



Dilaton black hole entropy from entropy function formalism

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Abstract It has been shown that the entropy function formalism is an efficient way to calculate the entropy of black holes in string theory. We check this formalism for the extremal charged dilaton black hole. We find the general four-derivative correction on the black hole entropy from the value of the entropy function at its extremum point.

1 Introduction

Black hole thermodynamics is a fascinating topic in theoretical physics. From the theoretical point of view, black hole thermodynamics provides an intriguing arena for quantum gravity researches. It is expected that a theory of quantum gravity must have an interpretation for the thermodynamic behavior of black holes observed in classical general relativity. So far, several approaches to quantum gravity and semi-classical gravity have discussed this issue [1]. One of them is the Wald's Noether charge approach that is applicable to general diffeomorphism invariant theories. A quantum field theory calculation of the Wald entropy formula has been presented in [2] for general diffeomorphism invariant theories. It shows how this general black hole entropy formula appears from a fundamental theory of quantum gravity. By explicitly comparing the direct counting of microstates with the Noether charge entropy, the Wald's approach has been confirmed in many examples in the string theory [3,4].

It is known that the microscopic description of the black hole entropy needs the existence of an attractor. The attractor mechanism states that, the radial dependence of the moduli fields given by the equations, whose solutions lead to definite values at the horizon, regardless of their boundary values at infinity. Motivated by this mechanism, Sen has defined the black hole entropy function in higher derivative gravity, in which the Wald formula can be written in terms of this function. Sen's proposal included a particular kind of extremal

black holes with the near horizon geometry $AdS_2 \times S^{D-2}$ [8]. The entropy of such black holes is given by the value of the entropy function at the extremum. Sen has found this function by integrating the Lagrangian density over the horizon and then carried out the Legendre transform of the result with respect to some parameters.

In [9–28], the thermodynamics of some extremal black holes have been investigated by this mechanism. It has been shown that the entropy function formalism works correctly both in ten and lower dimensions. The higher derivative corrections to entropy and also corrections to background at near horizon has been studied. Several related works are given in [29–33]. The higher derivative terms may modify the solution such that the near horizon is not $AdS_2 \times S^{D-2}$. In such cases, the entropy function formalism is not applicable [34].

In this paper we would like to show that in the context of $N = 4$ supergravity and for extremal dyonic dilaton black holes, the entropy function formalism is applicable. We apply this method to find the higher derivative correction to entropy coming from general four-derivative terms including the metric curvature and the gauge field strength, added to usual supergravity action.

An outline of the paper is as follows. In Sect. 2, we review the dyonic dilaton black holes as a special solution of a dimensionally reduced superstring theory. In Sect. 3, we compute the entropy of the extremal dyonic dilaton black holes using the entropy function formalism and we show that this formalism works. We implement this formalism in Sect. 4 to find the higher derivative correction to entropy and explicitly only check the Gauss–Bonnet contribution of these correction terms.

2 Dilaton black holes

Understanding the nature of singularities in gravitational theory and the quantum thermal properties of processes near black holes may be possible with the help of investigation of

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evaporation of black holes. Dilaton black holes could be the stable endpoints of the evaporation process.

It is shown that $N = 4$ supergravity can be consequence of a dimensionally reduced superstring theory in $d = 4$. A solution of this reduction can be the spherically symmetric electrically and magnetically charged dilaton black hole that includes the classical Schwarzschild and Reissner-Nordstrom black holes and dilaton black holes with either purely electric, purely magnetic, or both charges considering in the solution [5]. The bosonic part of the action that can be described the above theory is given by

$$I = \int d^4x \sqrt{-g} \left(-R + 2\partial^\mu \phi \cdot \partial_\mu \phi - e^{-2\phi} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \right), \quad (1)$$

where

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= F_{\mu\nu} + G_{\mu\nu}, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu. \end{aligned}$$

The dilaton ϕ is the real part of a complex scalar in which its imaginary part is the axion putting to constant. The equations of motion and their solutions have been discussed in [5,6]. The field strengths $F_{\mu\nu}$ and $G_{\mu\nu}$ satisfy the axion field equation provided that one considers each vector field A_μ and B_μ has to be either electric or magnetic.

