1, \ldots, \widehat{x}_n$, i.e. $U_{nH}(\lambda\widehat{x}_1, \ldots, \lambda\widehat{x}_n) = \lambda^k U_{nH}(\widehat{x}_1, \ldots, \widehat{x}_n)$ for every $\lambda \in \mathbb{R}$. As a particular instance, (7.1) covers many types of anharmonic quantum oscillators, which have been extensively studied in the literature [50–52].

Perelomov found some conditions ensuring the existence of a t-dependent change of variables mapping a classical analogue of (7.1) onto an autonomous Hamiltonian system, up to a trivial time-reparametrisation, related to the Hamiltonian

$$H_{nH} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + U_{nH}(x_1, \ldots, x_n), \tag{7.2}$$

where U_{nH} is now understood as a real homogeneous polynomial of order k on the position variables $x_1, \ldots, x_n$ (see [23] for details). As many Hamiltonians of the form (7.2) are known to be explicitly integrable, the t-dependent change of variables found by Perelomov relates the solutions of such Hamiltonians with the solutions of an associated non-autonomous one H_{nH} [23].

Perelomov left as an open problem to look for a quantum analogue of his results (cf. [23]). Additionally, the classical anharmonic oscillator related to the t-dependent Hamiltonian (7.1) was briefly analysed via quasi-Lie schemes in [3]. Subsequently, these results are extended to the quantum realm by means of a quantum quasi-Lie scheme, resulting in a solution to Perelomov's open problem.

Let us build up a quantum quasi-Lie scheme $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ to deal with the t-dependent Hamiltonian operator $-i\widehat{H}_{nH}(t)$. This demands to determine subspaces $\mathfrak{W}_{nH}, \mathfrak{V}_{nH}$ of skew-Hermitian operators satisfying (4.3) and such that $-i\widehat{H}_c(t)$ belongs to $(\mathfrak{V}_{nH})_{\mathbb{R}}$. The construction of $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ is accomplished in the following lemma.

Lemma 7.1 *The spaces* $\mathfrak{V}_{nH} := \langle i\widehat{H}_1, i\widehat{H}_2, i\widehat{H}_3, i\widehat{H}_4\rangle$ *and* $\mathfrak{W}_{nH} := \langle i\widehat{H}_2, i\widehat{H}_3\rangle$, *with*

$$\widehat{H}_1 := \sum_{i=1}^n \frac{\widehat{p}_i^2}{2}, \quad \widehat{H}_2 := \frac{1}{4}\sum_{i=1}^n(\widehat{x}_i\widehat{p}_i + \widehat{p}_i\widehat{x}_i), \quad \widehat{H}_3 := \sum_{i=1}^n \frac{\widehat{x}_i^2}{2}, \quad \widehat{H}_4 := U_{nH}(\widehat{x}_1, \ldots, \widehat{x}_n),$$

give rise to quantum quasi-Lie scheme $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ *such that* $-i\widehat{H}_{nH}(t)$ *belongs to* $(\mathfrak{V}_{nH})_{\mathbb{R}}$.

Proof Let us prove first that $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ is a quantum quasi-Lie scheme. It is immediate that $\mathfrak{V}_{nH}$ and $\mathfrak{W}_{nH}$ are finite-dimensional real linear spaces of skew-Hermitian operators.

Since $[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_3] = \mathrm{i}\widehat{H}_3$, the space $\mathfrak{W}_{nH}$ is a real Lie algebra of skew-Hermitian operators as demanded by the definition of a quantum quasi-Lie scheme. Let us verify the only left property of quantum quasi-Lie schemes, namely $[\mathfrak{W}_{nH}, \mathfrak{V}_{nH}] \subset \mathfrak{V}_{nH}$. It is immediate that

$$[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_1] = -\mathrm{i}\widehat{H}_1, \qquad [\mathrm{i}\widehat{H}_3, \mathrm{i}\widehat{H}_1] = -2\mathrm{i}\widehat{H}_2, \qquad [\mathrm{i}\widehat{H}_3, \mathrm{i}\widehat{H}_4] = 0.$$

The only non-immediate part to corroborate the condition $[\mathfrak{W}_{nH}, \mathfrak{V}_{nH}] \subset \mathfrak{V}_{nH}$ is to show that $[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_4] \subset \mathfrak{V}_{nH}$. As $[\widehat{x}_j, \widehat{p}_j] = \mathrm{i}I$ for $j = 1, \ldots, n$, then $\widehat{p}_j\widehat{x}_j = \widehat{x}_j\widehat{p}_j - \mathrm{i}\widehat{I}$ and

$$\sum_{j=1}^n(\widehat{x}_j\widehat{p}_j + \widehat{p}_j\widehat{x}_j) = 2\sum_{j=1}^n \widehat{x}_j\widehat{p}_j - \mathrm{i}n\widehat{I}, \tag{7.3}$$

where $\widehat{I}$ is the identity operator on $\mathcal{H}$. Consequently,

$$[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_4] = -\frac{1}{2}\left[\sum_{j=1}^n \widehat{x}_j\widehat{p}_j, U_{nH}(\widehat{x}_1, \ldots, \widehat{x}_n)\right] = \frac{\mathrm{i}}{2}\sum_{j=1}^n \widehat{x}^j\frac{\partial U_{nH}}{\partial x^j}(\widehat{x}_1, \ldots, \widehat{x}_n).$$

