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Cosmology in modified $f(R,T)$-gravity

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Abstract In the present paper we propose a further modification of $f(R,T)$-gravity (where T is trace of the energy-momentum tensor) by introducing higher derivatives matter fields. We discuss stability conditions in the proposed theory and find restrictions for the parameters to prevent appearance of main type of instabilities, such as ghost-like and tachyon-like instabilities. We derive cosmological equations for a few representations of the theory and discuss main differences with conventional $f(R,T)$-gravity without higher derivatives. It is demonstrated that in the theory presented inflationary scenarios appear quite naturally even in the dust-filled Universe without any additional matter sources. Finally, we construct an inflationary model in one of the simplest representation of the theory, calculate the main inflationary parameters and find that it may be in quite good agreement with observations.

1 Introduction

According to current knowledge, based on experimental data, there were (at least) two different epochs of dynamical evolution of our Universe when the key role was played dark energy (DE): an inflationary stage at the early times of evolution and a late time acceleration (l.t.a.) stage, which started recently (on cosmological scales) and continues till modern time. We know about the existence of modern DE (associated with l.t.a.) with high precision from the different experiments, the first of which relate to SNI data [1,2], whereas about primordial DE (associated with inflation) we know only by indirect detection such as general isotropy and flatness of observable part of Universe and the non-flatness spectrum of primordial scalar perturbations [3,4]. Nevertheless the true nature of both DEs is unknown yet and this fact stimulates researchers to find solutions of the DE problem outside of standard physics.

Modifications of the gravitational sector are well known from early times and still are very popular, because different corrections to the gravitational action follow for instance from string theory [5,6] and one-loop quantum effects [7–9] (see also [10,11] for cosmological applications). The number of different approaches in this way is actually huge and we only mention here examples such as $f(R)$-gravity [12–14], Horndeski theory [15], unimodular gravity [16], teleparallel gravity [17], theories with non-minimal kinetic coupling [18]; see also [19].

Nevertheless there is another possibility to solve DE problem: we can introduce some exotic matter or modify the right hand side (matter sector) of the equations. The activity in this direction is not so intensive, but we can mention such attempts as phenomenological higher derivative matter fields [20,21], bulk viscosity and imperfect fluids [22–24], theories with non-minimally coupled Ricci scalar with matter lagrangian [25,26] and one of the most popular subclasses of this model, $f(R,T)$-gravity [27], where T is the trace of the energy-momentum tensor (stress-energy tensor). Note that the dependence on T may be induced by exotic imperfect fluids or quantum effects (such as the conformal anomaly). Also we can study such kinds of models as some phenomenological models, which arise from some more general theories. Indeed it is well known that brane models can modify exactly the r.h.s. of the equations of motions on the brane [28–31]. For these reasons in our paper we try to discuss a wider class of $f(R,T)$-gravity models and incorporate a function dependence by the derivatives of T (models containing $\Box R$-terms also are known as possible modifications of $f(R)$-gravity [32,33]).

This paper is organized as follows: in Sect. 2 we derive general equations and discuss stability conditions; in Sect. 3 we study a few concrete examples of functions and find some cosmological solutions; in Sect. 4 we estimate inflationary parameters for one of the simplest shapes of the function; and in Sect. 5 we give some concluding remarks.

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2 General equations and stability conditions

Let us try to generalize the well-known modified gravity theory [27] in the following way:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}F(R,\ T,\Box T) + \epsilon\int d^4x\sqrt{-g}L_m, \tag{1}$$

where R is the Ricci scalar and T is the trace of the energy-momentum tensor; ϵ is equal to 1 or to 0. First of all let us ensure that this theory is ghost-free. For this task let us introduce Lagrange multipliers in the following way[1]:

$$\begin{aligned} S = \int d^4x\sqrt{-g}F(R,\ T,\Box T) = &\int d^4x\sqrt{-g}[F(\lambda_1,\lambda_2,\lambda_3) \\ &+\mu_1(R-\lambda_1)+\mu_2(T-\lambda_2)+\mu_3(\Box T-\lambda_3)], \end{aligned} \tag{2}$$

variation with respect to μ_i give us

$$\lambda_1 = R, \quad \lambda_2 = T, \quad \lambda_3 = \Box T, \tag{3}$$

and variation with respect to λ_i

$$\mu_1 = F_{\lambda_1}, \quad \mu_2 = F_{\lambda_2}, \quad \mu_3 = F_{\lambda_3}; \tag{4}$$

thus the initial action (1) may be rewritten in the form

$$\begin{aligned} S = \int d^4x\sqrt{-g}[&\mu_1 R + \mu_3\Box\lambda_2 \\ &+\{F(\lambda_1,\lambda_2,\lambda_3)-\mu_1\lambda_1-\mu_3\lambda_3\}]. \end{aligned} \tag{5}$$