The dilaton black holes with electric or magnetic charges have been investigated in [7]. In the electric and magnetic dilaton black hole solution, A_μ and B_μ are purely electric and magnetic, respectively. Asymptotically non-vanishing dilaton field ϕ_0 and the dilaton charge Σ are defined by the equation $\phi \sim \phi_0 + \Sigma/r$ at $r \rightarrow \infty$. The dilaton charge can be calculated in terms of black hole mass, electric and magnetic charges as $\Sigma = (P^2 - Q^2)/2M$ in which $Q = e^{-\phi_0} Q_{elec}$ and $P = e^{-\phi_0} P_{magn}$.

Consider the extremal solution of the equations of motion of the action (1) that can be given in the following form[5]

$$ds^2 = \frac{(r - M)^2}{r^2 - \Sigma^2} dt^2 - \frac{r^2 - \Sigma^2}{(r - M)^2} dr^2 - (r^2 - \Sigma^2) d\Omega_2^2. \quad (2)$$

This solution has a duality symmetry that exchanges the electric charge and magnetic charge, the dilaton charge and negative sign of dilaton charge, simultaneously. For the above extremal dilaton black hole, the following condition is valid between the independent parameters

$$M^2 + \Sigma^2 = P^2 + Q^2. \quad (3)$$

Kallosh et al. [5] have shown that the bound deriving from supersymmetry is exactly the lower bound on the dilaton black hole mass imposed by cosmic censorship (3). Other solutions can be derived if both F and G are considered to

have electric and magnetic charges. In this set of solutions, there are two electric and two magnetic parameters corresponding to the fields F and G . Considering the axion field equation with a constant axion, this set of solutions would depend on five parameters. In this study, we are interested in the solution of the purely electric extremal dilaton black hole.

3 Entropy of dilaton black hole

The entropy function formalism is well known as an applicable method for deriving the entropy of black holes with near horizon geometry $AdS_2 \times S^{D-2}$. Let us review this method here: consider an extremal black hole with the near horizon geometry $AdS_2 \times S^{D-2}$ in the space-time dimension D in which the AdS_2 part is proportional to $-r^2 dt^2 + dr^2/r^2$. The background of the black hole includes different scalar, electric and magnetic fields u_s, e_i, p_a , respectively. These fields with v_1 and v_2 (as the size of AdS_2 and S^{D-2} , respectively) characterize the background. Define an entropy function by integrating the Lagrangian density over the horizon S^{D-2} , then carry out the Legendre transform of the result with respect to e_i . Extremizing this function with respect to the scalar and size parameters would result the values of these parameters. Eventually, the value of the result function at the horizon will be corresponded to the entropy [8].

In this section, we calculate the entropy of extremal dilaton black hole with two electric charges using the entropy function formalism. Since the near horizon solution of metric equation of motion (2) has the geometry of $AdS_2 \times S^2$, the entropy function formalism could be applied. To implement this formalism, the near horizon solution (2) was written in terms of parameters v_1 and v_2 as

$$\begin{aligned} ds^2 &= v_1 \left(-\frac{r^2}{M^2 - \Sigma^2} dt^2 + \frac{M^2 - \Sigma^2}{r^2} dr^2 \right) \\ &\quad + v_2 (M^2 - \Sigma^2) d\Omega_2^2, \\ F &= \frac{Q e^{\phi_0}}{(M - \Sigma)^2} dt \wedge dr, \\ G &= P e^{\phi_0} \sin\theta d\theta \wedge d\phi, \quad e^{2\phi} = e^{2\phi_0} \frac{M + \Sigma}{M - \Sigma}. \end{aligned} \quad (4)$$

One can assume the following values for form fields and the constant value of the scalar near the horizon

$$F_{rt} = e_1, \quad \tilde{G}_{rt} = e_2, \quad e^{-2\phi} = S, \quad (5)$$

where we use the duality rotation¹

$$\tilde{G}^{\mu\nu} = \frac{1}{2} i (-g)^{-\frac{1}{2}} e^{-2\phi} \epsilon^{\mu\nu\alpha\beta} G_{\alpha\beta}.$$

¹ This implies that \tilde{B}_μ is also electric, and the calculations are often simpler when using the electric solution \tilde{B}_μ , rather than the magnetic B_μ .