In view of the *Euler's homogeneous function theorem* [53] and recalling that U_{nH} is a homogeneous function of degree k, it follows that

$$\sum_{j=1}^n x^j\frac{\partial U_{nH}}{\partial x^j} = kU_{nH} \implies [\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_4] = \frac{\mathrm{i}}{2}kU_{nH}(\widehat{x}_1, \ldots, \widehat{x}_n).$$

This yields $[\mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_4] = \frac{\mathrm{i}}{2}k\widehat{H}_4$ and

$[\mathfrak{W}_{nH}, \mathfrak{V}_{nH}] \subset \mathfrak{V}_{nH}$. Thus, $\mathfrak{W}_{nH}, \mathfrak{V}_{nH}$ give rise to a quantum quasi-Lie scheme $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$.

Finally, the form of $\mathfrak{V}_{nH}$ and $-\mathrm{i}\widehat{H}_{nH}(t)$, which is given by (7.1), ensures that $-\mathrm{i}\widehat{H}_{nH}(t)$ takes values in $\mathfrak{V}_{nH}$, which finishes the proof. □

Once it is been stated that $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ is a quantum quasi-Lie scheme and it can be used to describe the t-dependent Hamiltonian operators in (7.1), it is time to apply the group of $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ to analyse $-\mathrm{i}\widehat{H}_{nH}(t)$. In particular, a relevant case occurs when an element $U(t)$ of $\mathcal{G}_{\mathfrak{W}_{nH}}$ allows us to map $-\mathrm{i}\widehat{H}_{nH}(t)$ into a t-dependent skew-Hermitian operator $-\mathrm{i}\widehat{H}'_U(t) := -U(t)_\bigstar\mathrm{i}\widehat{H}_{nH}(t)$ that can be considered as a curve on a one-dimensional subspace of $\mathfrak{V}_{nH}$. In such a case, a time-reparametrisation allows us to solve the transformed t-dependent Schrödinger equation related to $-\mathrm{i}\widehat{H}'_U(t)$ and, applying $U^{-1}(t)$ to the general solution of the latter, we obtain the general solution of the initial t-dependent Schrödinger equation.

Lemma 7.2 *The t-dependent operator* $-U(t)_\bigstar i\widehat{H}_{nH}(t)$ *is a curve on a one-dimensional subspace of* $\mathfrak{V}_{nH}$ *if and only if* $U(t)$, *which is assumed without loss of generality to take the form*

$$U(t) = \exp\big(i\alpha(t)\widehat{H}_2\big)\exp\big(i\beta(t)\widehat{H}_3\big), \tag{7.4}$$

is such that $c(t)e^{\frac{\alpha(t)(k+2)}{2}}$ *is a nonzero constant.*

Proof In view of the Lie algebra representation of $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$, namely $\rho : w \in \mathfrak{W}_{nH} \mapsto [w, \cdot] \in \mathrm{End}(\mathfrak{V}_{nH})$, the quantum quasi-Lie scheme $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ admits two subrepresentations given by $\mathfrak{V}_1 := \langle \mathrm{i}\widehat{H}_1, \mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_3\rangle$ and $\mathfrak{V}_2 := \langle \mathrm{i}\widehat{H}_2, \mathrm{i}\widehat{H}_3, \mathrm{i}\widehat{H}_4\rangle$.

Since $c(t) \neq 0$, the t-dependent operator $-\mathrm{i}\widehat{H}_{nH}(t)$ has no zero projection neither onto $\mathfrak{V}/\mathfrak{V}_1$ nor onto $\mathfrak{V}/\mathfrak{V}_2$. Then, Theorem 6.2 yields that the t-dependent skew-Hermitian operator of the form $-U(t)_\bigstar\mathrm{i}\widehat{H}_{nH}(t)$ will never take values neither in $\mathfrak{V}_1$ nor in $\mathfrak{V}_2$ for any $U(t)$ in $\mathcal{G}_{\mathfrak{W}}$. In other words, if we write $-U(t)_\bigstar\mathrm{i}\widehat{H}_{nH}(t)$ in the basis $\mathrm{i}H_1, \ldots, \mathrm{i}H_4$ of $\mathfrak{V}_{nH}$, then the t-dependent coefficients relative to $\mathrm{i}H_1$ or $\mathrm{i}H_4$ will never vanish. Moreover, as we want $-U(t)_\bigstar\mathrm{i}\widehat{H}_{nH}(t)$ to take values in a one-dimensional subspace of $\mathfrak{V}_{nH}$, the t-dependent coefficients relative to $\mathrm{i}H_4$ and $\mathrm{i}H_1$ must be proportional. Let us determine their values exactly without determining the exact form of $-U(t)_\bigstar\mathrm{i}\widehat{H}_{nH}(t)$.