We can see that the field μ_2 is non-physical and as a consequence decouples from the equation. Now let us focus on the second term. It may be reorganized by introducing new fields, $\lambda_2 = \chi_2 + \psi_2$ and $\mu_3 = \chi_2 - \psi_2$:

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\mu_3\Box\lambda_2 \\ &= \int d^4x\sqrt{-g}[\chi_2\Box\chi_2 - \psi_2\Box\chi_2 + \chi_2\Box\psi_2 - \psi_2\Box\psi_2] \\ &= \int d^4x\sqrt{-g}\left[-\nabla^i\chi_2\nabla_i\chi_2 + \nabla^i\psi_2\nabla_i\psi_2\right], \end{aligned} \tag{6}$$

where we used integration by parts. Equation (6) tells us that independently of the type of the function $F(R,\ T,\Box T)$ this theory contains three scalar fields (one additionally will appear after $\mu_1 R$ decoupling) and at least one from it is ghost-like. But there is one special case,[2] which allows us to solve this problem: if we put

$$F(R,\ T,\Box T) = f(R,\ T) + h(T)\Box T. \tag{7}$$

Note that $h(T) \neq \text{const}$ because in this case the contribution to the equations will be trivial. In this case the theory will contain only two physical scalar fields and both of them may be non-ghost, depending on the signs of h' and f_R. Indeed, if we start from lagrangian (7) and introduce auxiliary fields as $\lambda = R,\ \mu = f_\lambda$ we gain the next action

$$S = \int d^4x\sqrt{-g}[\mu R + V(\mu,\lambda,T) - h' g^{ik}\nabla_i T\nabla_k T], \tag{8}$$

where the potential $V = f(\lambda, T) - \mu\lambda$ and the last term from (7) was integrated by parts. Further, producing a conformal transformation of the metric $\bar{g}_{ik} = e^\chi g_{ik}$, $\chi = \ln\mu$ we have the action in canonical form,

$$\begin{aligned} S = \int d^4x\sqrt{-\bar{g}}\Big[&\bar{R} - \frac{3}{2}\bar{g}^{ik}\bar{\nabla}_i\chi\bar{\nabla}_k\chi + e^{-2\chi}V(\mu,\lambda,T) \\ &-h'e^{-\chi}\bar{g}^{ik}\bar{\nabla}_i T\bar{\nabla}_k T\Big]. \end{aligned} \tag{9}$$

We can see that the last kinetic term contains a multiplier h'/f_R; thus we need $f_R > 0$ and $h' > 0$ for the ghost-free theory.

Now varying lagrangian (7) with respect to T we find the field equation

$$\begin{aligned} &h'\Box T + \Box h + f_T(R,T) \\ &\quad = 2h'\Box T + h''(\nabla^i T)(\nabla_i T) + f_T(R,T) = 0, \end{aligned} \tag{10}$$

and finally varying (7) with respect to the metric we have

$$\begin{aligned} &f_R R_{ik} - \frac{1}{2}Fg_{ik} + (g_{ik}\Box - \nabla_i\nabla_k)f_R \\ &\quad = 8\pi\epsilon T_{ik} - (f_T + h'\Box T + \Box h)(T_{ik} + \Theta_{ik}) + h'\nabla_i T\nabla_k T \\ &\quad\quad - \frac{1}{2}h'\nabla_m T\nabla^m T g_{ik}, \end{aligned} \tag{11}$$

where

$$\Theta_{ik} \equiv g^{lm}\frac{\delta T_{lm}}{\delta g^{ik}}. \tag{12}$$

We can see that if take into account the field equation (10), the Einstein-like equation (11) has an essential simplification

$$\begin{aligned} &f_R R_{ik} - \frac{1}{2}Fg_{ik} + (g_{ik}\Box - \nabla_i\nabla_k)f_R \\ &\quad = 8\pi\epsilon T_{ik} + h'\nabla_i T\nabla_k T - \frac{1}{2}h'\nabla_m T\nabla^m T g_{ik}. \end{aligned} \tag{13}$$

Let us consider the solution of Eq. (10) in the form $T = T_0 + \delta T$ and for the trivial solution $R_0 = 0$, $T_0 = 0$ (with the flat background) we find the additional restriction $f_{TT} \geqslant 0$ for the absence of tachyon-like effective particles in the theory (the case $f_{TT} = 0$ cannot be totally excluded). For more complicate cases of a non-flat background this relation will have a more complicate structure and will contain h'' as well, but it is clear that the theory may be free from the tachyon instability.