For this background, the non-vanishing components of the Riemann tensor are

$$\begin{aligned} R_{abcd} &= \frac{1}{v_1(M^2 - \Sigma^2)}(g_{ac}g_{bd} - g_{ad}g_{bc}), \quad a, b, c, d = r, t, \\ R_{ijkl} &= \frac{-1}{v_2(M^2 - \Sigma^2)}(g_{ik}g_{jl} - g_{il}g_{jk}), \quad i, j, k, l = \theta, \phi \end{aligned} \quad (6)$$

and the ricci scalar is $R = 2(v_2 - v_1)/(v_1 v_2 (M^2 - \Sigma^2))$.

The vanishing of covariant derivative of the fields (gauge field strengths, Rieman tensor and scalar field) in the near horizon geometry is a main property of the general form of the background used in the entropy function formalism.

The integral of Lagrangian density over the horizon, f , can be derived as follows:

$$\begin{aligned} f &= \frac{1}{16\pi} \int d\theta d\phi \sqrt{-g} \mathcal{L} \\ &= \frac{v_1 - v_2}{2} + \frac{v_2}{v_1} \frac{M^2 - \Sigma^2}{2} \left(S e_1^2 + S^{-1} e_2^2 \right). \end{aligned} \quad (7)$$

Extremizing the above function with respect to S and v results the dilaton and metric equations of motion at the limit of near horizon, respectively. The electric charges are given by the gauge field equations of motion $q_i = \partial f / \partial e_i$. By rescaling Riemann tensor as $R_{rtrt} \rightarrow \lambda R_{rtrt}$, the entropy can be derived as

$$S_{BH} = 2\pi(M^2 - \Sigma^2) \frac{\partial f_\lambda}{\partial \lambda} \Big|_{\lambda=1}, \quad (8)$$

where f_λ is a function similar to f but whit R_{rtrt} rescaling. The variation of Lagrangian density with respect to Riemann tensors results the following constraint equation:

$$\int d\theta d\phi \sqrt{-g} R_{abcd} \frac{\partial L}{\partial R_{abcd}} = 2 \frac{\partial f}{\partial \lambda} \Big|_{\lambda=1}, \quad (9)$$

which is exactly equal to $f - e_1 \partial f / \partial e_1 - e_2 \partial f / \partial e_2$. Therefore, the entropy function F , that is defined as the Legendre transform of f with respect to electric fields, results as:

$$\begin{aligned} F &= e_i \frac{\partial f}{\partial e_i} - f = -\frac{\partial f}{\partial \lambda} \Big|_{\lambda=1} \\ &= \frac{v_2 - v_1}{2} + \frac{v_1}{2v_2} \frac{1}{M^2 - \Sigma^2} \left(S^{-1} q_1^2 + S q_2^2 \right), \end{aligned} \quad (10)$$

where we have used the gauge field equations of motion. Solving the equation of motion for metric and scalar, results the parameter values as follow:

$$v_1 = v_2 = 1, \quad S = \frac{q_1}{q_2}. \quad (11)$$

Implementing the above values in entropy Eq. (8), would result the entropy in terms of entropy function as:

$$S_{BH} = -2\pi(M^2 - \Sigma^2)F = \pi(M^2 - \Sigma^2). \quad (12)$$

It is straightforward to find the above entropy by directly computing the area of the horizon. So, we have shown that the entropy function formalism works here.

4 Higher derivative correction

The area-entropy law can not be simply used to find the entropy of black holes when the higher derivative correction terms have been taken into account. It is common to organize the interactions by their dimension or alternatively by the number of derivatives [35]. The Lagrangian (1) contains covariant terms up to two derivatives. Therefore, the possible interactions at fourth order in derivatives need to be considered. The general four-derivative Lagrangian that the extremal black hole with near horizon geometry $AdS_2 \times S^2$ saturates the equation of motion, is as follow:

$$\begin{aligned} \Delta \mathcal{L} = \frac{S}{16\pi G} &\left[\alpha_1 \left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \right) \right. \\ &+ \alpha_2 \left(RF^2 - R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \right) \\ &+ \alpha_3 \left(R\tilde{G}^2 - R_{\mu\nu\alpha\beta}\tilde{G}^{\mu\nu}\tilde{G}^{\alpha\beta} \right) + \alpha_4 \left((F^2)^2 - 2F^4 \right) \\ &+ \alpha_5 \left((\tilde{G}^2)^2 - 2(\tilde{G})^4 \right) \\ &+ \alpha_6 \left(R^{\mu\nu}F_{\mu\alpha}F^\alpha_\nu + \frac{1}{2}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \right) \\ &\left. + \alpha_7 \left(R^{\mu\nu}\tilde{G}_{\mu\alpha}\tilde{G}^\alpha_\nu + \frac{1}{2}R_{\mu\nu\alpha\beta}\tilde{G}^{\mu\nu}\tilde{G}^{\alpha\beta} \right) \right], \end{aligned} \quad (13)$$