As in the proof of Theorem 6.2, the representation of $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$ induces a representation $\widehat{\rho}_{\mathfrak{V}_2} : \mathfrak{W}_{nH} \to \mathrm{End}(\mathfrak{V}/\mathfrak{V}_2)$ such that $(\widehat{\rho}_{\mathfrak{V}_2})_{\mathrm{i}\widehat{H}_3}(\mathrm{i}\widehat{H}_1) = 0$ i $(\widehat{\rho}_{\mathfrak{V}_2})_{\mathrm{i}\widehat{H}_2}(\mathrm{i}\widehat{H}_1) = -\mathrm{i}\widehat{H}_1$. Hence, $\theta \in \mathfrak{W}^*_{nH}$ given by $\theta(\mathrm{i}\widehat{H}_3) = 0$ and $\theta(\mathrm{i}\widehat{H}_2) = -1$ is the unique element of

$\mathfrak{W}^*_{nH}$ such that $\tau_{\mathfrak{V}_2}[\widehat{A},\widehat{B}]=\theta(\widehat{A})\tau_{\mathfrak{V}_2}(\widehat{B})$ for every $\widehat{A}\in\mathfrak{W}_{nH}$ and $\widehat{B}\in\mathfrak{V}_{nH}$. If $U(t)$ has the form (7.4), then Corollary 6.3 gives that $\tau_{\mathfrak{V}_2}(U(t)_\bigstar \mathrm{i}\widehat{H}(t))=e^{-\alpha(t)}\mathrm{i}\widehat{H}_1$.

Meanwhile, the induced representation $\widehat{\rho}_{\mathfrak{V}_1}:\mathfrak{W}_{nH}\to\mathrm{End}(\mathfrak{V}/\mathfrak{V}_1)$ by the second subrepresentation $\mathfrak{V}_1$ is such that $(\widehat{\rho}_{\mathfrak{V}_1})_{\mathrm{i}\widehat{H}_3}(\mathrm{i}\widehat{H}_4)=0$ i $(\widehat{\rho}_{\mathfrak{V}_1})_{\mathrm{i}\widehat{H}_2}(\mathrm{i}\widehat{H}_4)=\mathrm{i}k/2\widehat{H}_4$. Hence, θ is determined by $\theta(\mathrm{i}H_3)=0$ and $\theta(\mathrm{i}\widehat{H}_2)=k/2$. Thus, if $U(t)$ has the form (7.4), then Corollary 6.3 gives that $\tau_{\mathfrak{V}_1}(U(t)_\bigstar \mathrm{i}H(t))=e^{k\alpha(t)/2}\mathrm{i}\widehat{H}_4$.

Since the t-dependent coefficients of $\tau_{\mathfrak{V}_1}(U(t)_\bigstar \mathrm{i}\widehat{H}(t))$ and $\tau_{\mathfrak{V}_2}(U(t)_\bigstar \mathrm{i}\widehat{H}(t))$ must be proportional, it follows that $c(t)e^{\frac{\alpha(t)(k+2)}{2}}$ must be a constant. □

Lemma 7.2 provides an easily derivable necessary condition to map $-\mathrm{i}\widehat{H}_{nH}(t)$ into a new t-dependent operator $-\mathrm{i}\widehat{H}'_{nH}(t)$ giving rise to a curve on a one-dimensional subspace of $\mathfrak{V}_{nH}$ via an $U(t)$ belonging to $\mathcal{G}_{\mathfrak{W}_{nH}}$. Nevertheless, the determination of sufficient conditions to obtain such a $-\mathrm{i}\widehat{H}'_{nH}(t)$ demands to determine the action of a generic $U(t)$ on $-\mathrm{i}\widehat{H}_{nH}(t)$. If $U(t)$ is in $\mathcal{G}_{\mathfrak{W}_{nH}}$, then

$$-\mathrm{i}\widehat{H}'_{nH}(t)=U(t)_\bigstar(-\mathrm{i}\widehat{H}_{nH}(t))=\frac{dU}{dt}(t)U(t)^{-1}+\mathrm{Ad}(U(t))(-\mathrm{i}\widehat{H}_{nH}(t)),$$

and Theorem 5.5 ensures that $U(t)_\bigstar(-\mathrm{i}\widehat{H}_{nH}(t))$ takes values in $\mathfrak{V}_{nH}$. Every element $U(t)$ in $\mathcal{G}_{\mathfrak{W}_{nH}}$ can be written in the form

$$U(t):=\exp\big(\mathrm{i}\alpha(t)\widehat{H}_2\big)\exp\big(\mathrm{i}\beta(t)\widehat{H}_3\big). \tag{7.5}$$