[1] In this section we exclude L_m from our discussion and concentrate our attention on the function F.

[2] One may note that the more general case of infinite rows like $\sum c_i\Box^i T$ may produce a ghost-free theory, as has happened in the pure $f(R)$-gravity case [34].

Let us take the divergence of Eq. (11). The divergence of the l.h.s. reads

$$\nabla^i\left[f_R R_{ik}-\frac{1}{2}F g_{ik}+(g_{ik}\Box-\nabla_i\nabla_k)f_R\right]=(\nabla^i f_R)R_{ik}$$
$$+f_R\nabla^i R_{ik}-\frac{1}{2}f_R\nabla_k R-\frac{1}{2}f_T\nabla_k T+(\nabla_k\Box-\Box\nabla_k)f_R$$
$$-\frac{1}{2}\nabla_k(h\Box T)=(\nabla^i f_R)R_{ik}$$
$$+f_R\nabla^i G_{ik}-\frac{1}{2}f_T\nabla_k T-(\nabla^i f_R)R_{ik}$$
$$-\frac{1}{2}\nabla_k(h\Box T)=-\frac{1}{2}f_T\nabla_k T-\frac{1}{2}\nabla_k(h\Box T), \tag{14}$$

where we follow [35] using $\nabla^i G_{ik}=0$ and $(\nabla^i f_R)R_{ik}=(\Box\nabla_k-\nabla_k\Box)f_R$. Thus from the r.h.s. the conservation equation now reads

$$(8\pi\epsilon-f_T-h'\Box T-\Box h)\nabla^i T_{ik}$$
$$=-\frac{1}{2}f_T\nabla_k T+(T_{ik}+\Theta_{ik})\nabla^i(f_T+h'\Box T+\Box h)$$
$$+(f_T+h'\Box T+\Box h)\nabla^i\Theta_{ik}$$
$$-\frac{1}{2}\nabla_k(h\Box T)-\nabla^i\left[h'\nabla_i T\nabla_k T\right]+\frac{1}{2}\nabla_k(h'\nabla_l T\nabla^l T), \tag{15}$$

which also may be simplified by using (10) as

$$8\pi\epsilon\nabla^i T_{ik}=-\frac{1}{2}f_T\nabla_k T$$
$$-\frac{1}{2}\nabla_k(h\Box T)-\nabla^i[h'\nabla_i T\nabla_k T]$$
$$+\frac{1}{2}\nabla_k(h'\nabla_l T\nabla^l T). \tag{16}$$

Note here a very essential thing. Equations (13) and (16) do not contain true limits at $h=0$. If we want to find these limits, we must use Eqs. (11) and (15). The reason of such a kind of situation is quite understandable: for the limit $h=0$ the field equation (10) is just absent (trivial) and any simplification which would have been produced became impossible. Exactly for this reason the proposed theory has a very significant difference from the usual $F(R,T)$-gravity.

3 Some concrete examples for cosmological applications

Now let us discuss some particular cases of the gravitational field equations. For cosmological application we usually used

$$T_{ik}=(\rho+p)u_i u_k-p g_{ik}, \tag{17}$$

with $u_i u^i=1$ and $u^i\nabla_k u_i=0$. In this case the expression for Θ_{ik} takes the very simple form

$$\Theta_{ik}=-2T_{ik}-p g_{ik}. \tag{18}$$

3.1 The simplest case of functions: $f(R,T)=R+2f(T)$, $h(T)=\alpha T$

Let us discuss firstly the simplest case, $f(R,T)=R+2f(T)$, for the Universe with FLRW-metric,

$$ds^2=dt^2-a^2(t)(dx^2+dy^2+dz^2), \tag{19}$$

filled by dust matter ($p=0$, $T=\rho$). In this case Eq. (11) gives us

$$3H^2=8\pi\epsilon\rho+f(\rho)+\frac{1}{2}\alpha\dot\rho^2+\frac{1}{2}\alpha\rho(\ddot\rho+3H\dot\rho), \tag{20}$$
$$2\dot H+3H^2=f(\rho)-\frac{1}{2}\alpha\dot\rho^2+\frac{1}{2}\alpha\rho(\ddot\rho+3H\dot\rho), \tag{21}$$