where we consider the near horizon (4) as the solution of equations of motion for (1)+(13).² In above equation, $F^2 = F_{\mu\nu}F^{\mu\nu}$, $F^4 = F_\mu^\nu F_\nu^\alpha F_\alpha^\beta F_\beta^\mu$ (similarly for \tilde{G}^2 and \tilde{G}^4) and α_i are some unspecified coupling constants.³

This would result a change in f and F which consequently change the relation between q_i and e_i . This is in contrast to the situation in [9] where the function f corresponding to the correction term is independent of the parameters v_1 and v_2 and

² There may be some other solutions that have the same near horizon geometry as (2), but considering the Sen entropy function method, it is sufficient to have a near horizon geometry as $AdS^2 \times S^2$. In fact, we deal with a more general class of black hole solutions that have the same near horizon structure. We would like to thank the referee for his/her comment for pointing this issue.

³ In a string theory context, it might be expected all of these interaction terms to emerge in the low-energy effective Lagrangian as string-loop or α' corrections to the two-derivative supergravity Lagrangian [35]. In such a context, these terms would appear as a perturbative expansion where the higher order terms is suppressed by powers of, e.g., the ratio of the string scale over the curvature scale. It has been demonstrated in [35] that within a perturbative framework, one can use field redefinitions to reduce the most general four derivative action to include lower interaction terms than appearing in (13).

therefore the entropy contribution from the correction terms can be calculated separately. But in this case the entropy function F corresponding to the lagrangian $\mathcal{L} + \delta\mathcal{L}$ needs to be found first followed by the corresponding entropy. So, the contribution of the correction terms to the entropy could be easily determined. Therefore, the entropy function formalism can be applied for the black hole solutions obtained from the higher derivative lagrangian (13).

Based on the four-derivative Lagrangian (13), the contribution of the higher derivative terms to the function f would be:

$$\Delta f = -\frac{S}{v_1}(\alpha_2 e_1^2 + \alpha_3 e_2^2) - S \frac{v_2}{v_1^2}(\alpha_6 e_1^2 + \alpha_7 e_2^2) + 2 \frac{S\alpha_1}{M^2 - \Sigma^2}. \quad (14)$$

The electric charges carried by the black hole can be derived as follow:

$$\begin{aligned} q_1 &= \frac{\partial(f + \Delta f)}{\partial e_1} \\ &= -2Se_1 \left(\frac{v_2}{v_1^2} \alpha_6 + \frac{\alpha_2}{v_1} \right) + (M^2 - \Sigma^2)Se_1 \frac{v_2}{v_1}, \\ q_2 &= \frac{\partial(f + \Delta f)}{\partial e_2} \\ &= -2Se_2 \left(\frac{v_2}{v_1^2} \alpha_7 + \frac{\alpha_3}{v_1} \right) + (M^2 - \Sigma^2) \frac{e_2}{v_1}. \end{aligned} \quad (15)$$

Based on entropy function formalism, the entropy function \mathcal{F} , as the legendre transform of $f + \Delta f$ with respect to the electric fields e_1 and e_2 , can be calculated as follow:

$$\begin{aligned} \mathcal{F} &= (M^2 - \Sigma^2) \left(\frac{v_2}{2v_1} (Se_1^2 + S^{-1}e_2^2) - \frac{1}{2}(v_1 - v_2) \right. \\ &\quad \left. - 2S\alpha_1(M^2 - \Sigma^2)^{-1} - \frac{S}{v_1} (\alpha_2 e_1^2 + \alpha_3 e_2^2 + \frac{v_2}{v_1} (\alpha_6 e_1^2 + \alpha_7 e_2^2)) \right). \end{aligned} \quad (16)$$

The equations of motion could be found from the variation of \mathcal{F} with respect to v_1 , v_2 and S .