To obtain $-\mathrm{i}\widehat{H}'_{nH}=U(t)_\bigstar(-\mathrm{i}\widehat{H}_{nH}(t))$, it is necessary to use that

$$\begin{aligned}
\mathrm{Ad}\big[\exp\big(\mathrm{i}\lambda\widehat{H}_2\big)\big]\widehat{H}_1&=e^{-\lambda}\widehat{H}_1, &\quad \mathrm{Ad}\big[\exp\big(\mathrm{i}\lambda\widehat{H}_3\big)\big]\widehat{H}_1&=\widehat{H}_1-2\lambda\widehat{H}_2+\lambda^2\widehat{H}_3,\\
\mathrm{Ad}\big[\exp\big(\mathrm{i}\lambda\widehat{H}_2\big)\big]\widehat{H}_3&=e^{\lambda}\widehat{H}_3, &\quad \mathrm{Ad}\big[\exp\big(\mathrm{i}\lambda\widehat{H}_3\big)\big]\widehat{H}_2&=\widehat{H}_2-\lambda\widehat{H}_3,\\
\mathrm{Ad}\big[\exp\big(\mathrm{i}\lambda\widehat{H}_2\big)\big]\widehat{H}_4&=e^{\frac{k\lambda}{2}}\widehat{H}_4, &\quad \mathrm{Ad}\big[\exp\big(\mathrm{i}\lambda\widehat{H}_3\big)\big]\widehat{H}_4&=\widehat{H}_4.
\end{aligned}$$

Then,

$$-\widehat{H}'_{nH}(t)=-e^{-\alpha(t)}\widehat{H}_1+\left(\frac{d\alpha}{dt}(t)+2\beta(t)\right)\widehat{H}_2+e^{\alpha(t)}\left(\frac{d\beta}{dt}(t)-\beta^2(t)-\omega^2(t)\right)\widehat{H}_3-e^{\frac{k\alpha(t)}{2}}c(t)\widehat{H}_4, \tag{7.6}$$

where we recall that $\omega(t)$ is the t-dependent frequency of our initial t-dependent Hermitian operator (7.1).

Note also that Proposition 6.1 ensures that the group of $S(\mathfrak{W}_{nH},\mathfrak{V}_{nH})$ allows us to map $-\mathrm{i}\widehat{H}_{nH}(t)$ into a new $-\mathrm{i}\widehat{H}'_{nH}(t)$ giving rise to a curve on $\langle \mathrm{i}\widehat{H}_1,\mathrm{i}\widehat{H}_4\rangle$ for some $U(t)$ in $\mathcal{G}_{\mathfrak{W}_{nH}}$. For instance, suppose that $\beta(t)$ and $\alpha(t)$ are such that

$$\frac{d\beta}{dt}(t)=\beta^2(t)+\omega^2(t),\qquad \frac{d\alpha}{dt}(t)+2\beta(t)=0, \tag{7.7}$$

and we recall that $\alpha(0)=\beta(0)=0$. Under such conditions,

$$\widehat{H}'_{nH}(t)=e^{-\alpha(t)}\widehat{H}_1+e^{\frac{\alpha(t)k}{2}}c(t)\widehat{H}_4.$$

To ensure that $-\mathrm{i}\widehat{H}'_{nH}(t)$ takes values in a one-dimensional subspace of $\mathfrak{V}_{nH}$, we have to recall the condition in (7.2), i.e.

$$c(t)e^{\frac{k\alpha(t)}{2}}=e^{-\alpha(t)}l\Longrightarrow c(t)e^{\frac{\alpha(t)(k+2)}{2}}=l, \tag{7.8}$$

for a nonzero constant $l\in\mathbb{R}$. Note that we can assume without loss of generality that $l=1$ by redefining $\widehat{H}_4$.

Using (7.8) in (7.7), we obtain that if $k\neq 2$ and there exist $\alpha(t)$ and $\beta(t)$ satisfying the previous conditions, then

$$(k+2)\frac{d^2c}{dt^2}(t)=(k+3)\frac{(dc/dt)^2(t)}{c(t)}+\omega^2(t)(k+2)^2c(t). \tag{7.9}$$

Conversely, if (7.9) holds, then there exist $\alpha(t)$ and $\beta(t)$ satisfying (7.7) and (7.8).

It is worth noting that the definition of a new variable $v_c:=dv/dt$ allows us to transform (7.9) into a Lie system related to a VG Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. In fact, if $k=0$, this is indeed a type of Kummer–Schwarz equation of second-order studied in [54].