and Eq. (10) reads

$$\alpha(\ddot\rho+3H\dot\rho)=-f'(\rho), \tag{22}$$

and finally Eq. (15) gives us

$$8\pi\epsilon(\dot\rho+3H\rho)=-f'(\rho)\dot\rho-\frac{1}{2}\alpha\rho\frac{d}{dt}\Box\rho-\frac{3}{2}\alpha\dot\rho\Box\rho, \tag{23}$$

which may be transformed to

$$8\pi\epsilon(\dot\rho+3H\rho)=\frac{1}{2}f'\dot\rho+\frac{1}{2}f''\rho\dot\rho. \tag{24}$$

It is easy to test our system: let us take the time derivative from (20), add (21) multiplied by $-3H$ and cut from the result the $-9H^3$-term by using (20); as a result we find Eq. (23).

Now we can see that in this simplest case it is possible to write a Friedman-like equation in the form $H^2=b(\rho)$, where b is some function of ρ. Indeed, expressing $\dot\rho$ from (24) and substituting into (20) and taking into account (22) we find

$$3H^2\left[1-\frac{\alpha}{2}\frac{3(8\pi\epsilon\rho)^2}{(\frac{1}{2}f'(\rho)+\frac{1}{2}f''\rho-8\pi\epsilon)^2}\right]$$
$$=8\pi\epsilon\rho-\frac{1}{2}\rho f'+f, \tag{25}$$

which for instance for $f=2\lambda\rho$, $\epsilon=1$ reads

$$3H^2\left[1-\frac{3\alpha}{2}\frac{(8\pi\rho)^2}{(\lambda-8\pi)^2}\right]=(8\pi+\lambda)\rho, \tag{26}$$

and for $f=\lambda\rho^2$, $\epsilon=1$

$$3H^2\left[1-\frac{3\alpha}{2}\frac{(8\pi\rho)^2}{(2\lambda\rho-8\pi)^2}\right]=8\pi\rho. \tag{27}$$

Note that these expressions look like the expressions which arise from brane cosmological models.

Also it is useful calculate the value $w_{\rm eff}\equiv-1-2\dot H/H^2$:

$$w_{\rm eff}=\frac{-2f+\rho f'+\alpha\dot\rho^2}{16\pi\rho\epsilon+2f-\rho f'+\alpha\dot\rho^2}. \tag{28}$$

Comparison with the case $h = 0$ and analogy with scalar field inflation

First of all, note that Eqs. (20) and (21) are very similar to the equations that describe cosmology with a scalar field (we need to substitute (22) there). Indeed there is a kinetic term $\frac{1}{2}\alpha\dot{\rho}^2$ and some kind of potential $f - \frac{1}{2}\rho f'$. This fact is very well understandable from (28): if we can ignore by $16\pi\rho$ with respect to $2f - \rho f'$ (or put by hand $\epsilon = 0$) we obtain $w_{\rm eff} = -1$ in the slow-roll regime when $\dot{\rho}^2 \ll 2f - \rho f'$. This means we have classical inflation on the scalar field. Thus our new term shows a behavior absolutely identical to the scalar field one.

Now let us compare our theory with the limit case $h = 0$, which was studied in previous investigations. In this case the field equation (10) is absent and we need to use (11) instead of (13). Equations (20) and (21) now reads

$$3H^2 = 8\pi\epsilon\rho + f + 2\rho f', \tag{29}$$

$$2\dot{H} + 3H^2 = f, \tag{30}$$

which corresponds to the EoS

$$w_{\rm eff} = \frac{-f}{8\pi\epsilon\rho + f + 2\rho f'},$$

and this is not equal to -1 even for $\epsilon = 0$. Moreover, there is no kinetic term here, which might have provided an exit from inflation. Thus we can see that the difference is very significant.

Unification of inflation and l.t.a.