A symbolic computer algebra system was used to derive the solution of equations of motion as follow:

$$\begin{aligned} S &= q_1 \left(\alpha_2^2 \alpha_3^2 + q_2^2 \right)^{-\frac{1}{2}} (1 + \alpha_1^{-2})^{\frac{1}{2}} \\ &\quad + \frac{1}{24} (q_1^3 \alpha_2^4 + q_2^3 \alpha_3^4) \\ &\quad + \frac{1}{6} (\alpha_6^2 + \alpha_7^2 + \alpha_6^2 \alpha_7^2) (q_1^{\frac{1}{2}} - q_2^{\frac{1}{2}}), \\ v_2 &= (M^2 - \Sigma^2)^{-1} \left([q_2^2 + (4\alpha_2 \alpha_3 + 1)^2]^{\frac{1}{2}} + \left[\frac{q_1^2 q_2^4}{q_2^2 + 4} \right]^{\frac{1}{2}} \right), \\ v_1 &= \left(4q_1^{\frac{1}{2}} q_2^{\frac{3}{2}} \alpha_6 \alpha_7 [(v_2^2 - 1)^{-1} - 1] - 2v_2 [\alpha_2 (q_1^3 + 2q_1) \right. \end{aligned}$$

$$\begin{aligned} &\quad \left. + \alpha_3 (q_2^3 + 2q_2) - 1] \right)^{\frac{1}{2}} \\ &\quad \times \left(8q_1^3 q_2^3 (\alpha_6^2 + \alpha_7^2) (v_2^2 - 1)^{-1} + 2v_2^{-1} \right)^{-\frac{1}{2}}, \end{aligned} \quad (17)$$

where we use the relations in (15).

By considering the coupling constants α_i as dimensionless coefficients with real finite values, one can find that solving the above system of equations would result the following identities:

$$\begin{aligned} q_1^{\frac{5}{2}} \alpha_6^{\frac{1}{2}} + q_2^{\frac{5}{2}} \alpha_7^{\frac{1}{2}} &= 0, \\ \left(q_1^{\frac{1}{2}} + q_2^{-\frac{1}{2}} \right) \left(\alpha_6^2 + \alpha_7^2 + \frac{4}{3} \alpha_6 \alpha_7 \right) &= 0. \end{aligned} \quad (18)$$

By implementing S , v_2 and v_1 values in entropy function \mathcal{F} , the entropy can be derived as follow:

$$\begin{aligned} S_{BH} &= \frac{\pi}{12} \left((1 + 4\alpha_2 \alpha_3)^2 + 4q_2^{-2} \right) \\ &\quad \times \left(24q_1 q_2 + 8(\alpha_6^2 + \alpha_7^2 + 4\alpha_6 \alpha_7) \left(q_1^{\frac{1}{2}} - q_2^{\frac{1}{2}} \right) + \alpha_3^4 q_2^3 \right). \end{aligned} \quad (19)$$

The Gauss–Bonnet correction is the most popular candidate to imitate string corrections to the Einstein action. This correction can be deduced by only considering the first term of (13) which is the coefficient of α_1 .⁴ We calculate the solution of equations of motion and the entropy corresponding to the Gauss–Bonnet correction independently and find it as following:

$$\begin{aligned} v_1 = v_2 &= \frac{\sqrt{1 + q_2^2} + \sqrt{\frac{q_1^2 q_2^4}{q_2^2 + 4}}}{M^2 - \Sigma^2}, \\ S &= \frac{q_1}{q_2} \sqrt{1 + \alpha_1^{-2}}, \quad S_{BH}^{(GB)} = 2\pi q_1 q_2 \left(1 + 4q_2^{-2} \right)^{\frac{1}{2}}. \end{aligned} \quad (20)$$

By setting all coefficients equal to zero apart from α_1 , one can find the above solution of equations of motion and corresponding entropy from (17) and (19), respectively.

In this study, the entropy of extremal electrically charged dilaton black hole was calculated. The entropy at the presence of a general four derivative Lagrangian was derived from the entropy function formalism. It would be interesting to find these results by counting the number of degeneracy of the microstates. It is also interesting to do the same calculations for the non-extremal charge dilaton black hole solution

⁴ Considering the term $-4R_{\mu\nu}R^{\mu\nu}$ as the only non-zero term of (13), yields the heterotic correction that deduced from the heterotic-type I duality.

and find the correction to the tree-level solution by using the entropy function formalism.⁵

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⁵ In applying the non-extremal result to extremal case, one must define the entropy of an extremal black hole to be the limit of the entropy of the associated non-extremal black hole in which the non-extremal parameter goes to zero [36].