Assume that (7.9) is satisfied. Then, the t-dependent Schrödinger equation related to $\widehat{H}'_{nH}(t)$ reads

$$\mathrm{i}\frac{\partial\psi}{\partial t}=\widehat{H}'_{nH}(t)\psi=e^{-\alpha(t)}(\widehat{H}_1+\widehat{H}_4)\psi.$$

and by means of the time-reparametrisation

$$\tau(t):=\int^t e^{-\alpha(t')}dt'=\int^t[c(t')l^{-1}]^{\frac{2}{k+2}}dt',$$

the previous t-dependent Schrödinger equation can be mapped into

$$\frac{\partial\psi'}{\partial\tau}=-\mathrm{i}(\widehat{H}_1+\widehat{H}_4)\psi',$$

whose solution is

$$\psi'(t) = \exp\big(-\mathrm{i}\tau(t)(\widehat{H}_1 + \widehat{H}_4)\big)\psi'(0),$$

and ψ' is the transformed solution and $\psi'(0)$ is an arbitrary element of $\mathcal{L}^2(\mathbb{R}^n)$. Hence, the solution of the initial Schrödinger equation determined by the t-dependent Hamiltonian operator $\widehat{H}_{nH}(t)$ can be obtained by inverting the unitary transformation (7.5), namely

$$\psi(t) = \exp\big(-\mathrm{i}\beta(t)\widehat{H}_3\big)\exp\big(-\mathrm{i}\alpha(t)\widehat{H}_2\big)\exp\big(-\mathrm{i}\tau(t)(\widehat{H}_1 + \widehat{H}_4)\big)\psi'(0). \tag{7.10}$$

Therefore, we have mapped a non-autonomous Schrödinger equation into a new one determined by a t-independent Hermitian Hamiltonian operator, similarly to what it was done by Perelomov in [23], but in a quantum mechanical way. Our result is summarised in the proposition below.

Proposition 7.3 *Every t-dependent Schrödinger equation related to a t-dependent Hermitian Hamiltonian operator $\widehat{H}_{nH}(t)$ of the form (7.1), with a homogeneous potential of order $k \neq -2$ and a non-vanishing function $c(t)$ that is a particular solution of (7.9), has a general solution (7.10), where*

$$\alpha(t) = -\frac{2}{k+2}\log c(t), \qquad \beta(t) = \frac{dc/dt}{c(t)(k+2)}, \quad \tau(t) = \int^t c(t')^{\frac{2}{k+2}}dt'.$$

Note 7.4 *It was already noted that to transform $-\mathrm{i}\widehat{H}_{nH}(t)$ into an autonomous system up to a time-reparametrisation via the group of $S(\mathfrak{W}_{nH}, \mathfrak{V}_{nH})$, the condition (7.8) is necessary. Despite that, the condition (7.7) is not mandatory and other alternative ones could be developed. This would lead to new integration methods for (7.1).*

8 Homogeneous potentials of degree minus two

Proposition 7.3 cannot be applied to t-dependent Hamiltonian operators of the form (7.1) with a homogeneous potential of degree minus two. These potentials include the relevant potential

$$U_{nH}(\widehat{x}_1, \ldots, \widehat{x}_n) := \sum_{j<k} \frac{g}{(\widehat{x}_j - \widehat{x}_k)^2}, \qquad g \in \mathbb{R}\backslash\{0\},$$

of a fluid in a t-dependent homogeneous trapping potential [24], which is also a type of Calogero–Moser potential, or the celebrated Smorodinsky–Winternitz potentials

$$U_{nH}(\widehat{x}_1, \ldots, \widehat{x}_n) = \sum_{j=1}^n \frac{g_j}{\widehat{x}_j^2}, \qquad g_j \in \mathbb{R}, \qquad \sum_{j=1}^n g_j^2 \neq 0.$$

Nevertheless, the procedure to prove Proposition 7.3 can be modified to deal with these pathological potentials with $k = -2$. This is the main aim of this section.

Lemma 7.1, Lemma 7.2, and the expression for $-\mathrm{i}\widehat{H}'_{nH}(t) = U(t)_\bigstar(-\mathrm{i}\widehat{H}_{nH}(t))$ given by (7.6) are still valid when the potential under study is homogeneous of degree minus two. Hence, to map $-\mathrm{i}\widehat{H}_{nH}(t)$ into a new t-dependent skew-Hermitian operator $-\mathrm{i}\widehat{H}'_{nH}(t) = -U(t)_\bigstar \mathrm{i}H_{nH}(t)$ taking values in a one-dimensional subspace of $\mathfrak{V}_{nH}$, it is mandatory to apply the condition

$$c(t)e^{\alpha(t)(k+2)/2} = l,$$

for a certain nonzero constant $l \in \mathbb{R}$. Since $k = -2$, the function $c(t)$ becomes a constant c. If we assume that $\alpha(t)$ and $\beta(t)$ are particular solutions to (7.7), the transformed t-dependent Hamiltonian operator (7.6) takes the form $\widehat{H}'_{nH}(t) = e^{-\alpha(t)}(\widehat{H}_1 + c\widehat{H}_4)$. Hence, the general solution to the t-dependent Schrödinger equation related to $\widehat{H}'_{nH}(t)$, whose explicit t-dependence can be removed after a t-reparametrisation, reads

$$\psi'(x, t) = \exp\left(\int_0^t e^{-\alpha(t')}dt'(\widehat{H}_1 + \widehat{H}_4)\right)\psi'(x),$$

where $\psi'(x)$ is an arbitrary element of $\mathcal{L}^2(\mathbb{R}^n)$ and $\psi'(x, 0) = \psi(x)$. Hence, the general solution to (7.1) for a homogeneous potential of degree minus two reads as (7.10) for some solutions $\alpha(t)$, $\beta(t)$ to equations to (7.7), (7.8) and (7.9). This result is summarised in the following proposition.