Note also that it is possible to construct a cosmological model which will unify inflation and late time acceleration by a special shape of the function $f(\rho)$. Indeed let us put

$$f(\rho) = \frac{a_1\rho^n + b_1\rho^m}{a_2\rho^n + b_2\rho^m}, \tag{31}$$

where we imply all constants are positive and $n > m > 0$. This function has the limits

$$\lim_{\rho\to+\infty} f(\rho) = \frac{a_1}{a_2}, \quad \lim_{\rho\to+0} f(\rho) = \frac{b_1}{b_2}, \tag{32}$$

and we can see from Eqs. (20) and (21) that the term f will play the role of a cosmological constant in the beginning (the case of large values of ρ) and in the end (the case of small values of ρ) of the evolution of the Universe, whereas the existence of a kinetic term $\dot{\rho}^2$ may provide a transition between these two limit regimes. So in realistic cosmological models, the constants must satisfy the following conditions:

$$\frac{a_1}{a_2} \approx \Lambda_{\rm inf}, \quad \frac{b_1}{b_2} \approx \Lambda_0, \tag{33}$$

where $\Lambda_{\rm inf}$ is the cosmological constant in inflation epoch and Λ_0 is the cosmological constant in the present time.

3.2 Non-minimally coupling case: $f(R, T) = R + 2\gamma RT + 2f(T)$

In this case we have

$$\begin{aligned} 3H^2(1 + 2\gamma\rho) &= 8\pi\epsilon\rho \\ &\quad + f(\rho) + \frac{1}{2}h'(\rho)\dot{\rho}^2 + \frac{1}{2}h(\ddot{\rho} + 3H\dot{\rho}) - 6\gamma H\dot{\rho}, \end{aligned} \tag{34}$$

$$\begin{aligned} &(2\dot{H} + 3H^2)(1 + 2\gamma\rho) \\ &\quad = f(\rho) - \frac{1}{2}h'(\rho)\dot{\rho}^2 + \frac{1}{2}h(\ddot{\rho} + 3H\dot{\rho}) - 2\gamma(\ddot{\rho} + 2H\dot{\rho}), \end{aligned} \tag{35}$$

Eq. (10) reads

$$\begin{aligned} &h'(\rho)(\ddot{\rho} + 3H\dot{\rho}) + \frac{1}{2}h''(\rho)\dot{\rho}^2 \\ &\quad = -f'(\rho) + 6\gamma(\dot{H} + 2H^2), \end{aligned} \tag{36}$$

and finally Eq. (16) gives us

$$\begin{aligned} 8\pi\epsilon(\dot{\rho} + 3H\rho) = 6\gamma\dot{\rho}(\dot{H} + 2H^2) - f'\dot{\rho} \\ - \frac{3}{2}h'\dot{\rho}(\ddot{\rho} + 3H\dot{\rho}) \\ - \frac{1}{2}h''\dot{\rho}^3 - \frac{1}{2}h\frac{\mathrm{d}}{\mathrm{d}t}(\ddot{\rho} + 3H\dot{\rho}). \end{aligned} \tag{37}$$

It is easy to verify our system like in the previous case: taking the time derivative from (34), adding (35) multiplied by $-3H$ and cut from the result the $-9H^3$-term by using (34), we find Eq. (37). Note also that the energy conservation law (37) takes a very complicate form in this case.

Special solution

Now let us try to find some special solutions. We will find a solution in the form

$$\rho = \rho_0 \ln[1 + (t_c - t)],$$

which near the critical point $t = t_c$ may be decomposed as

$$\rho = \rho_0\left[(t_c - t) - \frac{1}{2}(t_c - t)^2 + \frac{1}{3}(t_c - t)^3\right] = \rho_0(t_c - t),$$

and therefore

$$\dot{\rho} = \rho_0[-1 + (t_c - t)], \quad \ddot{\rho} = \rho_0[-1 + 2(t_c - t)].$$

Now let us suppose for the sake of simplicity $h = \alpha\rho$ and $f(0) = 0$, which is quite natural if we do not want to introduce cosmological constant by hand. Substituting these expressions to (34) and taking into account that $f = f'(0)\rho$ we find near the critical point $t = t_c$ the algebraical equation for H

$$3H^2(1 + 2\gamma\rho_0 x) + H\left[\frac{3}{2}\alpha\rho_0^2 x - 6\gamma\rho_0(1 - x)\right]$$

$$+\left[\frac{3}{2}\alpha\rho_0^2 x - 8\pi\epsilon\rho_0 x - f'(0)\rho_0 x - \frac{1}{2}\alpha\rho_0^2\right] = 0, \quad (38)$$

where we denote $x \equiv (t_c - t)$. Equation (38) has the only positive solution (after linearization with respect to x)