Proposition 8.1 *Every t-dependent Hamiltonian operator $\widehat{H}(t)$ of the form (7.1) with a homogeneous potential of order minus two and $c(t) = c \in \mathbb{R}$ has a general solution (7.10) where $\beta(t)$ is a particular solution to the Riccati differential equation $d\beta/dt = \beta^2 + \omega^2(t)$ and*

$$\alpha(t) = -2\int^t \beta(t')dt', \quad c_0 \in \mathbb{R}\backslash\{0\}, \quad \tau(t) = \int^t e^{-\alpha(t')}dt'.$$

Let us apply Proposition 8.1 to the t-dependent Hamiltonian

$$\widehat{H}_{nH}(t) = \frac{1}{2}\sum_{j=1}^{n}\left(\widehat{p}_j^2 + K(t)\widehat{x}_j^2\right) + \sum_{j<k}\frac{\lambda(\lambda-1)}{(\widehat{x}_j - \widehat{x}_k)^2},$$

whose potential is homogeneous of degree minus two. This t-dependent Hamiltonian operator was analysed in [24] to study a quantum one-dimensional fluid in a Paul trap [55]. Sutherland provided an Ansatz [24] to get a particular solution. Here, we recover some of his results from an algorithmic point of view and show that more useful solutions can be derived to study the properties of the quantum system.

Suppose that we take a t-dependent transformation of $\mathcal{G}_{\mathfrak{W}}$ with functions $\beta(t)$ and $\alpha(t)$ satisfying the differential equations

$$\frac{d\beta}{dt}(t) = \beta^2(t) + K(t), \qquad \frac{d\alpha}{dt}(t) + 2\beta(t) = 0. \tag{8.1}$$

In view of Proposition 8.1, the t-dependent Schrödinger equation related to $\widehat{H}_{nH}(t)$ reads

$$\psi(x,t) = \exp(-\beta(t)\mathrm{i}\widehat{H}_3)\exp(-\alpha(t)\mathrm{i}\widehat{H}_2)\exp\left(\int_0^t e^{-\alpha(t')}dt'(\widehat{H}_1 + \widehat{H}_4)\right)\psi'(x,0).$$

In particular, if we assume

$$\psi(x,0) = \prod_{j<k}(x_j - x_k)^{\lambda},$$

we obtain

$$(\widehat{H}_1 + \widehat{H}_4)\prod_{j<k}(x_j - x_k)^{\lambda} = 0.$$

and hence

$$\psi(x,t) = \exp(-\beta(t)\mathrm{i}\widehat{H}_3)\exp(-\alpha(t)\mathrm{i}\widehat{H}_2)\prod_{j<k}(x_j - x_k)^{\lambda}.$$

Since $\prod_{j<k}(x_j - x_k)^{\lambda}$ is a homogeneous function with degree $\lambda n(n-1)/2$ and in view of (7.3), it follows that

$$\mathrm{i}\widehat{H}_2\prod_{j<k}(x_j - x_k)^{\lambda} = \left(\frac{n}{4} + \frac{\lambda n(n-1)}{4}\right)\prod_{j<k}(x_j - x_k)^{\lambda},$$

and

$$\psi(x,t) = \exp(-\beta(t)\mathrm{i}\widehat{H}_3)\exp\left(-\alpha(t)n\frac{(1+\lambda(n-1))}{4}\right)\prod_{j<k}(x_j - x_k)^{\lambda}.$$

In view of the expression of $\widehat{H}_3$, it follows that

$$\psi(x,t) = \exp\left(-\mathrm{i}\,\beta(t)\frac{\sum_{j=1}^{n}x_j^2}{2}\right)\exp\left(-\alpha(t)n\frac{(1+\lambda(n-1))}{4}\right)\prod_{j<k}(x_j - x_k)^{\lambda}.$$

From equation (8.1), it turns out that $d\phi/dt(t) = \beta(t)\phi(t)$, with $\phi(t) := e^{-\alpha(t)/2}$. Hence,

$$\psi(x,t) = \exp\left(-\mathrm{i}\,\beta(t)\frac{\sum_{j=1}^{n}x_j^2}{2}\right)\phi(t)^{-n\frac{1+\lambda(n-1)}{2}}\prod_{j<k}(x_k - x_j)^{\lambda},$$

recovering, in this way, the particular solution given in [24]. Nevertheless, our approach is more general and our solution is not retrieved through a particular Ansatz as in previous literature.

9 Conclusions and relevance of achieved results

We have proposed the theory of quantum quasi-Lie schemes as a way to transform an initial t-dependent Schrödinger equation compatible with a quasi-Lie scheme into another one that is compatible with the same quasi-Lie scheme. In particular, the transformation has been accomplished so that the transformed t-dependent Schrodinger equation will be simpler or could be described by means of the usual theory of quantum Lie systems. Our new theory enables us to investigate a larger set of t-dependent Schrödinger equations than just those related to quantum Lie systems using methods generalising those for quantum Lie systems. We have also shown that the theory of quasi-Lie systems admits a generalisation to the quantum framework.

As a particular instance, we have applied in Sect. 7 our theory to solve a problem posed by Perelomov [23] in 1978 about the possibility of relating a quantum t-dependent Hamiltonian for a nonlinear oscillator to a quantum t-independent one of the same type. We have also obtained in Sect. 8 the solution of a quantum one-dimensional fluid in a Paul trap.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. J.F. Cariñena, J. Grabowski, G. Marmo, *Lie-Scheffers Systems: A Geometric Approach* (Bibliopolis, Naples, 2000)
2. J.F. Cariñena, J. Grabowski, G. Marmo, Superposition rules, Lie theorem and partial differential equations. Rep. Math. Phys. **60**, 237–258 (2007)
3. J.F. Cariñena, J. de Lucas, Lie systems: theory, generalizations and applications. Dissertationes Math. **479**, 1–162 (2011)
4. P. Winternitz, Lie groups and solutions of nonlinear differential equations, in: Nonlinear Phenomena, Vol 189, (Springer-Verlag, New York, 1983)
5. J. de Lucas, C. Sardón, *A Guide to Lie Systems with Compatible Geometric Structures* (World Scientific, Singapore, 2020)
6. S. Lafortune, P. Winternitz, Superposition formulas for pseudounitary matrix Riccati equations. J. Math. Phys. **37**, 1539–1550 (1996)
7. J.F. Cariñena, J. Clemente-Gallardo, J.A. Jover-Galtier, J. de Lucas, Application of Lie Systems to Quantum Mechanics: Superposition rules, in Proceedings of the „60 Years Alberto Ibort Fest Classical and Quantum Physics: Geometry, Dynamics and Control, (Springer 2019)
8. X. Gràcia, J. de Lucas, M.C. Muñoz-Lecanda, S. Vilariño, Multisymplectic structures and invariant tensors for Lie systems. J. Phys. A **52**, 215201 (2019)
9. A. González-López, N. Kamran, P.J. Olver, Lie algebras of vector fields in the real plane. Proc. Lond. Math. Soc. **64**, 339–368 (1990)
10. R. Hermann, M. Ackerman, *Sophus Lie's 1880 Transformation Group Paper* (Math. Sci. Press, Brookline, 1975)
11. F. Avram, J.F. Cariñena, J. de Lucas, A Lie system's approach for the first time-passage of piece-wise deterministic processes, in Modern trends in controlled stochastic processes: Theory Appl., (Luniver Press, Rome, 2010) pp. 144–160
12. R. Flores-Espinoza, J. de Lucas, Y.M. Vorobiev, Phase splitting for periodic Lie systems. J. Phys. A **43**, 205208 (2010)
13. J.F. Cariñena, G. Marmo, J. Nasarre, The nonlinear superposition principle and the Wei-Norman method. Int. J. Mod. Phys. A **13**, 3601–3627 (1988)
14. V..K. Chandrasekar, M. Senthilvelan, M. Lakshmanan, On the complete integrability and linearization of certain second-order nonlinear ordinary differential equations. Proc. R. Soc. Ser A Math. Phys. Eng. Sci **461**, 2451–2476 (2005)
15. S. Charzyński, M. Kuś, Wei-Norman equations for classical groups. J. Different. Equat. **259**, 1542–1559 (2015)
16. A. Ballesteros, A. Blasco, F.J. Herranz, J. de Lucas, C. Sardón, Lie-Hamilton systems on the plane: properties, classification and applications. J. Differ Equat. **258**, 2873–2907 (2015)
17. J.J. Levin, On the matrix Riccati equation. Proc. Amer. Math. Soc. **10**, 519–524 (1959)
18. M. Sorine, P. Winternitz, Superposition rules for solution of differential matrix equations arising in Control Theory. IEEE Trans. Automat. Control **30**, 266–272 (1985)
19. O. Esen, E. Fernández-Saiz, C. Sardón, M. Zając, A generalization of a SIS epidemic model with fluctuations. Math. Med. Appl. Sci. **45**, 3718–3731 (2021). https://doi.org/10.1002/mma.8013
20. J.F. Cariñena, J. de Lucas, Quasi-Lie families, schemes, invariants and their applications to Abel equations. J. Math. Anal. App. **430**, 648–671 (2015)
21. J.F. Cariñena, J. Clemente-Gallardo, A. Ramos, Motion of Lie groups and its application in control theory. Rep. Math. Phys. **51**, 159–170 (2003)
22. J..F. Cariñena, A. Ramos, Applications of Lie systems in quantum mechanics and control theory, in *Classical and Quantum Integrability*. Vol. 59 (Banach Center Publications, 2003) pp. 143–162
23. A. Perelomov, The simple relations between certain dynamical systems. Comm. Math. Phys. **63**, 9–11 (1978)
24. B. Sutherland, Exact coherent states of a one-dimensional quantum fluid in a t-dependent trapping potential. Phys. Rev. Lett. **80**, 3678–3681 (1997)
25. J. Grabowski, M. Kuś, G. Marmo, Geometry of quantum systems: density states and entanglement. J. Phys. A **38**, 10217–10244 (2005)
26. B.C. Hall, *Quantum Theory for Mathematicians, Graduate Texts in Mathematics 267* (Springer, New York, 2013)
27. S. Lang, *Introduction to Differentiable Manifolds* (Interscience publishers, New York, 1962)
28. J.M. Lee, *Introduction to Smooth Manifolds, Graduate Texts in Mathematics 218* (Springer, New York, 2013)
29. D. Beltiţă, T. Goliński, A.B. Tumpach, Queer Poisson brackets. J. Geom. Phys. **132**, 358–362 (2018)