$$H = \gamma\rho_0\left(1 + \sqrt{1 + \frac{\alpha}{6\gamma}}\right) + xF = H_0 + xF, \quad (39)$$

where F is the some function of parameters. Of course, such a kind of solution may not exist for arbitrary set of parameters, so let us ensure that the solution which we found satisfies all equations from the system (34)–(37). First of all note that all these equations will have a finite part, so for verifying we can neglect all terms $\sim x$. Substituting our solution into (36) we find

$$6\gamma\dot{H}_0 = f'(0) - 12\gamma H_0^2 - \alpha\rho_0(1 + 3H_0), \quad (40)$$

where $\dot{H}_0$ and H_0 denote the finite part of $\dot{H}$ and H, respectively. Substituting Eq. (40) into (37) we find

$$8\pi\rho_0 = \frac{1}{2}\alpha\rho_0^2(1 + 3H_0), \quad (41)$$

and this expression provides us with the relation between ρ_0 and the parameters of the theory, which is quite natural because ρ_0 is an integration constant which must be determined from the equations (the constraint equation). Finally, combining (40) and (35) we have

$$H_0^2 - \frac{1}{3\gamma}f'(0) + \frac{\alpha}{3\gamma}\rho_0(1 + 3H_0)$$
$$-\frac{1}{2}\alpha\rho_0^2 + 2\gamma\rho_0 + 4\gamma\rho_0 H_0 = 0, \quad (42)$$

which may be transformed by using (34) to

$$f'(0) = \alpha\rho_0(1 + 3H_0) + 6\gamma^2\rho_0 + 18\gamma\rho_0 H_0 - \alpha\rho_0^2, \quad (43)$$

which tells us that the solution found exists only for some specific shape of the function f.[3] We have a future solution near which for zero energy density ρ we have non-zero Hubble parameter H.[4]

Special case $\gamma = 0,\ f(T) = 0$

In this special case Eqs. (34)–(35) read

$$3H^2 = 8\pi\epsilon\rho + \frac{1}{2}h'(\rho)\dot{\rho}^2 + \frac{1}{2}h(\ddot{\rho} + 3H\dot{\rho}), \quad (44)$$

$$(2\dot{H} + 3H^2) = -\frac{1}{2}h'(\rho)\dot{\rho}^2 + \frac{1}{2}h(\ddot{\rho} + 3H\dot{\rho}), \quad (45)$$

Eq. (36) tells us

$$h'(\rho)(\ddot{\rho} + 3H\dot{\rho}) + \frac{1}{2}h''(\rho)\dot{\rho}^2 = 0, \quad (46)$$

and Eq. (37) takes the form

$$8\pi(\dot{\rho} + 3H\rho) = -\frac{3}{2}h'\dot{\rho}(\ddot{\rho} + 3H\dot{\rho})$$
$$-\frac{1}{2}h''\dot{\rho}^3 - \frac{1}{2}h\frac{\mathrm{d}}{\mathrm{d}t}(\ddot{\rho} + 3H\dot{\rho}), \quad (47)$$

and we have the effective EoS

$$w_{\mathrm{eff}} = \frac{h'\dot{\rho}^2 + \frac{hh''}{2h'}\dot{\rho}^2}{16\pi\epsilon\rho + h'\dot{\rho}^2 - \frac{hh''}{2h'}\dot{\rho}^2}. \quad (48)$$

We can see that in the case when $\dot{\rho}^2 \ll \rho$, which is quite natural for the power-law solutions $H \propto 1/t$, $w_{\mathrm{eff}} = 0$, and the dust stage is realized in the Universe. In some sense this is the limit case without potential, but only with kinetic term. Note also that this term may play a key role near the Big Rip solution.

Future singularities in special case $\gamma = 0,\ f(T) = 0$

Let us discuss the possibility of future singularities in our theory. Since we interested in future singularities which may appear due to the new kinetic term, we discuss this question in the special case $\gamma = 0$, $f(T) = 0$. According to conventional classification [36] there are four types of future singularities, which may be parameterized as follows (for more details see the original paper). If we have a Hubble parameter H near singularity, which occurs at the moment t_s,

$$H = H_0 + h_0(t_s - t)^{-\beta}, \quad (49)$$

the values of the parameter $\beta \geqslant 1$ correspond to Type I of singularities, values $-1 < \beta < 0$ to Type II, values $0 < \beta < 1$ to Type III and $\beta < -1$ to Type IV [37]. Let us study the possibility of the realization for every type consistently.

Type I: Let us suppose $h \propto \rho^n$ and $\rho \propto (t_s - t)^{\alpha}$. Substituting these relations and (49) to (44) and taking into account (46), we find that such a kind of particular solution may be realized for $\alpha = (1 - \beta)/(n + 1)$ and $\beta > 1$. Thus we can see that Type I of future singularity, which is also known as "Big Rip" may appear due to our new terms.

Type II: For this type of singularity we have $\dot{H} \gg H^2 = H_0^2$ near the t_s point. It means that terms from the r.h.s. of Eq. (45) also are much more higher than H^2 and the only possibility to satisfy Eq. (44) is to put $2h'^2 = h''h$, which leads to a very specific shape of the function $h = -1/(C_1\rho + C_2)$ and contradicts our basic requirements for stability. Thus we can see that this type of future singularity cannot be realized due to our new terms.

Type III: In this case the situation is very similar to the previous one: we have $\dot{H} \gg H^2 \gg 1$ and realization of this type of singularity is impossible.

[3] This situation may be changed if we use a more general function h.

[4] For this solution we have non-zero $\dot{H}$ as well, thus it does not correspond to an exact dS-solution.

Type IV: In this case we have $H = H_0$, $\dot{H} = 0$, while higher derivatives of H diverge. From (45) we can see that the only possibility to satisfy this equation is to have $\dot{\rho} = \mathrm{const} \neq 0$ near the point t_s. It implies the only possible solution $\rho = \Lambda_0 + \rho_0(t_s - t)$, but even in this case we need the additional condition $8\pi\Lambda_0 + h'\rho_0^2 = 0$ to have a consistent system (44) and (45). To satisfy this condition we need to put $h' < 0$, which contradicts the general stability condition, or to put $\Lambda_0 < 0$, which breaks the null energy condition. Thus we can see that this type of singularity also cannot be realized due to our new terms.

Finally, we can see that only Type I future singularities may appear in our theory, but it is also quite clear that in the most general case of non-minimal coupling $f(R, T)$ any types of future singularities may appear due to the non-trivial dependence of the function from R, as happens in the usual $f(R)$-gravity. We shall address this question in future investigations.

4 Basic inflationary model and its parameters

In this section let us try to calculate the parameters of inflation, which may be constructed by using the models previously described. The tensor–scalar ratio r and the spectral index of the primordial curvature perturbations n_s may be expressed by using the slow-roll parameters in the following way:

$$n_s = 1 - 6\epsilon + 2\eta,$$
$$r = 16\epsilon,$$

where the slow-roll indices are defined in terms of the Hubble rate as follows:

$$\epsilon = -\frac{\dot{H}}{H^2},$$
$$\eta = \epsilon - \frac{\ddot{H}}{2H\dot{H}}.$$

As an example of the calculations let us take the model described in Sect. 3.1 with arbitrary function $h(\rho)$.[5] Since during the inflation stage we have a slow-roll approximation, we can put the next relations $\ddot{\rho} \ll H\dot{\rho}$ and $\dot{\rho}^2 \ll f$ and now Eqs. (20)–(22) take the form

$$3H^2 = 8\pi\rho + f - \frac{h}{2h'}f', \tag{50}$$

$$3H^2 + 2\dot{H} = f - \frac{h}{2h'}f', \tag{51}$$

$$\dot{\rho} = -\frac{f'}{3Hh'}. \tag{52}$$

[5] The simplest case discussed above cannot produce viable inflationary parameters.

It is clear that in the slow-roll approximation the Hubble parameter changes slowly, so we have $\dot{H} \ll H^2$ and from comparison of Eqs. (50) and (51) we can see that we have equality only if we can neglect the $8\pi\rho$ term with respect to f. Thus such a kind of regime may be realized for any $f \propto \rho^n$ with $n > 1$, because in this regime we have large values of ρ. Finally, instead of (50) and (51) we have now

$$3H^2 = f - \frac{h}{2h'}f'. \tag{53}$$

Expressions for $\dot{H}$ and $\ddot{H}$, which are also needed for the slow-roll parameters, may be calculated by consistent differentiating of Eq. (53). Indeed we have for instance for $\dot{H}$

$$\begin{aligned}6\dot{H} &= \frac{1}{H}\left(f' - \frac{h'}{2h'}f' - \frac{h}{2h'}f'' + \frac{hh''}{2h'^2}f'\right)\dot{\rho} \\ &= \frac{-f'}{3H^2h'}\left(\frac{1}{2}f' - \frac{h}{2h'}f'' + \frac{hh''}{2h'^2}f'\right).\end{aligned} \tag{54}$$