30. A. Kriegl, P. Michor, *The Convenient Setting for Global Analysis* (AMS, Providence, 1997)
31. J.M. Lee, *Introduction to Topological Manifolds, Graduate Texts in Mathematics 202* (Springer, New York, 2011)
32. M. Schottenloher, The unitary group in its strong topology, arXiv:1309.5891
33. B.C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction, Graduate Texts in Mathematics 222* (Springer, Cham, 2015)
34. M.H. Stone, On one-parameter unitary groups in Hilbert Space. Ann. of Math. **33**, 643–648 (1932)
35. M..H. Stone, Linear transformations in Hilbert space III. Operational methods and Group Theory. Proceedings of National Academy of Sciences of the United States of America. Nat. Acad. Sci. **16**, 172–175 (1932)
36. L.J. Boya, J.F. Cariñena, J.M. Gracia-Bondía, Symplectic structure of the Aharonov-Anandan geometric phase. Phys. Lett. A **161**, 30–34 (1991)
37. B. Hamprecht, Exact solutions of the t-dependent Schrödinger equation in one space dimension, ArXiv:quant-ph/0211040
38. P. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, New-York, 1993)
39. M. Flato, J. Simon, H. Stellman, D. Sternheimer, Simple factors about analytic vectors and integrability. Ann. Sci. École Norm. Sup. **5**, 423–434 (1972)
40. E. Nelson, Analytic vectors. Ann. of Math. **70**, 572–615 (1959)
41. J. Simons, On the integrability of representations of finite-dimensional real Lie algebras. Comm. Math. Phys. **28**, 39–46 (1972)
42. J.F. Cariñena, A. Ramos, *Lie systems and connections in fibre bundles: Applications in quantum mechanics,* 9th Int (Matfyzpress, Prague, Conf. Differential Geometry and Applications, 2005), pp.437–452
43. H. Bateman, On dissipative systems and related variational principles. Phys. Rev. **38**, 815–819 (1931)
44. P. Caldirola, Forze non conservative nella meccanica quantistica. Nuovo Cim. **18**, 393–400 (1941)
45. A.N. Ikot, L.E. Akpabio, I.O. Akpan, M.I. Umo, E.E. Ituen, Quantum damped mechanical oscillator. Int. J. Optics **2010**, 275910 (2010)
46. E. Kanai, On the quantization of dissipative systems. Prog. Theor. Phys. **3**, 440–442 (1948)
47. D. Gómez-Vergel, E.J.S. Villaseñor, The time-dependent quantum harmonic oscillator revisited: Applications to Quantum Field Theory. Ann. Phys. **324**, 1360–1385 (2009)
48. R. Abraham, J.E. Marsden, *Foundations of Mechanics, Benjamin-Cummings* (Pub. Co, Reading, Ma, 1978)
49. A.S. Schwarz, *Topology for physicists, Fundamental Principles of Mathematical Sciences 308* (Springer-Verlag, Berlin, 1994)
50. L. Landau, E. Lifshitz, *Mecanique* (Mir, Moscow, 1965)
51. L. Skala, J. Cizek, V. Kapsa, E.J. Weniger, Large-order analysis of the convergent renormalized strong-coupling perturbation theory for the quartic anharmonic oscillator. Phys. Rev. A **56**, 4471 (1997)
52. L. Skala, J. Cizek, E.J. Weniger, J. Zarnastil, Large-order behavior of the convergent perturbation theory for anharmonic oscillators. Phys. Rev. A **59**, 102–106 (1999)
53. J.C. Burkill, *A First Course in Mathematical Analysis* (Cambridge University Press, New York, 1978)
54. J.F. Cariñena, J. Grabowski, J. de Lucas, Superposition rules for higher order systems and their applications. J. Phys. A **45**, 185202 (2012)
55. W. Paul, Electromagnetic traps for charged and neutral particles. Proceedings of the International School of physics "Enrico Fermi". Course CXVIII, Laser manipulation of atoms and ions. (North Holland, New York, 1992), PP.497–517