Now according to the definition the number of e-foldings before the end of inflation $N_e = \ln\frac{a_e}{a_*}$, where a_e is the scale factor related to the end of inflation and a_* is the scale factor related to N_e. By using the definition of the Hubble rate this formula may be transformed as follows:

$$N_e = \int_{t_*}^{t_e} H(t)\mathrm{d}t = \int_{\rho_*}^{\rho_e} H(\rho)\frac{\mathrm{d}\rho}{\dot{\rho}} = \int_{\rho_e}^{\rho_*} 3H^2h'\frac{\mathrm{d}\rho}{f'},$$

where we used (52), and finally we obtain

$$N_e = \int_{\rho_e}^{\rho_*} \frac{h'}{f'}\left(f - \frac{h}{2h'}f'\right)\mathrm{d}\rho, \tag{55}$$

where $\rho_* \gg \rho_e$. Now let us focus on the case of power functions and put $f = \mu\rho^m$, $h = \nu\rho^n$. In this case Eq. (55) may easily be integrated and we have

$$N_e = \frac{\nu(2n - m)}{2m(n+1)}\rho_*^{n+1}. \tag{56}$$

Equation (52) takes the form

$$\dot{\rho} = \frac{-\mu m}{\nu n}\frac{\rho^{m-n}}{3H}, \tag{57}$$

and for Hubble rate and its derivatives we have

$$\begin{aligned}3H^2 &= \mu\rho^m\frac{2n - m}{2n}, \\ \dot{H} &= \frac{-m^2\mu}{6n\nu}\rho^{m-n-1}, \\ \ddot{H} &= \frac{m^3\mu^2(m - n - 1)}{6n^2\nu^2}\frac{\rho^{2m-2n-2}}{3H};\end{aligned}$$

finally collecting all terms we find for the slow-roll parameters

$$\begin{aligned}\epsilon &= \frac{m}{2(n+1)N_e}, \\ \eta &= \frac{2m - n - 1}{2(n+1)N_e} = \frac{1}{N_e}\left(\frac{m}{n+1} - \frac{1}{2}\right),\end{aligned}$$

where we take into account solution (56).

Now let us take for example $n = m = 2$. In this case we have, for $N_e = 50$, $r = 0.107$, $n_s = 0.9667$; and for $N_e = 60$, $r = 0.09$, $n_s = 0.9778$. We can see that the inflationary parameters lie near the boundary of viable region and taking more complicated functions may move them deeper into this region.

Finally, let us ensure that all variables have physical values. From (56) we can see that ρ_* has actually a large value. $\dot{\rho} < 0$ and $\dot{H} < 0$; it means that these variables both decrease during inflation, as they must. $\ddot{H}$ may has a different sign, depending on the parameters, we see that for $n = m = 2$ it has negative values. Finally all derivatives $\dot{\rho}$, $\dot{H}$, $\ddot{H}$ must be small in comparison with ρ and H; this fact puts some additional restrictions for the parameters m and n (otherwise our slow-roll approximation will broken). For instance in the case $n = m = 2$ we have according to our formulas $\dot{\rho} \propto \rho^{-1}$, $\dot{H} \propto \rho^{-1}$ and $\ddot{H} \propto \rho^{-3}$ and since the energy density ρ has a large value all time derivatives actually are small.

5 Conclusions

In this paper we discuss the possibility of a further generalization of $f(R, T)$-gravity by incorporating higher derivative terms $\Box T$ in the action. First of all we find that in the proposed theory inflationary scenarios appear quite naturally and may produce viable inflationary parameters. Moreover, higher derivative terms decrease more rapidly than the classic ones, but this may lead to future singularities of Type I. Another important thing: since new terms produce a contribution to the inflationary parameters it may resurrect such inflationary models as R^n with $n > 2$, which are already closed by modern observational data. We shall address this question in further investigations. It may be interesting also to generalize our theory by incorporating terms like $\sum c_i \Box^i T$, which may produce a ghost-free theory for some specific sets of coefficients c_i. Thus we propose a theory which is free from standard pathologies and promising for cosmological applications.

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